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Existence and regularity of mild solutions in some interpolation spaces for functional partial differential equations with nonlocal initial conditions

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Abstract: This paper is devoted to study the existence and regularity of mild solutions in some interpolation spaces for a class of functional partial differential equations with nonlocal initial conditions. The linear part is assumed to be a sectorial operator in Banach space X. The fractional power theory and α-norm are used to discuss the problem so that the obtained results can be applied to equations with terms involving spatial derivatives. Moreover, we present an example to illustrate the application of main results.

Keywords: Functional partial differential equation, Nonlocal condition, Strong solution, Analytic compact semigroup, Fractional power operator, α-norm

MSC: 34G20, 34K30, 35D35

1 Introduction

Let X be a Banach space with norm $|\cdot|^*$ and α be a constant such that $0 < \alpha < 1$. We denote by $C(J, D(A^\alpha))$ the Banach space of all the continuous functions from J to $D(A^\alpha)$ provided with the uniform norm topology and $D(A^\alpha)$ the domain of the linear operator A^α to be defined later.

In this paper, by using fractional power of operators and Schauder's fixed point theorem, we study the existence and regularity of mild solutions in some interpolation to the following functional partial differential equations with nonlocal initial conditions

$$u'(t) + Au(t) = f(t, u(t), u(a_1(t)), u(a_2(t)), \cdots, u(a_m(t))), \quad t \in (0, a], \tag{1}$$

$$u(0) = \sum_{i=1}^{p} c_i u(t_i), \tag{2}$$

where $u(\cdot)$ takes values in a subspace spaces of Banach space X, $A : D(A) \subset X \to X$ is a sectorial operator, m is a positive integer number, $J = [0, a]$, $a > 0$ is a constant, $a_j : J \to J$ are continuous functions such that $0 \le a_j(t) \le t$ for $j = 1, 2, \cdots, m$, $f : J \times (D(A^\alpha))^{m+1} \to X$ is Carathéodory continuous nonlinear function, $0 < t_1 < t_2 < \cdots < t_p < a$, c_i are real numbers, $c_i \ne 0$, $i = 1, 2, \cdots, p$.

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Nonlocal initial conditions can be applied in physics with better effect than the classical initial condition $u(0) = u_0$. For example, in [1] Deng used the nonlocal condition (2) to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (2) allows additional measurements at t_i, $i = 1, 2, \cdots, p$, which is more precise than the measurement just at $t = 0$. In [2], Byszewski pointed out that if $c_i \neq 0$, $i = 1, 2, \cdots, p$, then the results can be applied to kinematics to determine the location evolution $t \to u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \cdots, u(t_p)$, but we know that the nonlocal condition (2) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(0) = u_0$. The importance of nonlocal conditions have also been discussed in [3-6].

In [7], Fu and Ezzinbi studied the following neutral functional evolution equation with nonlocal conditions

$$\begin{cases} \frac{d}{dt}[x(t) + F(t, x(t)), x(b_1(t)), \cdots, x(b_m(t)))] + Ax(t) \\ \qquad = G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))), \quad 0 \le t \le a, \\ x(0) + g(x) = x_0 \in X, \end{cases} \tag{3}$$

where the operator $-A : D(A) \subset X \to X$ generates an analytic compact semigroup $T(t)$ $(t \ge 0)$ of uniformly bounded linear operators on a Banach space X, $F : [0, a] \times X^{m+1} \to X$, $G : [0, a] \times X^{n+1} \to X$, a_i, b_j, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$ and g are given functions satisfying some assumptions. The authors have proved the existence and regularity of mild solutions. In the subsequent years, various similar results have been established by many authors, see for example [8,9].

Recently, Chang and Liu [10] studied the existence of mild and strong solutions in some interpolation spaces between X and the domain of the linear part for the following semilinear evolution problem with nonlocal initial conditions:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \\ u(0) + g(u) = u_0 \in X, \end{cases} \tag{4}$$

where $T > 0$, the linear part A is a sectorial operator in X, f and g are given X-valued functions.

The aim of this paper is to establish some existence results of (1) and (2) without assuming Lipschitz condition on the nonlinear term and complete continuity on the nonlocal condition. The result obtained is a partial continuation of some results reported in [2,7,10-12]. It is worth mentioning that the theory of fractional power and α-norm are used to discuss the problems so that the results obtained in this chapter can be applied to the systems in which the nonlinear terms involve derivatives of spatial variables, and therefore, they have broader applicability.

The rest of this paper is organized as follows: We introduce some basic definitions and preliminary facts which will be used throughout this paper in section 2. The existence results of mild solutions are discussed in Section 3 by applying fixed point theorem. In Section 4, we provide some sufficient conditions to guarantee the regularity of solutions, that is, we obtain the existence of strong solutions. Finally, an example is presented in Section 5 to show the applications of the abstract results obtained.

2 Preliminaries

Assume $A : D(A) \subset X \to X$ be a sectorial operator and $-A$ generates an analytic compact semigroup $T(t)$ $(t \ge 0)$ on X. It is easy to see that $T(t)$ $(t \ge 0)$ is exponentially stable, i.e. there exist constants $M \ge 1$ and $\delta < 0$ such that

$$|T(t)|^* \le Me^{\delta t}, \quad \text{for each } t \ge 0, \tag{5}$$

the infimum of δ

$$\nu_0 = \inf \{\delta < 0 : \exists M \ge 1, \ |T(t)|^* \le Me^{\delta t}, \ \forall t \ge 0\}$$

is called a growth index of semigroup $T(t)$ $(t \geq 0)$, and at this point $\nu_0 < 0$. For each $\nu \in (0, |\nu_0|)$, by the definition of ν_0, there exists a constant $M \geq 1$, such that

$$|T(t)|^* \leq Me^{-\nu t}, \quad \text{for each } t \geq 0. \tag{6}$$

Define an equivalent norm in X by

$$\|x\| = \sup_{t\geq 0} |e^{\nu t}T(t)x|^*, \tag{7}$$

then $|x|^* \leq \|x\| \leq M|x|^*$. We denoted by $\|T(t)\|$ the norm of the operator $T(t)$ in space $(X, \|\cdot\|)$. By (6), we have

$$\|T(t)\| \leq e^{-\nu t} \leq 1, \quad \text{for each } t \geq 0. \tag{8}$$

and

$$\|T(t_i)\| \leq e^{-\nu t_i} \leq e^{-\nu t_1} < 1, \quad i = 1, 2, \cdots, p. \tag{9}$$

It is well known [13, Chapter 4, Theorem 2.9] that for any $u_0 \in D(A)$ and $h \in C^1(J, X)$, the initial value problem of linear evolution equation (LIVP)

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, \\ u(0) = u_0, \end{cases} \tag{10}$$

has a unique classical solution $u \in C^1((0, a], X) \cap C(J, D(A))$ expressed by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)h(s)ds. \tag{11}$$

If $u_0 \in X$ and $h \in L^1(J, X)$, the function u given by (11) belongs to $C(J, X)$, which is known as a mild solution of the LIVP (10). If a mild solution u of the LIVP (10) belongs to $W^{1,1}(J, X) \cap L^1(J, D(A))$ and satisfies the equation for a.e. $t \in J$, we call it a strong solution.

Throughout this paper, we assume that:

(P_0) $\sum_{i=1}^{p} |c_i| < 1$.

Applying (9) and assumption (P_0), we get $\left\| \sum_{i=1}^{\infty} c_i T(t_i) \right\| \leq \sum_{i=1}^{p} |c_i| e^{-\nu t_1} < 1$. Combining this with the operator spectrum theorem, we know that

$$B := \Big(I - \sum_{i=1}^{p} c_i T(t_i)\Big)^{-1} \tag{12}$$

exists and it is bounded. Furthermore, by Neumann expression, B can be expressed by

$$B = \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{p} c_i T(t_i)\Big)^n. \tag{13}$$

Therefore,

$$\|B\| \leq \sum_{n=0}^{\infty} \left\| \sum_{i=1}^{p} c_i T(t_i) \right\|^n = \frac{1}{1 - \left\| \sum_{i=1}^{p} c_i T(t_i) \right\|} \leq \frac{1}{1 - \sum_{i=1}^{p} |c_i|}. \tag{14}$$

To prove our main results, for any $h \in C(J, X)$, we consider the linear evolution equation nonlocal problem (LNP)

$$u'(t) + Au(t) = h(t), \quad t \in J, \tag{15}$$

$$u(0) = \sum_{i=1}^{p} c_i u(t_i). \tag{16}$$

Lemma 2.1. *If the condition (P_0) holds, then the LNP* (15)-(16) *has a unique mild solution $u \in C(J, X)$ given by*

$$u(t) = \sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) h(s) ds + \int_0^t T(t - s) h(s) ds, \qquad t \in J. \tag{17}$$

Proof. By (10) and (11), we know that Eq. (15) has a unique mild solution $u \in C(J, X)$ which can be expressed by

$$u(t) = T(t)u(0) + \int_0^t T(t - s)h(s)ds. \tag{18}$$

It follows from (18) that

$$u(t_i) = T(t_i)u(0) + \int_0^{t_i} T(t_i - s)h(s)ds, \qquad i = 1, 2, \cdots, p. \tag{19}$$

Combining (16) with (19), we have

$$u(0) = \sum_{i=1}^{p} c_i T(t_i)u(0) + \sum_{i=1}^{p} c_i \int_0^{t_i} T(t_i - s)h(s)ds. \tag{20}$$

Since $I - \sum_{i=1}^{p} c_i T(t_i)$ has a bounded inverse operator B,

$$u(0) = \sum_{i=1}^{p} c_i B \int_0^{t_i} T(t_i - s)h(s)ds. \tag{21}$$

By (18) and (21), we know that u satisfies (17).

Inversely, we can verify directly that the function $u \in C(J, X)$ given by (17) is a mild solution of the LNP (15)-(16). This completes the proof. □

We recall some concepts and conclusions on the fractional powers of A. Because $A : D(A) \subset X \to X$ is a sectorial operator, it is possible to define the fractional powers A^α for $0 < \alpha \le 1$. Now we define (see [13]) the operator $A^{-\alpha}$ by

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t)x dt, \quad x \in X,$$

where Γ denotes the gamma function. The operator $A^{-\alpha}$ is bijective and the operator A^α is defined by

$$A^\alpha = (A^{-\alpha})^{-1}.$$

We denote by $D(A^\alpha)$ the domain of the operator A^α.

Furthermore, we have the following properties which appeared in [13].

Lemma 2.2. *Let $0 < \alpha < 1$. Then:*

(i) $T(t) : X \to D(A^\alpha)$ for each $t > 0$.

(ii) $A^\alpha T(t)x = T(t)A^\alpha x, \quad$ for each $x \in D(A^\alpha)$ and $t \ge 0$.

(iii) For every $t > 0$, the linear operator $A^\alpha T(t)$ is bounded on X and there exist M_α such that

$$\|A^\alpha T(t)\| \le \frac{M_\alpha}{t^\alpha}.$$

(iv) For every $t > 0$, there exists a constant C'_α such that

$$\|(T(t) - I)A^{-\alpha}\| \le C'_\alpha t^\alpha.$$

(v) For $0 < \alpha < \beta \le 1$, we get $D(A^\beta) \hookrightarrow D(A^\alpha)$.

$D(A^\alpha)$ endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for all $x \in D(A^\alpha)$ is a Banach space. We denote it by X_α.

From now on, for the sake of brevity, we rewrite that

$$(t, u(t), u(a_1(t)), \cdots, u(a_m(t))) := (t, x(t)). \tag{22}$$

Definition 2.3. *A function $f : J \times X_\alpha^{m+1} \to X$ is said to be Carathéodory continuous provided that*
(i) for all $x \in X_\alpha^{m+1}$, $f(\cdot, x) : J \to X$ is measurable,
(ii) for a.e. $t \in [0, a]$, $f(t, \cdot) : \times X_\alpha^{m+1} \to X$ is continuous.

3 Existence of mild solutions

This section is devoted to the study of the existence of mild solutions for a class of functional partial differential equations with nonlocal initial conditions (1)-(2). In what follows, we will make the following hypotheses on the data of our problem (1)-(2).

(P_1) The function $f : J \times X_\alpha^{m+1} \to X$ is Carathéodory continuous and for some positive constant r, there exist constants $q \in [0, 1-\alpha)$, $\gamma > 0$ and function $\varphi_r \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$ such that for any $t \in J$ and $u_j \in X_\alpha$ satisfying $\|u_j\|_\alpha \le r$ for $j = 0, 1, 2, \cdots, m$,

$$\|f(t, u_0, u_1, \cdots, u_m)\| \le \varphi_r(t), \quad \liminf_{r \to +\infty} \frac{\|\varphi_r\|_{L^{\frac{1}{q}}([0,a])}}{r} := \gamma < +\infty.$$

Theorem 3.1. *Assume that the hypotheses (P_0) and (P_1) are satisfied. Then the problem* (1)-(2) *has at least one mild solution on $C(J, X_\alpha)$ provided that*

$$\frac{a^{1-\alpha-q} M_\alpha \gamma}{1 - \sum\limits_{i=1}^{p} |c_i|} \Big(\frac{1-q}{1-\alpha-q} \Big)^{1-q} < 1. \tag{23}$$

Proof. We consider the operator Q on $C(J, X_\alpha)$ defined by

$$(Qu)(t) = \sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds + \int_0^t T(t-s) f(s, x(s)) ds, \qquad t \in J. \tag{24}$$

With the help of Lemma 2.2, we know that the mild solution of the problem (1)-(2) is equivalent to the fixed point of the operator Q defined by (24). In what follows, we shall prove that the operator Q has at least one fixed point by applying the famous Schauder's fixed point theorem.

For this purpose, we first prove that there exists a positive constant R such that the operator Q defined by (24) maps the bounded closed convex set

$$D_R = \{u \in C(J, X_\alpha) : \ \|u(t)\|_\alpha \le R, \ t \in J\}$$

to D_R. If this is not true, there would exist $u_r \in D_r$ and $t_r \in J$ such that $\|(Qu_r)(t_r)\|_\alpha > r$ for each $r > 0$. However, by (9), the condition (P_1), Lemma 2.1 and Hölder inequality, we get that

$$\begin{aligned} r <& \|(Qu_r)(t_r)\|_\alpha \\ \le& \Big\| \sum_{i=1}^{p} c_i T(t_r) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds \Big\|_\alpha + \Big\| \int_0^{t_r} T(t_r - s) f(s, x(s)) ds \Big\|_\alpha \\ \le& \|B\| \sum_{i=1}^{p} |c_i| \int_0^{t_i} \|A^\alpha T(t_i - s) f(s, x(s))\| ds + \int_0^{t_r} \|A^\alpha T(t_r - s) f(s, x(s))\| ds \end{aligned} \tag{25}$$

$$
\begin{aligned}
&\leq \frac{\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\int_0^{t_i}\frac{M_\alpha}{(t_i-s)^\alpha}\varphi_r(s)ds+\int_0^{t_r}\frac{M_\alpha}{(t_r-s)^\alpha}\varphi_r(s)ds\\
&\leq \frac{M_\alpha\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\Big(\int_0^{t_i}(t_i-s)^{\frac{-\alpha}{1-q}}ds\Big)^{1-q}\Big(\int_0^{t_i}\varphi_r^{\frac{1}{q}}(s)ds\Big)^q\\
&\quad + M_\alpha\Big(\int_0^{t_r}(t_r-s)^{\frac{-\alpha}{1-q}}ds\Big)^{1-q}\Big(\int_0^{t_r}\varphi_r^{\frac{1}{q}}(s)ds\Big)^q\\
&\leq \frac{M_\alpha\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}a^{1-\alpha-q}\|\varphi_r\|_{L^{\frac{1}{q}}([0,a])}\\
&\quad + M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}a^{1-\alpha-q}\|\varphi_r\|_{L^{\frac{1}{q}}([0,a])}\\
&\leq \frac{a^{1-\alpha-q}M_\alpha}{1-\sum_{i=1}^{p}|c_i|}\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\|\varphi_r\|_{L^{\frac{1}{q}}([0,a])}.
\end{aligned}
$$

Divided by r on both sides of (25) and then take the lower limits as $r \to +\infty$ we get

$$\frac{a^{1-\alpha-q}M_\alpha\gamma}{1-\sum_{i=1}^{p}|c_i|}\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\geq 1,$$

which contradicts with the inequality (23). Therefore, there exists a positive constant R such that the operator Q maps D_R to D_R.

Below we will verify that $Q : D_R \to D_R$ is a completely continuous operator. From the definition of operator Q and the assumption (P_1) we note that Q is obviously continuous on D_R. Next, we shall prove that $\{Qu : u \in D_R\}$ is a family of equi-continuous functions. Let $u \in D_R$ and $t', t'' \in J$, $t' < t''$. By (24) one get that

$$
\begin{aligned}
(Qu)(t'')-(Qu)(t') &\leq (T(t'')-T(t'))\sum_{i=1}^{p}c_iB\int_0^{t_i}T(t_i-s)f(s,x(s))ds\\
&\quad+\int_{t'}^{t''}T(t''-s)f(s,x(s))ds\\
&\quad+\int_0^{t'}[T(t''-s)-T(t'-s)]f(s,x(s))ds\\
&:= B_1+B_2+B_3.
\end{aligned}
$$

It is obvious that

$$\|(Qu)(t'')-(Qu)(t')\|_\alpha \leq \sum_{k=1}^{3}\|B_k\|_\alpha.$$

Therefore, we only need to check $\|B_k\|_\alpha$ tend to 0 independently of $u \in D_R$ when $t''-t' \to 0$ for $k = 1, 2, 3$.

For B_1, by the condition (P_1), Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\Big\|\sum_{i=1}^{p} c_i B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds\Big\|_\alpha \\
&\le \frac{\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\int_0^{t_i}\|A^\alpha T(t_i - s) f(s, x(s))\| ds \\
&\le \frac{\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\int_0^{t_i}\frac{M_\alpha}{(t_i - s)^\alpha}\varphi_R(s) ds \\
&\le \frac{M_\alpha\sum_{i=1}^{p}|c_i|}{1-\sum_{i=1}^{p}|c_i|}\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])}.
\end{aligned} \tag{26}$$

Combining (26) and the strong continuity of the semigroup $T(t)$ $(t \ge 0)$, one can easily get that $\|B_1\|_\alpha \to 0$ as $t'' - t' \to 0$.

For B_2, taking assumption (P_1), Lemma 2.1 and Hölder inequality into account, we obtain

$$\begin{aligned}
\|B_2\|_\alpha &\le \int_{t'}^{t''}\|A^\alpha T(t'' - s) f(s, x(s))\| ds \\
&\le \int_{t'}^{t''}\frac{M_\alpha}{(t'' - s)^\alpha}\varphi_R(s) ds \\
&\le M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}(t'' - t')^{1-\alpha-q}\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\to 0 \ \ as \ \ t'' - t' \to 0.
\end{aligned}$$

For $t' = 0$, $0 < t'' \le a$, it is easy to see that $\|B_3\| = 0$. For $t' > 0$ and $0 < \epsilon < t'$ small enough, by the condition (P_1), Lemma 2.1, Hölder inequality and the equi-continuity of $T(t)$ $(t > 0)$, we know that

$$\begin{aligned}
\|B_3\|_\alpha &\le \int_0^{\epsilon}\|[T(t'' - t' + \frac{s}{2}) - T(\frac{s}{2})]A^\alpha T(\frac{s}{2}) f(t' - s, x(t' - s))\| ds \\
&\quad + \int_{\epsilon}^{t'}\|[T(t'' - t' + \frac{s}{2}) - T(\frac{s}{2})]A^\alpha T(\frac{s}{2}) f(t' - s, x(t' - s))\| ds \\
&\le 2\int_0^{\epsilon}\|A^\alpha T(\frac{s}{2}) f(t' - s, x(t' - s))\| ds \\
&\quad + \sup_{s\in[\epsilon,t']}\|T(t'' - t' + \frac{s}{2}) - T(\frac{s}{2})\|\int_{\epsilon}^{t'}\|A^\alpha T(\frac{s}{2}) f(t' - s, x(t' - s))\| ds \\
&\le 2M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\Big(\frac{\epsilon}{2}\Big)^{1-\alpha-q}\|\varphi_R\|_{L^{\frac{1}{q}}[0,a]} + \sup_{s\in[\epsilon,t']}\|T(t'' - t' + \frac{s}{2}) - T(\frac{s}{2})\| \\
&\quad M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\Big(\Big(\frac{t'}{2}\Big)^{\frac{1-\alpha-q}{1-q}} - \Big(\frac{\epsilon}{2}\Big)^{\frac{1-\alpha-q}{1-q}}\Big)^{1-q}\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\le 2M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\Big(\frac{\epsilon}{2}\Big)^{1-\alpha-q}\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} + \sup_{s\in[\epsilon,t']}\|T(t'' - t' + \frac{s}{2}) - T(\frac{s}{2})\| \\
&\quad M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}\Big(\frac{t' - \epsilon}{2}\Big)^{1-\alpha-q}\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])}
\end{aligned}$$

$$\to 0 \text{ as } t'' - t' \to 0 \text{ and } \epsilon \to 0.$$

As a result, $\|(Qu)(t'') - (Qu)(t')\|_\alpha \to 0$ independently of $u \in D_R$ as $t'' - t' \to 0$, which means that Q maps D_R into a family of equi-continuous functions.

It remains to prove that $V(t) = \{(Qu)(t) : u \in D_R\}$ is relatively compact in X_α. Obviously it is true in the case $t = 0$. Fix $t \in (0, a]$, for each $\epsilon \in (0, t)$ and $u \in D_R$, define

$$\begin{aligned}(Q_\epsilon u)(t) &= \sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds + \int_0^{t-\epsilon} T(t-s) f(s, x(s)) ds \\ &= T(t) \sum_{i=1}^{p} c_i B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon) f(s, x(s)) ds.\end{aligned}$$

The compactness of $T(t)$ $(t > 0)$ ensures that the sets $V_\epsilon(t) = \{(Q_\epsilon u)(t) : u \in D_R\}$ are relatively compact in X_α. Since

$$\begin{aligned}\|(Qu)(t) - (Q_\epsilon u)(t)\|_\alpha &\le \int_{t-\epsilon}^{t} \|T(t-s) f(s, x(s))\|_\alpha ds \\ &\le \int_{t-\epsilon}^{t} \|A^\alpha T(t-s) f(s, x(s))\| ds \\ &\le \int_{t-\epsilon}^{t} \frac{M_\alpha}{(t-s)^\alpha} \varphi_R(s) ds \\ &\le M_\alpha \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} \epsilon^{1-\alpha-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])}\end{aligned}$$

for every $u \in D_R$. Therefore, there are relatively compact sets $V_\epsilon(t)$ arbitrarily close to $V(t)$ for $t > 0$. Hence, $V(t)$ is also relatively compact in X_α for $t \ge 0$.

Thus, the Ascoli-Arzela theorem guarantees that $Q : D_R \to D_R$ is a completely continuous operator. According to the famous Schauder's fixed point theorem we know that the operator Q has at least one fixed point $u \in D_R$, and this fixed point is the desired mild solution of the problem (1)-(2) on $C(J, X_\alpha)$. This completes the proof. □

If we replace the condition (P_1) by the following condition:

(P_2) The function $f : J \times X_\alpha^{m+1} \to X$ is Carathéodory continuous and there exist a function $\phi \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$ $(q \in [0, 1-\alpha))$ and a nondecreasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, u_0, u_1, \cdots, u_m)\| \le \phi(t) \psi\Big(\sum_{j=0}^{m} \|u_j\|_\alpha\Big),$$

for all $u_j \in C(J, X_\alpha)$, $j = 0, 1, 2, \cdots, m$, and $t \in J$,

then we have the following existence result.

Theorem 3.2. *Assume that the hypotheses (P_0) and (P_2) are satisfied. Then the problem* (1)-(2) *has at least one mild solution on $C(J, X_\alpha)$ provided that there exists a positive constant R such that*

$$\frac{a^{1-\alpha-q} \psi((m+1)R) M_\alpha}{1 - \sum_{i=1}^{p} |c_i|} \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} \|\phi\|_{L^{\frac{1}{q}}([0,a])} \le R. \tag{27}$$

Proof. From the proof of Theorem 3.1, we know that the mild solution of the problem (1)-(2) is equivalent to the fixed point of the operator Q defined by (24). In what follows, we prove that there exists a positive constant R such that the operator Q maps the set D_R to itself. For any $u_j \in D_R$, $j = 0, 1, 2, \cdots, m$, and $t \in J$, by (8), (14), (24), (27), the hypothesis (P_2) and Hölder inequality, we have

$$\begin{aligned}
&\|(Qu)(t)\|_\alpha \\
\leq\ & \Big\|\sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds\Big\|_\alpha + \int_0^t \|A^\alpha T(t-s) f(s, x(s))\| ds \\
\leq\ & \frac{\sum_{i=1}^{p} |c_i|}{1 - \sum_{i=1}^{p} |c_i|} \int_0^{t_i} \frac{\psi((m+1)R) M_\alpha}{(t_i - s)^\alpha} \phi(s) ds + \int_0^t \frac{\psi((m+1)R) M_\alpha}{(t-s)^\alpha} \phi(s) ds \\
\leq\ & \frac{\psi((m+1)R) M_\alpha \sum_{i=1}^{p} |c_i|}{1 - \sum_{i=1}^{p} |c_i|} \Big(\int_0^{t_i} (t_i - s)^{\frac{-\alpha}{1-q}} ds\Big)^{1-q} \Big(\int_0^{t_i} \phi^{\frac{1}{q}}(s) ds\Big)^q \\
& + \psi((m+1)R) M_\alpha \Big(\int_0^t (t-s)^{\frac{-\alpha}{1-q}} ds\Big)^{1-q} \Big(\int_0^t \phi^{\frac{1}{q}}(s) ds\Big)^q \\
\leq\ & \frac{a^{1-\alpha-q} \psi((m+1)R) M_\alpha}{1 - \sum_{i=1}^{p} |c_i|} \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} \|\phi\|_{L^{\frac{1}{q}}([0,a])} \\
\leq\ & R,
\end{aligned}$$

which implies $Q(D_R) \subset D_R$. By adopting a completely similar method which used in the proof of Theorem 3.1, we can prove that the problem (1)-(2) has at least one mild solution on $C(J, X_\alpha)$. This completes the proof. □

Similarly to Theorem 3.2, we have the following result.

Corollary 3.3. *Assume that the hypotheses (P_0) and (P_2) are satisfied. Then the problem* (1)-(2) *has at least one mild solution on $C(J, X_\alpha)$ provided that*

$$\liminf_{r \to +\infty} \frac{\psi((m+1)r)}{r} < \frac{1 - \sum_{i=1}^{p} |c_i|}{a^{1-\alpha-q} M_\alpha \|\phi\|_{L^{\frac{1}{q}}[0,a]}} \Big(\frac{1-q}{1-\alpha-q}\Big)^{q-1}. \tag{28}$$

4 The regularity of solutions

In this section, we discuss the existence of strong solutions for the problem (1)-(2), that is, we shall provide conditions to allow the differential for mild solutions of the problem (1)-(2). To do this, we need the following lemma:

Lemma 4.1 ([12])**.** *If X is a reflexive Banach space, then X_α is also a reflexive Banach space.*

Theorem 4.2. *Let X be a reflexive Banach space. If there exists $\alpha' \in (\alpha, 1)$, such that the hypothesis (P_0), (P_3)there exists a positive constant $\overline{L}$, such that for any t'', $t' \in J$ and x_j, $y_j \in X_\alpha$, $j = 0, 1, 2, \cdots, m$,*

$$\|f(t'', x_0, x_1, \cdots, x_m) - f(t', y_0, y_1, \cdots, y_m)\| \leq \overline{L}\Big(|t'' - t'|^{\alpha' - \alpha} + \sum_{j=0}^{m} \|x_j - y_j\|_\alpha\Big)$$

(P_4)There exist constants $l_j > 0$, $j = 1, 2, \cdots, m$, such that for any t'', $t' \in J$,

$$|a_j(t'') - a_j(t')| \leq l_j |t'' - t'|, \quad j = 1, 2, \cdots, m,$$

$(P_5) \frac{a^{1-\alpha}\overline{L}M_\alpha}{1-\alpha}\left(1+\sum_{j=1}^{m} l_j^{\alpha'-\alpha}\right) < 1,$

hold, then the problem (1)-(2) *has a strong solution.*

Proof. Let Q be the operator defined in the proof of Theorem 3.1. By the conditions (P_0), (P_3) and (P_5), one can use the same argument as in the proof of Theorem 3.1 to deduce that there exists a constant $R > 0$, such that $Q(D_R) \subset D_R$. For this R, consider the set

$$D = \{u \in C(J, X_\alpha) : \ \|u\|_\alpha \le R, \ \|u(t'') - u(t')\|_\alpha \le L^*|t'' - t'|^{\alpha'-\alpha}, \ t'', t' \in J, \ |t'' - t'| < 1\} \tag{29}$$

for some L^* large enough. It is clear that D is a convex, closed and nonempty set. We shall prove that Q has a fixed point on D. For any $u \in D$ and $t', t'' \in J$, $0 < t'' - t' < 1$, we have

$$\begin{aligned} \|(Qu)(t'') - (Qu)(t')\|_\alpha \le & \left\|(T(t'') - T(t'))\sum_{i=1}^{p} c_i B \int_0^{t_i} T(t_i - s)f(s, x(s))ds\right\|_\alpha \\ & + \left\|\int_0^{t'} T(s)(f(t'' - s, x(t'' - s)) - f(t' - s, x(t' - s)))ds\right\|_\alpha \\ & + \left\|\int_0^{t''-t'} T(t'' - s)f(s, x(s))ds\right\|_\alpha \\ := & I_1 + I_2 + I_3. \end{aligned} \tag{30}$$

By (8), (30), Lemma 2.1 and the condition (P_3), we know that

$$\begin{aligned} I_1 = & \left\|(T(t'') - T(t'))\sum_{i=1}^{p} c_i B \int_0^{t_i} T(t_i - s)f(s, x(s))ds\right\|_\alpha \\ = & \left\|T(t')(T(t'' - t') - I)A^{-(\alpha'-\alpha)}\sum_{i=1}^{p} c_i B \int_0^{t_i} A^{\alpha'} T(t_i - s)f(s, x(s))ds\right\| \\ \le & C'_{\alpha'-\alpha}(t'' - t')^{\alpha'-\alpha}\left\|\sum_{i=1}^{p} c_i B \int_0^{t_i} A^{\alpha'} T(t_i - s)f(s, x(s))ds\right\| \\ \le & \frac{C'_{\alpha'-\alpha}\sum_{i=1}^{p}|c_i|}{1 - \sum_{i=1}^{p}|c_i|}\int_0^{t_i} M_{\alpha'}(t_i - s)^{-\alpha'}\|f(s, x(s))\|ds \cdot (t'' - t')^{\alpha'-\alpha} \\ \le & \frac{a^{1-\alpha'}M_{\alpha'}C'_{\alpha'-\alpha}\sum_{i=1}^{p}|c_i|\{\overline{L}(a + (1 + m)R) + \|f(0, 0)\|\}}{(1 - \alpha')\left(1 - \sum_{i=1}^{p}|c_i|\right)}(t'' - t')^{\alpha'-\alpha}. \end{aligned} \tag{31}$$

According to the assumptions (P_3) and (P_4), Lemma 2.1, (29) and (30), we have

$$\begin{aligned} I_2 =& \Big\| \int_0^{t'} T(s)[f(t''-s,x(t''-s)) - f(t'-s,x(t'-s))]ds \Big\|_\alpha \\ =& \Big\| \int_0^{t'} A^\alpha T(s)(f(t''-s,x(t''-s)) - f(t'-s,x(t'-s)))ds \Big\| \\ \leq& M_\alpha \int_0^{t'} s^{-\alpha} \|f(t''-s,x(t''-s)) - f(t'-s,x(t'-s))\| ds \\ \leq& \overline{L} M_\alpha \int_0^{t'} s^{-\alpha} \Big(|t''-t'|^{\alpha'-\alpha} + \|u(t''-s) - u(t'-s)\|_\alpha \\ &+ \sum_{j=1}^m \|u(a_j(t''-s)) - u(a_j(t'-s))\|_\alpha \Big) ds \\ \leq& \overline{L} M_\alpha \int_0^{t'} s^{-\alpha} \Big(|t''-t'|^{\alpha'-\alpha} + L^* |t''-t'|^{\alpha'-\alpha} \\ &+ \sum_{j=1}^m L^* |a_j(t''-s) - a_j(t'-s)|^{\alpha'-\alpha} \Big) ds \\ \leq& \frac{a^{1-\alpha} \overline{L} M_\alpha}{1-\alpha} \Big[|t''-t'|^{\alpha'-\alpha} + L^* \Big(1 + \sum_{j=1}^m l_j^{\alpha'-\alpha}\Big) |t''-t'|^{\alpha'-\alpha} \Big]. \end{aligned} \tag{32}$$

Using the condition (P_3), Lemma 2.1, (30) and (29), we get that

$$I_3 = \Big\| \int_0^{t''-t'} A^\alpha T(t''-s) f(s,x(s))ds \Big\| \leq \frac{M_\alpha \{\overline{L}(a + (1+m)R) + \|f(0,0)\|\}}{1-\alpha} |t''-t'|^{\alpha'-\alpha}. \tag{33}$$

Thus, from (30)-(33) we get that

$$\|(Qu)(t'') - (Qu)(t')\|_\alpha \leq \Big\{ K_0 + L^* \Big[\frac{a^{1-\alpha} \overline{L} M_\alpha}{1-\alpha} \Big(1 + \sum_{j=1}^m l_j^{\alpha'-\alpha}\Big) \Big] \Big\} |t''-t'|^{\alpha'-\alpha}, \tag{34}$$

where K_0 is a constant independent of L^*. Since the condition (P_5) implies that

$$K^* := \frac{a^{1-\alpha} \overline{L} M_\alpha}{1-\alpha} \Big(1 + \sum_{j=1}^m l_j^{\alpha'-\alpha}\Big) < 1,$$

then

$$\|(Qu)(t'') - (Qu)(t')\|_\alpha \leq L^* |t''-t'|^{\alpha'-\alpha},$$

whenever

$$L^* \geq \frac{K_0}{1-K^*}.$$

Therefore, Q has a fixed point u which is a mild solution of the problem (1)-(2). By the above calculation, we see that for this $u(\cdot)$ and the following function

$$F(t) := \int_0^t T(t-s) f(s,x(s))ds$$

are Hölder continuous. Since the space X_α is reflexive by assumption and Lemma 4.1, $u(\cdot)$ is almost everywhere differentiable on $(0,a]$ and $u'(\cdot) \in L^1(J,X)$. A similar argument shows that F also have this property. Furthermore, we can obtain that

$$F'(t) = f(t,x(t)) - A \int_0^t T(t-s) f(s,x(s))ds. \tag{35}$$

Therefore, by (35) we get for almost all $t \in J$ that

$$\begin{aligned} \frac{d}{dt}u(t) &= \frac{d}{dt}\Big(\sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds + \int_0^{t} T(t-s) f(s, x(s)) ds\Big) \\ &= -A\Big(\sum_{i=1}^{p} c_i T(t) B \int_0^{t_i} T(t_i - s) f(s, x(s)) ds\Big) \\ &\quad + f(t, x(t)) - A \int_0^{t} T(t-s) f(s, x(s)) ds \\ &= -Au(t) + f(t, x(t)). \end{aligned}$$

This shows that u is a strong solution of the problem (1)-(2). This completes the proof. □

5 Example

In this section we apply some of the results established in this paper to the following first order parabolic partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition

$$\begin{cases} \frac{\partial}{\partial t} w(x,t) - \frac{\partial^2}{\partial x^2} w(x,t) = f(x, t, w(x,t), \frac{\partial}{\partial x} w(x,t), w(x, a_1(t)), \frac{\partial}{\partial x} w(x, a_1(t))), \\ \qquad\qquad (x,t) \in [0,\pi] \times J, \\ w(0,t) = w(\pi, t) = 0, \quad t \in J, \\ w(x,0) = \sum_{i=1}^{p} \arctan \frac{1}{2i^2} w(x,i), \quad x \in [0,\pi], \end{cases} \tag{36}$$

where the functions f and a_1 will be described below.

Set $X = L^2([0,\pi], \mathbb{R})$ with the norm $\|\cdot\|_{L^2}$. Then X is reflexive Banach space. Define an operator A in reflexive Banach space X by

$$D(A) = W^{2,2}(0,\pi) \cap W_0^{1,2}(0,\pi), \quad Au = -\frac{\partial^2}{\partial x^2} u.$$

From [13] we know that $-A$ generates a strong continuous semigroup $T(t)$ $(t \geq 0)$, which is compact, analytic and exponentially stable in X. Furthermore, A has discrete spectrum with eigenvalues $x_n = n^2, n \in \mathbb{N}$, associated normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Then the following properties hold:

(a) If $u \in D(A)$, then

$$Au = \sum_{n=0}^{\infty} n^2 \langle u, e_n \rangle e_n.$$

(b) For each $u \in X$,

$$A^{-\frac{1}{2}} u = \sum_{n=1}^{\infty} \frac{1}{n} \langle u, e_n \rangle e_n.$$

In particular, $\|A^{-\frac{1}{2}}\| = 1$. Hence, it follows that $\|A^{-1}\| \leq 1$.

(c) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} = \sum_{n=1}^{\infty} n \langle u, e_n \rangle e_n$$

on the space $D(A^{\frac{1}{2}}) = \{u(\cdot) \in X, \sum_{n=1}^{\infty} n \langle u, e_n \rangle e_n \in X\}$.

To prove the main result of this section, we need the following lemma.

Lemma 5.1 ([14]). *If $u \in D(A^{\frac{1}{2}})$, then u is absolutely continuous with $u' \in X$ and $\|u'\|_{L^2} = \|A^{\frac{1}{2}}u\|_{L^2}$.*

According to Lemma 5.1, we can define the Banach space $X_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$. Then for $u \in X_{\frac{1}{2}}$, we have

$$\|u\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}u\|_{L^2} = \|u'\|_{L^2}.$$

We assume that the nonlinear function g satisfies the following assumption:
(P_6)The function $g : [0, \pi] \times J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and there is a function $h \in L^\infty(J, \mathbb{R})$ such that $|g(x, t, \zeta, \xi, \eta, \rho)| \le h(t)$, for $t \in J$ and $\zeta, \xi, \eta, \rho \in \mathbb{R}$.

For each $t \in J$ and $u \in X_{\frac{1}{2}}$, we define

$$\alpha = \frac{1}{2}, \qquad m = 1, \qquad u(t) = w(\cdot, t),$$

$$f(t, u(t), u(a_1(t)))(x) = g(x, t, w(x, t), \frac{\partial}{\partial x} w(x, t), w(x, a_1(t)), \frac{\partial}{\partial x} w(x, a_1(t))),$$

$$c_i = \sum_{i=1}^{p} \arctan \frac{1}{2i^2}, \ t_i = i, \ i = 1, 2, \cdots, p.$$

Then $f : [0, 1] \times X_{\frac{1}{2}} \times X_{\frac{1}{2}} \to X$, and the parabolic partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial conditions (36) can be rewritten into the abstract form of problem (1)-(2) for $m = 1$. Since $\sum_{i=1}^{p} |c_i| \le \sum_{i=1}^{p} \arctan \frac{1}{2i^2} = \frac{\pi}{4} < 1$, the condition (P_0) is satisfied. Below we will verify that f satisfies the condition (P_1). In fact, it follows from assumption (P_6) that

$$\sup_{\|u\|_{\frac{1}{2}} \le r} \|f(t, u(t), u(a_1(t)))\| \le h(t) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\|h\|_{L^{\frac{1}{q}}([0,a])}}{r} = 0 < +\infty.$$

Therefore, Theorem 3.1 ensures the following existence result.

Theorem 5.2. *If the nonlinear function g satisfies the assumption* (P_6), *then the parabolic partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial conditions* (36) *has at least one mild solution.*

In order to obtain the existence of strong solutions to the parabolic partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial conditions (36), the following assumptions are also needed.
(P_7) The function $g : [0, \pi] \times J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and there is constant $L > 0$ and $\alpha' \in (\alpha, 1)$ such that

$$\begin{aligned} &|g(x, t'', \zeta_2, \xi_2, \eta_2, \rho_2) - g(x, t', \zeta_1, \xi_1, \eta_1, \rho_1)| \\ &\le L(|t'' - t'|^{\alpha' - \alpha} + |\zeta_2 - \zeta_1| + |\xi_2 - \xi_1| + |\eta_2 - \eta_1| + |\rho_2 - \rho_1|), \end{aligned}$$

(P_8) There exist constants $l_1 > 0$ such that

$$|a_1(t'') - a_1(t')| \le l_1|t'' - t'|, \quad \text{for} \quad t'', t' \in J.$$

For each $\phi_j, \psi_j, \in X_{\frac{1}{2}}$, $j = 1, 2$ and $t', t'' \in J$, we have

$$\begin{aligned} &\|f(t'', \phi_2, \psi_2) - f(t', \phi_1, \psi_1)\|_{L^2} \\ &= \Big[\int_0^\pi \Big(g(x, t'', \phi_2(x, t''), \frac{\partial}{\partial x}\phi_2(x, t''), \psi_2(x, t''), \frac{\partial}{\partial x}\psi_2(x, t'')) \\ &\quad - g(x, t', \phi_1(x, t''), \frac{\partial}{\partial x}\phi_1(x, t''), \psi_1(x, t''), \frac{\partial}{\partial x}\psi_2(x, t''))\Big)^2 dx \Big]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq \Big[\int_0^{\pi} L^2\Big(|t''-t'|^{\alpha'-\alpha}+|\phi_2(x,t'')-\phi_1(x,t')|+|\frac{\partial}{\partial x}\phi_2(x,t'')-\frac{\partial}{\partial x}\phi_2(x,t')| \\
&\quad +|\psi_2(x,t'')-\psi_1(x,t')|+|\frac{\partial}{\partial x}\psi_2(x,t'')-\frac{\partial}{\partial x}\psi_2(x,t')|\Big)^2 dx\Big]^{\frac{1}{2}} \\
&\leq L\Big(\sqrt{\pi}|t''-t'|^{\alpha'-\alpha}+\|\phi_2-\phi_1\|_{L^2}+\|(\phi_2-\phi_1)'\|_{L^2} \\
&\quad +\|\psi_2-\psi_1\|_{L^2}+\|(\psi_2-\psi_1)'\|_{L^2}\Big) \\
&\leq L\Big(\sqrt{\pi}|t''-t'|^{\alpha'-\alpha}+\|(\phi_2-\phi_1)'\|_{L^2} \\
&\quad +\|(\phi_2-\phi_1)'\|_{L^2}+\|(\psi_2-\psi_1)'\|_{L^2}+\|(\psi_2-\psi_1)'\|_{L^2}\Big) \\
&\leq 2L(|t''-t'|^{\alpha'-\alpha}+\|\phi_2-\phi_1\|_{\frac{1}{2}}+\|\psi_2-\psi_1\|_{\frac{1}{2}}).
\end{aligned}
$$

Hence (P_3) holds with $\overline{L} = 2L$. Therefore, it from Theorem 4.2, we have the following result.

Theorem 5.3. *If the assumptions (P_7) and (P_8) are satisfied, then the parabolic partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial conditions* (36) *has a strong solution solution provided that* $4a^{1/2}LM_\alpha\Big(1+l_1^{\alpha'-1/2}\Big) < 1$.

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References

[1] Deng K., Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, J. Math. Anal. Appl., 1993, 179, 630-637.

[2] Byszewski L., Existence and uniqueness of a classical solution to a functional differential abstract nonlocal Cauchy problem, J. Math. Appl. Stoch. Anal., 1999, 12, 91-97.

[3] Boucherif A., Semilinear evolution inclutions with nonlocal conditions, Appl. Math. Letters, 2009, 22, 1145-1149.

[4] Byszewski L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Appl. Anal., 1991, 12, 494-505.

[5] Ezzinbi K., Fu X., Hilal K., Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions, Nonlinear Anal., 2007, 67, 1613-1622.

[6] Xiao T.J., Liang J., Existence of classical solutions to nonautonomous nonlocal parabolic problems, Nonlinear Anal., 2005, 63, 225-232.

[7] Fu X., Ezzinbi K., Existence of solutions for neutral functional differential evolution equations with nonlocal conditions, Nonlinear Anal., 2003, 54, 215-227.

[8] Bahuguna D., Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential problems, Nonlinear Anal., 2004, 57, 1021-1028.

[9] Bahuguna D., Agarwal S., Approximations of solutions to neutral didderential equations with nonlocal history conditions, J. Math. Anal. Appl., 2006, 317, 583-602.

[10] Chang J.C., Liu H., Existence of solutions in some interpolation spaces for a class of semilinear evolutions with nonlocal initial conditions, J. Funct. Spaces Appl., 2013, Art. ID 451252, 11 pp.

[11] Liang J., Van Casteren J., Xiao T.J., Nonlocal Cauchy problems for semilinear evolution equations, Nonlinear Anal., 2002, 50, 173-189.

[12] Liu H., Chang J.C., Existence for a class of partial differential equations with nonlocal conditions, Nonlinear Anal., 2009, 70, 3076-3083.

[13] Pazy A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Berlin: Springer-Verlag, 1983.

[14] Travis C.C., Webb G.F., Existence, stability, and compactness in the α-norm for partial functional differential equations, Trans. Amer. Math. Soc., 1978, 240, 129-143.