DE GRUYTER Open Math. 2018; 16: 127–132

**Open Mathematics**

**Research Article**

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# The log-concavity of the $q$-derangement numbers of type $B$

https://doi.org/10.1515/math-2018-0009
Received September 22, 2017; accepted January 11, 2018.

**Abstract:** Recently, Chen and Xia proved that for $n \geq 6$, the $q$-derangement numbers $D_n(q)$ are log-concave except for the last term when $n$ is even. In this paper, employing a recurrence relation for $D_n^B(q)$ discovered by Chow, we show that for $n \geq 4$, the $q$-derangement numbers of type $B$ $D_n^B(q)$ are also log-concave.

**Keywords:** The $q$-derangement numbers of type $B$, Unimodality, Log-concavity

**MSC:** 05A15, 05A19, 05A20

## 1 Introduction

Let $\mathcal{D}_n$ denote the set of derangements on $\{1, 2, \ldots, n\}$ and let $D(\pi) := \{i | 1 \leq i \leq n-1, \pi(i) > \pi(i+1)\}$ denote the descent set of a permutation $\pi$. Define the major index of $\pi$ by

$$\mathrm{maj}(\pi) := \sum_{i \in D(\pi)} i. \tag{1}$$

The $q$-derangement number $D_n(q)$ is defined by

$$D_n(q) := \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{maj}(\pi)}. \tag{2}$$

Gessel [1] (see also [2]) discovered the following formula

$$D_n(q) := [n]! \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \frac{1}{[k]!}, \tag{3}$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2]\cdots[n]$. Combinatorial proofs of (3) have been found by Wachs [3] and Chen and Xu [4]. Chen and Rota [5] showed that the $q$-derangement numbers are unimodal, and conjectured that the maximum coefficient appears in the middle. Zhang [6] confirmed this conjecture by showing that the $q$-derangement numbers satisfy the spiral property. Recently, Chen and Xia [7] introduced the notion of ratio monotonicity for polynomials with nonnegative coefficients, and they proved that, for $n \geq 6$, the $q$-derangement numbers $D_n(q)$ are strictly ratio monotone except for the last term when $n$ is even. The ratio monotonicity implies the spiral property and log-concavity.

Let $B_n$ denote the hyperoctahedral group of rank $n$, consisting of the signed permutations of $\{1, 2, \ldots, n\}$. Let $\mathcal{D}_n^B$ denote the set of derangements on $B_n$, which is defined as

$$\mathcal{D}_n^B := \{\pi | \pi \in B_n, \pi(i) \neq i \text{ for all } i \in \{1, 2, \ldots, n\}\}. \tag{4}$$

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Let $N(\pi) := \#\{i|1 \le i \le n, \pi(i) < 0\}$ be the number of negative letters of $\pi$ and let $\mathrm{maj}(\pi)$ be defined as before. In [8], Chow considered the $q$-derangement number of type $B$ $D_n^B(q)$, which is defined as

$$D_n^B(q) := \sum_{\pi \in \mathcal{D}_n^B} q^{\mathrm{fmaj}(\pi)}, \tag{5}$$

where $\mathrm{fmaj}(\pi) := 2\mathrm{maj}(\pi) + N(\pi)$. Chow [8] (see also [9]) established the following formula

$$D_n^B(q) := [2][4]\cdots[2n] \sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2][4]\cdots[2k]}, \tag{6}$$

where $[n]$ is defined as before. Furthermore, Chow [8] discovered that for all integers $n \ge 1$,

$$D_{n+1}^B(q) = (1 + q + \cdots + q^{2n+1})D_n^B(q) + (-1)^{n+1} q^{n^2+n}. \tag{7}$$

Chen and Wang [10] proved the normality of the limiting distribution of the coefficients of the usual $q$-derangement numbers of type $B$.

Recall that a positive sequence $a_0, a_1, \ldots, a_n$ or the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ is called log-concave if the ratios

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n} \tag{8}$$

form an increasing sequence. Clearly, if a positive sequence is log-concavity, then it is unimodality. In this paper, we prove that for $n \ge 4$, the $q$-derangement numbers of type $B$ $D_n^B(q)$ are log-concave.

Suppose that $n$ is given. It is easy to prove that the degree of $D_n^B(q)$ is $n^2$ and the coefficient of $q^{n^2}$ is 1. Set

$$D_n^B(q) = B_n(1)q + B_n(2)q^2 + \cdots + B_n(n^2)q^{n^2}. \tag{9}$$

The log-concavity of $D_n^B(q)$ can be stated as the following theorem.

**Theorem 1.1.** *For all integers $n \ge 4$, the $q$-derangement numbers of type $B$ $D_n^B(q)$ are log-concave, namely,*

$$\frac{B_n(1)}{B_n(2)} < \frac{B_n(2)}{B_n(3)} < \cdots < \frac{B_n(n^2-2)}{B_n(n^2-1)} < \frac{B_n(n^2-1)}{B_n(n^2)}. \tag{10}$$

For example, by (6), we have

$$\begin{aligned} D_4^B(q) =& q + 4q^2 + 8q^3 + 13q^4 + 18q^5 + 22q^6 + 26q^7 + 28q^8 + 28q^9 + 25q^{10} \\ &+ 21q^{11} + 17q^{12} + 11q^{13} + 7q^{14} + 3q^{15} + q^{16}. \end{aligned}$$

It is easy to check that

$$\frac{1}{4} < \frac{4}{8} < \frac{8}{13} < \frac{13}{18} < \frac{18}{22} < \frac{22}{26} < \frac{26}{28} < \frac{28}{28} < \frac{28}{25} < \frac{25}{21} < \frac{21}{17} < \frac{17}{11} < \frac{11}{7} < \frac{7}{3} < \frac{3}{1}.$$

## 2 Some lemmas

To prove Theorem 1.1, we first present some lemmas. By (7), it is easy to check that

**Lemma 2.1.** *For $n \ge 4$,*

$$B_{n+1}(k) = \begin{cases} \sum_{i=1}^{k} B_n(i), & 1 \le k \le 2n+2, \\ \sum_{i=k-2n-1}^{k} B_n(i), & 2n+2 < k \le n^2, \\ \sum_{i=n^2-n-1}^{n^2} B_n(i) + (-1)^{n+1}, & k = n^2 + n, \\ \sum_{i=k-2n-1}^{n^2} B_n(i), & n^2 \le k \le (n+1)^2 \text{ and } k \ne n^2 + n. \end{cases} \tag{11}$$

Based on recurrence relation (11), it is easy to verify the following lemma.

**Lemma 2.2.** *Let $n \geq 4$ be an integer. Then $B_n(i)$ are positive integers for $1 \leq i \leq n^2$ and*

$$B_n(n^2) = 1, \qquad B_n(n^2-1) = n-1, \tag{12}$$

$$B_n(n^2-2) = \frac{n^2-n+2}{2}, \quad B_n(n^2-3) = \frac{n^3+5n-18}{6}. \tag{13}$$

To prove Theorem 1.1, we require the following lemma.

**Lemma 2.3.** *For positive integers $a_1, a_2, \ldots, a_{k+1}, a_{k+2}$ $(k \geq 1)$ satisfying*

$$\frac{a_i}{a_{i+1}} < \frac{a_{i+1}}{a_{i+2}}, \quad 1 \leq i \leq k, \tag{14}$$

*we have*

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k+1} a_i} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+2} a_i}, \tag{15}$$

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k+1} a_i} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=2}^{k+2} a_i}, \tag{16}$$

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=2}^{k+1} a_i} < \frac{\sum_{i=2}^{k+1} a_i}{\sum_{i=3}^{k+2} a_i}, \tag{17}$$

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=2}^{k+1} a_i} < \frac{\sum_{i=2}^{k+1} a_i}{\sum_{i=3}^{k+1} a_i}, \tag{18}$$

$$\frac{\sum_{i=1}^{k} a_i}{\sum_{i=2}^{k} a_i} < \frac{\sum_{i=2}^{k} a_i}{\sum_{i=3}^{k} a_i}. \tag{19}$$

*Proof.* We only prove (15). The rest can be proved similarly and the details are omitted. Based on (14),

$$a_i a_{k+2} < a_{i+1} a_{k+1} \qquad (1 \leq i \leq k),$$

and

$$a_{k+2}(a_1 + a_2 + \cdots + a_k) < a_{k+1}(a_2 + a_3 + \cdots + a_{k+1}).$$

Therefore,

$$\begin{aligned}
&(a_1 + a_2 + \cdots + a_k)(a_1 + a_2 + \cdots + a_k + a_{k+1} + a_{k+1})\\
=&(a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+2}(a_1 + a_2 + \cdots + a_k)\\
<&(a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+1}(a_2 + a_3 + \cdots + a_{k+1})\\
<&(a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+1}(a_2 + a_3 + \cdots + a_{k+1}) + a_1 a_{k+1}\\
=&(a_1 + a_2 + \cdots + a_k + a_{k+1})^2,
\end{aligned}$$

which yields (15). This completes the proof of this lemma.

# 3 Proof of Theorem 1.1

We prove Theorem 1.1 by induction on $n$. It is easy to check that Theorem 1.1 holds for $4 \leq n \leq 12$. Thus, we always assume that $n \geq 13$ in the following proof. Suppose that Theorem 1.1 holds for $n = m$, namely,

$$\frac{B_m(i)}{B_m(i+1)} < \frac{B_m(i+1)}{B_m(i+2)}, \quad 1 \leq i \leq m^2 - 2. \tag{20}$$

We proceed to show that Theorem 1.1 holds for $n = m + 1$, that is,

$$\frac{B_{m+1}(k)}{B_{m+1}(k+1)} < \frac{B_{m+1}(k+1)}{B_{m+1}(k+2)}, \quad 1 \le k \le (m+1)^2 - 2. \tag{21}$$

Employing (11), (15) and (20), we see that (21) holds for $1 \le k \le 2m$. It follows from (11), (16) and (20) that (21) is true for the case $k = 2m + 1$. In view of (11), (17) and (20), we find that (21) holds for $2m + 2 \le k \le m^2 - 2$. From (11), (18) and (20), we deduce that (21) is true for the case $k = m^2 - 1$. By (11), (19) and (20), we can prove that (21) holds for $m^2 \le k \le m^2 + m - 3$ and $m^2 + m + 1 \le k \le (m+1)^2 - 2$.

Now, special attentions should be paid to three cases $k = m^2 + m - 2$, $k = m^2 + m - 1$ and $k = m^2 + m$.

By (12) and (13), it is easy to check that for $m \ge 4$,

$$\begin{aligned}&\frac{B_m(m^2-2) + B_m(m^2-1) + B_m(m^2) - m - 3}{B_m(m^2-1) + B_m(m^2) + 1} - \frac{B_m(m^2-3)}{B_m(m^2-2)}\\ &\quad = \frac{m^4 - 13m^2 + 72m - 132}{6(m-3)(m^2-m+2)} > 0.\end{aligned} \tag{22}$$

In view of (20) and (22),

$$\frac{B_m(m^2-m-3)}{B_m(m^2-m-2)} < \frac{B_m(m^2-3)}{B_m(m^2-2)} < \frac{B_m(m^2-2) + B_m(m^2-1) + B_m(m^2) - m - 3}{B_m(m^2-1) + B_m(m^2) + 1}. \tag{23}$$

From (11), it is easy to prove that for $m \ge 4$,

$$B_m(m^2-m-2) > B_m(m^2-m-1) > \cdots > B_m(m^2-1) > B_m(m^2). \tag{24}$$

Thanks to (23) and (24),

$$\begin{aligned}&B_m(m^2-m-3)(B_m(m^2-1) + B_m(m^2) + (-1)^{m+1}) + (-1)^{m+1}\sum_{i=0}^{m+2} B_m(m^2-i)\\ &\quad < B_m(m^2-m-3)(B_m(m^2-1) + B_m(m^2) + 1) + (m+3)B_m(m^2-m-2)\\ &\quad < B_m(m^2-m-2)(B_m(m^2-2) + B_m(m^2-1) + B_m(m^2)).\end{aligned} \tag{25}$$

By (20),

$$B_m(m^2-m-3)\sum_{i=2}^{m+1} B_m(m^2-i) < B_m(m^2-m-2)\sum_{i=3}^{m+2} B_m(m^2-i). \tag{26}$$

Combining (25) and (26) yields

$$\frac{\sum\limits_{i=m^2-m-3}^{m^2} B_m(i)}{\sum\limits_{i=m^2-m-2}^{m^2} B_m(i)} < \frac{\sum\limits_{i=m^2-m-2}^{m^2} B_m(i)}{\sum\limits_{i=m^2-m-1}^{m^2} B_m(i) + (-1)^{m+1}}, \tag{27}$$

which can be rewritten as

$$\frac{B_{m+1}(m^2+m-2)}{B_{m+1}(m^2+m-1)} < \frac{B_{m+1}(m^2+m-1)}{B_{m+1}(m^2+m)}. \tag{28}$$

Therefore, (21) holds for the case $k = m^2 + m - 2$.

Based on (12) and (13), we deduce that for $m \ge 13$,

$$\begin{aligned}&\frac{B_m(m^2-2) + B_m(m^2-1) + B_m(m^2) - 2(m+2)}{B_m(m^2-1) + B_m(m^2)} - \frac{B_m(m^2-3)}{B_m(m^2-2)}\\ &\quad = \frac{m^4 - 12m^3 - 13m^2 + 36m - 36}{6m(m^2-m+2)} > 0.\end{aligned} \tag{29}$$

By (20) and (29),

$$\frac{B_m(m^2-m-2)}{B_m(m^2-m-1)} < \frac{B_m(m^2-3)}{B_m(m^2-2)} < \frac{B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)-2(m+2)}{B_m(m^2-1)+B_m(m^2)}. \tag{30}$$

It follows from (24) and (30) that

$$\begin{aligned} &B_m(m^2-m-2)(B_m(m^2-1)+B_m(m^2)) \\ &\quad < B_m(m^2-m-1)(B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)-2(m+2)) \\ &\quad < B_m(m^2-m-1)(B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)) - 2\sum_{i=0}^{m+1} B_m(m^2-i) \\ &\quad \le B_m(m^2-m-1)(B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)) \\ &\qquad + 2\times(-1)^{m+1}\sum_{i=0}^{m+1} B_m(m^2-i). \end{aligned} \tag{31}$$

In view of (20),

$$B_m(m^2-m-2)\sum_{i=2}^{m} B_m(m^2-i) < B_m(m^2-m-1)\sum_{i=3}^{m+1} B_m(m^2-i). \tag{32}$$

It follows from (31) and (32) that

$$\frac{\sum\limits_{i=m^2-m-2}^{m^2} B_m(i)}{\sum\limits_{i=m^2-m-1}^{m^2} B_m(i)+(-1)^{m+1}} < \frac{\sum\limits_{i=m^2-m-1}^{m^2} B_m(i)+(-1)^{m+1}}{\sum\limits_{i=m^2-m}^{m^2} B_m(i)}. \tag{33}$$

By (11), we can rewrite (33) as follows

$$\frac{B_{m+1}(m^2+m-1)}{B_{m+1}(m^2+m)} < \frac{B_{m+1}(m^2+m)}{B_{m+1}(m^2+m+1)}, \tag{34}$$

which implies that (21) holds for the case $k=m^2+m-1$.

In view of (12) and (13), we see that for $m \ge 4$,

$$\begin{aligned} &\frac{B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)-m}{B_m(m^2-1)+B_m(m^2)} - \frac{B_m(m^2-3)}{B_m(m^2-2)} \\ &\quad = \frac{(m+1)(m^3-7m^2+12m+12)}{6m(m^2-m+2)} > 0. \end{aligned} \tag{35}$$

By (20) and (35), we find that for $m \ge 4$,

$$\frac{B_m(m^2-m-1)}{B_m(m^2-m)} < \frac{B_m(m^2-3)}{B_m(m^2-2)} < \frac{B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)-m}{B_m(m^2-1)+B_m(m^2)}. \tag{36}$$

It follows from (24) and (36) that

$$\begin{aligned} &B_m(m^2-m-1)(B_m(m^2-1)+B_m(m^2)) + (-1)^{m+1}\sum_{i=0}^{m-1} B_m(m^2-i) \\ &\quad < B_m(m^2-m-1)(B_m(m^2-1)+B_m(m^2)) + mB_m(m^2-m) \\ &\quad \le B_m(m^2-m)(B_m(m^2-2)+B_m(m^2-1)+B_m(m^2)). \end{aligned} \tag{37}$$

By (20),

$$B_m(m^2-m-1)\sum_{i=2}^{m-1} B_m(m^2-i) < B_m(m^2-m)\sum_{i=3}^{m} B_m(m^2-i). \tag{38}$$

In view of (37) and (38), we can prove that

$$\frac{\sum\limits_{i=m^2-m-1}^{m^2} B_m(i) + (-1)^{m+1}}{\sum\limits_{i=m^2-m}^{m^2} B_m(i)} < \frac{\sum\limits_{i=m^2-m}^{m^2} B_m(i)}{\sum\limits_{i=m^2-m+1}^{m^2} B_m(i)}. \tag{39}$$

By (11), we can rewrite (39) as follows

$$\frac{B_{m+1}(m^2+m)}{B_{m+1}(m^2+m+1)} < \frac{B_{m+1}(m^2+m+1)}{B_{m+1}(m^2+m+2)}, \tag{40}$$

which implies that (21) holds for the case $k = m^2 + m$. This completes the proof.

**Acknowledgement:** The authors would like to thank the anonymous referee for valuable corrections and comments. This work was supported by the National Science Foundation of China (11701362).

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