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Oscillation of first order linear differential equations with several non-monotone delays

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Abstract: Consider the first-order linear differential equation with several retarded arguments

$$x^{'}(t)+\sum_{k=1}^{n}p_k(t)x(\tau_k(t))=0,\quad t\geq t_0,$$

where the functions $p_k, \tau_k \in C([t_{0,}\infty), \mathbb{R}^+)$, $\tau_k(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty}\tau_k(t)=\infty$, for every $k=1,2,\ldots,n$. Oscillation conditions which essentially improve known results in the literature are established. An example illustrating the results is given.

Keywords: Oscillation, Differential equations, Non-monotone delays

MSC: 34K11, 34K06

1 Introduction

This paper is concerned with the oscillation of the first order differential equation with several delays of the form

$$x^{'}(t)+\sum_{k=1}^{n}p_k(t)x(\tau_k(t))=0,\qquad t\geq t_0, \tag{1}$$

where $p_k, \tau_k \in C([t_0,\infty),[0,\infty))$, such that $\tau_k(t) < t$ and $\lim\limits_{t\to\infty}\tau_k(t)=\infty$, $k=1,2,\ldots,n$.

Let $T_0 \in [t_0,\infty)$, $\tau(t)=\min\{\tau_k(t) : k=1,\ldots,n\}$ and $\tau_{(-1)}(t)=\inf\{\tau(s) : s\geq t\}$. By a solution of Eq. (1) we understand a function $x\in C([t_0,\infty),\mathbb{R})$ continuously differentiable on $[\tau_{(-1)}(T_0),\infty)$ which satisfies (1) for $t\geq\tau_{(-1)}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

We assume throughout this work that there exist $t_1 \geq t_0$, a family of nondecreasing continuous functions $\{g_k(t)\}_{k=1}^n$ and a nondecreasing continuous function $g(t)$ such that

$$\tau_k(t)\leq g_k(t)\leq g(t)\leq t,\quad t\geq t_1,\quad k=1,2,\ldots,n.$$

To simplify the notations, we denote by $\lambda(\xi)$ the smaller root of the equation $\mathrm{e}^{\xi\lambda}=\lambda$, $\xi\geq 0$ and

$$c(\xi)=\frac{1-\xi-\sqrt{1-2\xi-\xi^2}}{2},\quad 0\leq\xi\leq\frac{1}{\mathrm{e}}.$$

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Next, we mention some known oscillation criteria for Eq. (1).

In the case that the arguments $\tau_k(t)$ are monotone the following sufficient oscillation conditions have been established. In 1978, Ladde [1] and in 1982 Ladas and Stavroulakis [2] obtained the sufficient oscillatory criterion

$$\liminf_{t\to\infty} \int_{\tau_{\max}(t)}^{t} \sum_{k=1}^{n} p_k(s)ds > \frac{1}{\mathrm{e}},$$

where $\tau_{\max}(t) = \max_{1\le k\le n}\{\tau_k(t)\}$.

In 1984, Hunt and Yorke [3] established the condition

$$\liminf_{t\to\infty} \sum_{k=1}^{n} p_k(t)(t-\tau_k(t)) > \frac{1}{\mathrm{e}},$$

where $(t-\tau_k(t)) \le \tau_0$, for some $\tau_0 > 0$, $1 \le k \le n$.

Also, in 1984, Fukagai and Kusano [4] established the following result.

Assume that there is a continuous nondecreasing function $\tau^*(t)$ *such that* $\tau_k(t) \le \tau^*(t) \le t$ *for* $t \ge t_0$, $1 \le k \le n$. *If*

$$\liminf_{t\to\infty} \int_{\tau^*(t)}^{t} \sum_{k=1}^{n} p_k(s)ds > \frac{1}{e}, \tag{2}$$

then all solutions of Eq.(1) *oscillate. If, on the other hand, there exists a continuous nondecreasing function* $\tau_\star(t)$ *such that* $\tau_\star(t) \le \tau_k(t)$ *for* $t \ge t_0$, $1 \le k \le n$, $\lim_{t\to\infty} \tau_\star(t) = \infty$ *and* for all sufficiently large t,

$$\int_{\tau_\star(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \le \frac{1}{e},$$

then Eq.(1) *has a non-oscillatory solution.*

In 2003, Grammatikopoulos, Koplatadze and Stavroulakis [5] improved the above results as follows:

Assume that the functions τ_k *are nondecreasing for all* $k \in \{1, \dots, n\}$,

$$\int_{0}^{\infty} \left|p_i(t) - p_j(t)\right| dt < +\infty, \quad i, j = 1, \dots, n,$$

and

$$\liminf_{t\to\infty} \int_{\tau_k(t)}^{t} p_k(s)ds > 0, \quad k = 1, \dots, n.$$

If

$$\sum_{k=1}^{n} \left(\liminf_{t\to\infty} \int_{\tau_k(t)}^{t} p_k(s)ds \right) > \frac{1}{e},$$

then all solutions of Eq. (1) *oscillate.*

In the case of non-monotone arguments we mention the following known oscillation results. In 2015 Infante, Koplatadze and Stavroulakis [6] obtained the following sufficient oscillation conditions:

$$\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{g_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{g_i(t)} \sum_{k=1}^{n} p_k(u) \exp\Big(\int_{\tau_k(u)}^{u} \sum_{\ell=1}^{n} p_\ell(v)dv \Big) du \right) ds \right]^{\frac{1}{n}} > \frac{1}{n^n},$$

and

$$\limsup_{\epsilon\to 0^+} \left[\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{g_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{g_i(t)} \sum_{k=1}^{n} (\lambda(q_k) - \epsilon) p_k(u) du \right) ds \right]^{\frac{1}{n}} \right] > \frac{1}{n^n}, \tag{3}$$

where

$$q_k = \liminf_{t\to\infty} \int_{\tau_k(t)}^{t} p_k(s)ds > 0, \quad k = 1, 2, \ldots, n. \tag{4}$$

Also, in 2015 Koplatadze [7] derived the following three conditions. The first one takes the form

$$\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{g_j(t)}^{t} p_i(s) \exp\left(n \int_{\tau_i(s)}^{g_i(t)} \left(\prod_{\ell=1}^{n} p_\ell(\xi) \right)^{\frac{1}{n}} \psi_k(\xi) d\xi \right) ds \right]^{\frac{1}{n}} > \frac{1}{n^n} \left(1 - \prod_{i=1}^{n} c(\beta_i) \right),$$

where $k \in \mathbb{N}$ and

$$\psi_1(t) = 0, \quad \psi_i(t) = \exp\left(\sum_{k=1}^{n} \int_{\tau_k(t)}^{t} \left(\prod_{\ell=1}^{n} p_\ell(s) \right)^{\frac{1}{n}} \psi_{i-1}(s) ds \right), \quad i = 2, 3, \ldots,$$

and

$$\beta_i = \liminf_{t\to\infty} \int_{g_i(t)}^{t} p_i(s)ds, \quad i = 1, 2, \ldots, n. \tag{5}$$

The second condition is

$$\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{g_j(t)}^{t} p_i(s) \exp\left(n(\lambda(\bar{p}_\star) - \epsilon) \int_{\tau_i(s)}^{g_i(t)} \left(\prod_{\ell=1}^{n} p_\ell(\xi) \right)^{\frac{1}{n}} d\xi \right) ds \right]^{\frac{1}{n}} > \frac{1}{n^n} \left(1 - \prod_{\ell=1}^{n} c(\beta_\ell) \right),$$

where $\epsilon \in (0, \lambda(\bar{p}_\star))$, and

$$0 < \bar{p}_\star := \liminf_{t\to\infty} \sum_{i=1}^{n} \int_{\tau_i(t)}^{t} \left(\prod_{\ell=1}^{n} p_\ell(s) \right)^{\frac{1}{n}} ds \le \frac{1}{\mathrm{e}}.$$

The third condition is

$$\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{g_j(t)}^{t} p_i(s) \int_{\tau_i(s)}^{g_i(t)} \left(\prod_{\ell=1}^{n} p_\ell(\xi) \right)^{\frac{1}{n}} d\xi ds \right]^{\frac{1}{n}} > 0, \tag{6}$$

and $\bar{p}_\star > \frac{1}{\mathrm{e}}$.

In 2016, Braverman, Chatzarakis and Stavroulakis [8] obtained the sufficient condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(u) a_r(h(t), \tau_i(u)) du > 1,$$

where

$$h(t) = \max_{1\le i\le n} h_i(t), \quad \text{and} \quad h_i(t) = \sup_{t_0\le s\le t} \tau_i(s), \quad i = 1, 2, \ldots, n, \tag{7}$$

and

$$a_1(t,s) = \exp\left\{ \int_{s}^{t} \sum_{i=1}^{n} p_i(u) du \right\},$$

$$a_{r+1}(t,s) = \exp\left\{ \int_{s}^{t} \sum_{i=1}^{n} p_i(u) a_r(u, \tau_i(u)) du \right\}, \quad r \in \mathbb{N}.$$

Also, in 2016 Akca, Chatzarakis and Stavroulakis [9] obtained the sufficient condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(u) a_r(h(u), \tau_i(u)) du > \frac{1+\ln(\lambda(\alpha))}{\lambda(\alpha)}, \tag{8}$$

where

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau_{\max}(t)}^{t} \sum_{i=1}^{n} p_i(s) ds \le \frac{1}{\mathrm{e}}.$$

2 Main results

To obtain our main results we need the following lemmas:

Lemma 2.1 ([6]). *Let $x(t)$ be an eventually positive solution of Eq. (1). Then*

$$\liminf_{t\to\infty} \frac{x(\tau_k(t))}{x(t)} \ge \lambda(q_k), \quad k = 1, 2, \dots, n,$$

where q_k is defined by (4).

Lemma 2.2 ([10]). *Assume that*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} P(s) ds = \alpha^*,$$

and $x(t)$ is an eventually positive solution of the first order delay differential inequality

$$x'(t) + P(t) x(g(t)) \le 0, \quad t \ge t_1,$$

where $P \in C([t_1, \infty), [0, \infty))$. If $0 \le \alpha^* \le \frac{1}{\mathrm{e}}$, then

$$\liminf_{t\to\infty} \frac{x(t)}{x(g(t))} \ge c(\alpha^*).$$

Theorem 2.3. *Assume that*

$$\rho := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{k=1}^{n} p_k(s) ds, \quad 0 < \rho \le \frac{1}{\mathrm{e}},$$

and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v) dv + c(\rho) \mathrm{e}^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s) ds} \right) > 1, \tag{9}$$

where

$$Q(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) \mathrm{e}^{\int_{g_k(t)}^{t} \sum_{j=1}^{n} p_j(v) dv + (\lambda(\rho) - \epsilon) \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u) du} ds,$$

and $\epsilon \in (0, \lambda(\rho))$. Then all solutions of Eq.(1) are oscillatory.

Proof. Assume the contrary, i.e., there exists a nonoscillatory solution $x(t)$ of (1). Because of the linearity of (1), we assume that $x(t)$ is eventually positive. Therefore, there exists a sufficiently large $t_2 \ge t_1$ such that

$x(\tau_k(t)) > 0$, for all $t \geq t_2$, $k = 1, 2, \ldots, n$. Thus, equation (1) implies that $x(t)$ is nonincreasing for all $t \geq t_2$. Integrating (1) from $\tau_i(t)$ to t, we obtain

$$x(t) - x(\tau_i(t)) + \sum_{k=1}^{n} \int_{\tau_i(t)}^{t} p_k(s)x(\tau_k(s))ds = 0. \tag{10}$$

Also, dividing (1) by $x(t)$ and integrating the resulting equation from $\tau_k(s)$ to $g_k(t)$, we get

$$x(\tau_k(s)) = x(g_k(t))\mathrm{e}^{\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du}.$$

Substituting this into (10),

$$x(t) - x(\tau_i(t)) + \sum_{k=1}^{n} x(g_k(t)) \int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du} ds = 0.$$

Multiplying the above equation by $p_i(t)$, and taking the sum over i, it follows that

$$x'(t) + x(t)\sum_{i=1}^{n} p_i(t) + \sum_{k=1}^{n} x(g_k(t)) \sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du} ds = 0.$$

The substitution $y(t) = \mathrm{e}^{\int_{t_0}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} x(t)$, reduces this equation to

$$y'(t) + \sum_{k=1}^{n} y(g_k(t))\mathrm{e}^{\int_{g_k(t)}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} \sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du} ds = 0, \tag{11}$$

which in turn, by integrating from $g(t)$ to t, leads to

$$y(t) - y(g(t)) + \int_{g(t)}^{t} \sum_{k=1}^{n} y(g_k(v))\mathrm{e}^{\int_{g_k(v)}^{v} \sum_{\ell=1}^{n} p_\ell(s)ds} \sum_{i=1}^{n} p_i(v) \int_{\tau_i(v)}^{v} p_k(s)\mathrm{e}^{\int_{\tau_k(s)}^{g_k(v)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du} ds\, dv = 0.$$

Hence, the nonincreasing nature of $y(t)$ implies that

$$y(t) - y(g(t)) + y(g(t)) \int_{g(t)}^{t} \sum_{k=1}^{n} \mathrm{e}^{\int_{g_k(v)}^{v} \sum_{\ell=1}^{n} p_\ell(s)ds} \sum_{i=1}^{n} p_i(v) \int_{\tau_i(v)}^{v} p_k(s)\mathrm{e}^{\int_{\tau_k(s)}^{g_k(v)} \sum_{\ell=1}^{n} p_\ell(u)\frac{x(\tau_\ell(u))}{x(u)}du} ds\, dv \leq 0, \tag{12}$$

for all $t \geq t_3$ and some $t_3 \geq t_2$.

On the other hand, using the nonincreasing nature of $x(t)$, equation (1) implies that

$$x'(t) + x(g(t))\sum_{k=1}^{n} p_k(t) \leq 0, \qquad \text{for all } t \geq t_3. \tag{13}$$

Therefore, from [11, Lemma 2.1.2], we obtain $\liminf_{t\to\infty} \frac{x(g(t))}{x(t)} \geq \lambda(\rho)$. Thus, for sufficiently small $\epsilon > 0$, we have

$$\frac{x(\tau_\ell(t))}{x(t)} \geq \frac{x(g(t))}{x(t)} > \lambda(\rho) - \epsilon, \qquad \text{for all } t \geq t_4,\ 1 \leq \ell \leq n, \tag{14}$$

for some $t_4 \geq t_3$. This together with (12) implies that

$$y(t) - y(g(t)) + y(g(t)) \int_{g(t)}^{t} \sum_{k=1}^{n} \mathrm{e}^{\int_{g_k(v)}^{v} \sum_{\ell=1}^{n} p_\ell(s)ds} \sum_{i=1}^{n} p_i(v) \int_{\tau_i(v)}^{v} p_k(s)\mathrm{e}^{(\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(v)} \sum_{\ell=1}^{n} p_\ell(u)du} ds\, dv \leq 0,$$

for all $t \geq t_5$, where $t_5 \geq t_4$. That is

$$\int_{g(t)}^{t} \sum_{k=1}^{n} e^{\int_{g_k(v)}^{v} \sum_{\ell=1}^{n} p_\ell(s)ds} \sum_{i=1}^{n} p_i(v) \int_{\tau_i(v)}^{v} p_k(s) e^{(\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(v)} \sum_{\ell=1}^{n} p_\ell(u)du} ds\, dv \leq 1 - \frac{y(t)}{y(g(t))}, \tag{15}$$

for all $t \geq t_5$. Also, in view of Lemma 2.2, for sufficiently small $\epsilon^\star > 0$ and some $t_6 \geq t_5$, inequality (13) leads to

$$\frac{x(t)}{x(g(t))} > c(\rho) - \epsilon^\star, \quad \text{for all } t \geq t_6.$$

Therefore,

$$\begin{aligned}\frac{y(t)}{y(g(t))} &= \frac{x(t)}{x(g(t))} e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds} \\ &> \left(c(\rho) - \epsilon^\star\right) e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds}, \quad \text{for all } t \geq t_6.\end{aligned}$$

Combining this inequality with (15), it follows that

$$\int_{g(t)}^{t} Q(v)dv \leq 1 - \left(c(\rho) - \epsilon^\star\right) e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds}, \quad \text{for all } t \geq t_6.$$

Then

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v)dv + c(\rho) e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds} \right) \leq 1 + \epsilon^\star \limsup_{t\to\infty} e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds}. \tag{16}$$

Notice that, by integrating (13) from $g(t)$ to t and using the nonincreasing nature of $x(t)$, we obtain

$$\int_{g(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \leq 1 - \frac{x(t)}{x(g(t))} \leq 1, \quad \text{for all } t \geq t_3.$$

Now, letting $\epsilon^\star \to 0$ in (16), we arrive at a contradiction with (9). The proof is complete. □

Remark 2.4. *Theorem 2.3 is proved using the core idea of the proof of [12, Theorem 2.1] which is given for Equation (1) when $n = 1$. However, Theorem 2.3 produces a new oscillation criterion even for equations with only one non-monotone delay.*

Using Lemma 2.1 instead of [11, Lemma 2.1.2], similar reasoning as in the proof of Theorem 2.3 implies the following result:

Theorem 2.5. *Assume that*

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q_1(v)dv + c(\rho) e^{\int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)ds} \right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) e^{\int_{g_k(t)}^{t} \sum_{j=1}^{n} p_j(v)dv + \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} (\lambda(q_\ell)-\epsilon_\ell) p_\ell(u)du} ds,$$

q_ℓ is defined by (4), ρ is defined as in Theorem 2.3 and $\epsilon_\ell \in (0, \lambda(q_\ell))$. Then all solutions of Equation (1) oscillate.

Theorem 2.6. *Assume that*

$$\limsup_{t\to\infty} \left(\prod_{j=1}^{n} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n} e^{\sum_{k=1}^{n} \int_{g_k(t)}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} \right) > \frac{1}{n^n},$$

where β_k is defined by (5) with $0 < \beta_k \le \frac{1}{e}$ and

$$R_k(s) = e^{\int_{g_k(s)}^{s} \sum_{j=1}^{n} p_j(u)du} \sum_{i=1}^{n} p_i(s) \int_{\tau_i(s)}^{s} p_k(u) e^{(\lambda(\rho)-\epsilon)\int_{\tau_k(u)}^{g_k(s)} \sum_{\ell=1}^{n} p_\ell(v)dv} du,$$

ρ is defined as in Theorem 2.3 and $\epsilon \in (0, \lambda(\rho))$. Then all solutions of Equation (1) oscillate.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). As usual, we assume that $x(t)$ is an eventually positive solution. Substituting (14) into (11), we obtain

$$y'(t) + \sum_{k=1}^{n} y(g_k(t)) e^{\int_{g_k(t)}^{t} \sum_{j=1}^{n} p_j(s)ds} \sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) e^{(\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} p_\ell(u)du} ds \le 0,$$

for all $t \ge t_2$ and some $t_2 \ge t_1$ where $\epsilon \in (0, \lambda(\rho))$ and $y(t) = e^{\int_{t_0}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} x(t)$. Integrating from $g_j(t)$ to t and using the nonincreasing nature of $y(t)$, we obtain

$$y(t) - y(g_j(t)) + \sum_{k=1}^{n} y(g_k(t)) \int_{g_j(t)}^{t} R_k(s)ds \le 0, \quad \text{for all } t \ge t_3,$$

and some $t_3 \ge t_2$. Using the relation between arithmetic and geometric mean, it follows that

$$y(g_j(t)) \ge n \left(\prod_{k=1}^{n} y(g_k(t)) \right)^{\frac{1}{n}} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + y(t), \quad \text{for all } t \ge t_3.$$

Taking the product on both sides,

$$\left(\prod_{j=1}^{n} y(g_j(t)) \right) \ge n^n \left(\prod_{k=1}^{n} y(g_k(t)) \right) \prod_{j=1}^{n} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + y(t)^n, \quad \text{for all } t \ge t_3.$$

Therefore,

$$\prod_{j=1}^{n} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + \frac{y(t)^n}{n^n \prod_{k=1}^{n} y(g_k(t))} \le \frac{1}{n^n}, \quad \text{for all } t \ge t_3. \tag{17}$$

Since $y(t) = e^{\int_{t_0}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} x(t)$, then

$$\frac{y(t)^n}{\prod_{k=1}^{n} y(g_k(t))} = e^{\sum_{k=1}^{n} \int_{g_k(t)}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} \frac{x(t)^n}{\prod_{k=1}^{n} x(g_k(t))}.$$

Substituting in (17),

$$\prod_{j=1}^{n} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + e^{\sum_{k=1}^{n} \int_{g_k(t)}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} \frac{x(t)^n}{n^n \left(\prod_{k=1}^{n} x(g_k(t)) \right)} \le \frac{1}{n^n}, \quad \text{for all } t \ge t_3. \tag{18}$$

On the other hand, the nonincreasing nature of $x(t)$ and (1) imply that

$$x'(t) + p_k(t)x(g_k(t)) \le 0, \quad \text{for all } t \ge t_3, \quad k = 1, 2, \ldots, n, \tag{19}$$

and hence, by Lemma 2.2, it follows that

$$\frac{x(t)^n}{\prod_{k=1}^{n} x(g_k(t))} > \prod_{k=1}^{n} c(\beta_k) - \epsilon^\star, \quad \text{for all } t \ge t_4,$$

for some $t_4 \geq t_3$ and sufficiently small $\epsilon^* > 0$. This together with (18) leads to

$$\prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t} R_k(s)ds\right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n} e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds} \leq \frac{1}{n^n} + \frac{\epsilon^*}{n^n} e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds},$$

for all $t \geq t_4$. Consequently,

$$\limsup_{t\to\infty}\left(\prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t} R_k(s)ds\right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n} e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds}\right)$$
$$\leq \frac{1}{n^n} + \frac{\epsilon^*}{n^n}\limsup_{t\to\infty} e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds}. \tag{20}$$

On the other hand, integrating (1) from $g_k(t)$ to t, we get

$$x(t) - x(g_k(t)) + \int_{g_k(t)}^{t}\sum_{\ell=1}^{n} x(\tau_\ell(s))p_\ell(s)ds = 0, \quad k = 1, 2, \ldots, n.$$

Then, using the nonincreasing nature of $x(t)$, we obtain

$$\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds \leq \frac{x(g_k(t))}{x(t)} - 1, \quad \text{for all } t \geq t_3, \quad k = 1, 2, \ldots, n.$$

But applying Lemma 2.2 to (19), we obtain

$$\limsup_{t\to\infty}\frac{x(g_k(t))}{x(t)} < +\infty, \quad k = 1, 2, \ldots, n.$$

Therefore, $e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds}$ is bounded. Now, allowing $\epsilon^* \to 0$ in (20), it follows that

$$\limsup_{t\to\infty}\left(\prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t} R_k(s)ds\right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n} e^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds}\right) \leq \frac{1}{n^n}.$$

This contradiction completes the proof. □

The following example shows that Theorem 2.3 can be applied but conditions (2), (3) and (6) fail to apply as well as (8) when $r = 1$.

Example 2.7. *Consider the first order delay differential equation*

$$x'(t) + a(b+\sin(t))x\left(t - \frac{\pi}{2}\right) + a(b+\cos(t))x\left(\bar{\tau}(t)\right) = 0, \qquad t \geq 0, \tag{21}$$

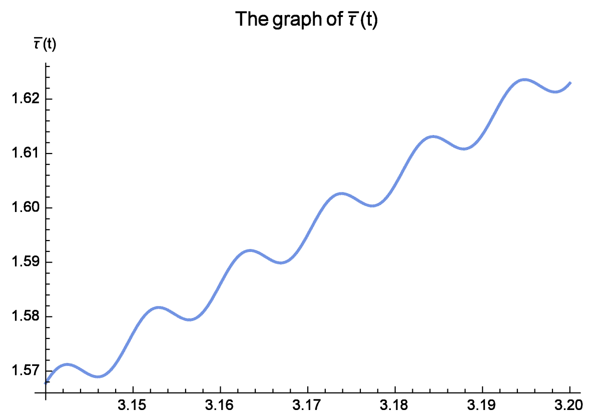
where $a = \frac{0.4}{1.137\pi+\sqrt{2}}$, $b = 1.784$ *and*

$$\bar{\tau}(t) = t - \frac{\pi}{2} - \sigma\sin^2(300t), \quad \sigma = \frac{1}{150}.$$

Clearly

$$t - \frac{\pi}{2} - \sigma \leq \bar{\tau}(t) \leq t - \frac{\pi}{2}.$$

Fig. 1



Equation (21) has the form (1) with $p_1(t) = a(b + \sin(t))$, $p_2(t) = a(b + \cos(t))$, $\tau_1(t) = t - \frac{\pi}{2}$ *and* $\tau_2(t) = t - \frac{\pi}{2} - \sigma \sin^2(300t)$. *Therefore, we can choose* $g_1(t) = g_2(t) = g(t) = t - \frac{\pi}{2}$ *and* $\epsilon = 0.001$. *Then*

$$\int_{g(t)}^{t} \sum_{k=1}^{2} p_k(s)ds = ab\pi + 2a\sin(t).$$

Consequently,

$$\rho = \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{k=1}^{2} p_k(s)ds = ab\pi - 2a \approx 0.2891659465,$$

$$c(\rho) = \frac{1-\rho-\sqrt{1-2\rho-\rho^2}}{2} \approx 0.06470619, \quad \text{and} \quad \lambda(\rho) - \epsilon \approx 1.577422807.$$

Let $I(t) = \int_{g(t)}^{t} Q(v)dv + c(\rho)e^{\int_{g(t)}^{t} \sum_{i=1}^{2} p_i(s)ds}$. *Then, the property that* $e^x \geq ex$ *for* $x \geq 0$ *leads to*

$$I(t) \geq e\left(\int_{g(t)}^{t} \sum_{i=1}^{2} p_i(v) \int_{g(v)}^{v} \sum_{k=1}^{2} p_k(s)\Big(\int_{g(v)}^{v} \sum_{\ell=1}^{2} p_\ell(s)ds + (\lambda(\rho)-\epsilon)\int_{g(s)}^{g(v)} \sum_{l=1}^{2} p_l(u)du\Big)ds\,dv\right)$$
$$+c(\rho)e^{\int_{g(t)}^{t} \sum_{i=1}^{2} p_i(s)ds}.$$

Now, using Maple, we get

$$\begin{aligned} I(t) \geq{} & 0.49727 + 0.0029516\sin(t)\cos^2(t) + 0.27514\sin(t) - 0.23059\cos(t) \\ & -0.015664\cos^2(t) - 0.083083\cos(t)\sin(t) + 0.0037423\cos^3(t) \\ & +0.10144e^{0.16044\sin(t)}. \end{aligned}$$

Choose $T_k = \frac{3\pi}{4} + 2\pi k$, *then*

$$I(T_k) \approx 1.0019 > 1, \quad \text{for all } k \in \mathbb{N},$$

which means that

$$\limsup_{t\to\infty} \Big(\int_{g(t)}^{t} Q(v)dv + c(\rho)e^{\int_{g(t)}^{t} \sum_{i=1}^{2} p_i(s)ds}\Big) > 1.$$

That is, condition (9) of Theorem 2.3 is satisfied and therefore all solutions of Eq. (21) oscillate.

We will show, however, that none of the conditions (2), (3), (6), and (8) (with $r = 1$*) is satisfied. Indeed, notice that*

$$\int_{\tau^*(t)}^{t} \sum_{k=1}^{2} p_k(s)ds \leq \int_{t-\frac{\pi}{2}}^{t} \sum_{k=1}^{2} p_k(s)ds = ab\pi + 2a\sin(t).$$

Hence,

$$\liminf_{t\to\infty}\int_{\tau^*(t)}^{t}\sum_{k=1}^{2}p_k(s)ds \le ab\pi-2a\approx 0.2891659465 < \frac{1}{e}.$$

On the other hand, since $h(t)=g(t)$, $p_i(t)\le a(b+1)$ for $i=1,2$, where $h(t)$ is defined by (7), we have

$$a_1(h(s),\tau_i(s)) = \exp\left(\int_{\tau_i(s)}^{g(s)}\sum_{i=1}^{2}p_i(u)du\right) \le \exp\left(\int_{s-\frac{\pi}{2}-\sigma}^{s-\frac{\pi}{2}}\sum_{i=1}^{2}p_i(u)du\right) \le \exp\left(\int_{s-\frac{\pi}{2}-\sigma}^{s-\frac{\pi}{2}}2a(b+1)du\right) = \exp(2a(b+1)\sigma).$$

Therefore,

$$\begin{aligned}\int_{h(t)}^{t}\sum_{i=1}^{2}p_i(s)a_1(h(s),\tau_i(s))ds &\le \int_{t-\frac{\pi}{2}}^{t}a(b+\sin(s)+b+\cos(s))\exp\left(2a(b+1)\sigma\right)ds\\ &= \left[\int_{t-\frac{\pi}{2}}^{t}2abds+\int_{t-\frac{\pi}{2}}^{t}a\sin(s)ds+\int_{t-\frac{\pi}{2}}^{t}a\cos(s)ds\right]\exp\left(2a(b+1)\sigma\right)\\ &= [ab\pi+2a\sin(t)]\exp\left(2a(b+1)\sigma\right)\le (ab\pi+2a)\exp\left(2a(b+1)\sigma\right)\\ &< 0.61188.\end{aligned}$$

Since $\lambda(\rho)\approx 1.578422807$, then

$$\frac{1+\ln(\lambda(\rho))}{\lambda(\rho)} > 0.92.$$

Therefore, none of conditions (2) and (8) (with $r=1$) is satisfied.

Also, since

$$\int_{g(t)}^{t}a(b+\sin(s))ds = \frac{ab\pi}{2}+a(\sin(t)-\cos(t)),$$

and

$$\begin{aligned}\int_{\tilde{\tau}(t)}^{t}a(b+\cos(s))ds &\le \int_{t-\frac{\pi}{2}-\sigma}^{t}a(b+\cos(s))ds\\ &\le a(b+1)\int_{t-\frac{\pi}{2}-\sigma}^{t-\frac{\pi}{2}}ds+\int_{t-\frac{\pi}{2}}^{t}a(b+\cos(s))ds\\ &= a\sigma(b+1)+\frac{ab\pi}{2}+a(\cos(t)+\sin(t)).\end{aligned}$$

Then, $q_1=\frac{ab\pi}{2}-a\sqrt{2}$ and $q_2\le\frac{ab\pi}{2}-a\sqrt{2}+a\sigma(b+1)$, where q_i is defined by (4), i=1,2. Therefore, $\lambda(q_1)\approx 1.134680932$ and $\lambda(q_2)\approx 1.136881841$. Let

$$I_1(t)=\prod_{j=1}^{2}\left[\prod_{i=1}^{2}\int_{g_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{g_i(t)}\sum_{k=1}^{2}(\lambda(q_k)-\epsilon)p_k(u)du\right)ds\right]^{\frac{1}{2}}.$$

Then

$$
\begin{aligned}
I_1(t) &< \prod_{j=1}^{2}\left[\prod_{i=1}^{2}\int_{g_j(t)}^{t} p_i(s)\exp\left(\lambda(q_2)\int_{\tau_i(s)}^{g_i(t)}\sum_{l=1}^{2}p_l(u)du\right)ds\right]^{\frac{1}{2}}\\
&\le \prod_{j=1}^{2}\left[\prod_{i=1}^{2}\int_{g_j(t)}^{t} p_i(s)\exp\left(2a\lambda(q_2)(1+b)(t-s+\sigma)\right)ds\right]^{\frac{1}{2}}\\
&\le \prod_{i=1}^{2}\left(\int_{t-\frac{\pi}{2}-\sigma}^{t} p_i(s)\exp\left(2a\lambda(q_2)(1+b)(t-s+\sigma)\right)ds\right)\\
&\le \prod_{i=1}^{2}\left[\int_{t-\frac{\pi}{2}-\sigma}^{t-\frac{\pi}{2}} a(b+1)\exp\left(2a\lambda(q_2)(1+b)(t-s+\sigma)\right)ds\right.\\
&\qquad\left.+\int_{t-\frac{\pi}{2}}^{t} p_i(s)\exp\left(2a\lambda(q_2)(1+b)(t-s+\sigma)\right)ds\right]\\
&\approx 0.1363159006+0.08561813338\sin(t)-0.00925235722\cos(t)\\
&\quad -0.02983977134\cos^2(t)-0.00652549885\sin(t)\cos(t)<0.23771189<\frac{1}{4}.
\end{aligned}
$$

Therefore, $\limsup_{\epsilon\to0^+}\left(\limsup_{t\to\infty} I_1(t)\right) < \frac{1}{4}$, *which implies that condition (3) is not satisfied.*

Finally, since

$$
\begin{aligned}
\sum_{i=1}^{2}\int_{\tau_i(t)}^{t}\left(\prod_{l=1}^{2}p_l(s)\right)^{\frac{1}{2}}ds &\le \sum_{i=1}^{2}\left(\frac{1}{2}\int_{\tau_i(t)}^{t}\sum_{l=1}^{2}p_l(s)ds\right)\\
&\le \frac{1}{2}\int_{t-\frac{\pi}{2}}^{t}\sum_{l=1}^{2}p_l(s)ds+\frac{1}{2}\int_{t-\frac{\pi}{2}-\sigma}^{t}\sum_{l=1}^{2}p_l(s)ds\\
&= \frac{1}{2}\int_{t-\frac{\pi}{2}}^{t}\sum_{l=1}^{2}p_l(s)ds+\frac{1}{2}\left(\int_{t-\frac{\pi}{2}-\sigma}^{t-\frac{\pi}{2}}\sum_{l=1}^{2}p_l(s)ds+\int_{t-\frac{\pi}{2}}^{t}\sum_{l=1}^{2}p_l(s)ds\right)\\
&\le \int_{t-\frac{\pi}{2}}^{t}\sum_{l=1}^{2}p_l(s)ds+\int_{t-\frac{\pi}{2}-\sigma}^{t-\frac{\pi}{2}}a(b+1)ds\\
&\le \int_{t-\frac{\pi}{2}}^{t}\sum_{l=1}^{2}p_l(s)ds+a\sigma(b+1),
\end{aligned}
$$

then

$$
\begin{aligned}
\bar{p_\star} &= \liminf_{t\to\infty}\sum_{i=1}^{2}\int_{\tau_i(t)}^{t}\left(\prod_{l=1}^{2}p_l(s)\right)^{\frac{1}{2}}ds\\
&\le ab\pi-2a+a\sigma(b+1)<0.2906549<\frac{1}{\mathrm{e}}.
\end{aligned}
$$

Therefore, condition (6) fails to apply.

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