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Bounded solutions of self-adjoint second order linear difference equations with periodic coefficients

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Abstract: In this work we obtain easy characterizations for the boundedness of the solutions of the discrete, self–adjoint, second order and linear unidimensional equations with periodic coefficients, including the analysis of the so-called discrete *Mathieu equations* as particular cases.

Keywords: Discrete Schrödinger operator, Mathieu operator, Periodic coefficients, Bounded solutions

MSC: 39A12, 39A70

1 Introduction

Discrete Schrödinger operators over finite or infinite paths have been subject of an intensive research over the last four decades. They represent the discrete analogs of one–dimensional self–adjoint operators on a bounded or unbounded interval on the real line, see for instance [1]. In addition, those operators are in relation with *Jacobi matrices* and hence with the classical theory of orthogonal polynomials.

The particular case of the so–called *almost Mathieu operator* has deserved special attention not only by its connections to physics but for its rich spectral theory. In fact, one of the main problems in this area, related to the topological structure of the spectra and popularized as the *Ten Martini Problem*, has been recently solved by concatenating the work of many outstanding researchers, see [2–4]. The problem is closely related to the determination of those *energies* for which the corresponding Schrödinger equation has non trivial bounded eigenfunctions.

The aim of this communication is by far much more modest. We use recent advances in the study of linear difference equations with periodic coefficients, see [5], to provide easy characterizations for the boundedness of the solutions of the *Mathieu equations*, that correspond to some specific Schrödinger equations with periodic potential, see [6]. Moreover, we also extend the results to general second order linear difference equations with periodic coefficients.

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2 Preliminaries

Throughout the paper, $\ell(\mathbb{Z})$ denotes the vector space of real sequences; that is, $\ell(\mathbb{Z}) = \{z\colon \mathbb{Z} \longrightarrow \mathbb{R}\}$, whereas $\ell^*(\mathbb{Z})$ is the set of sequences $z \in \ell(\mathbb{Z})$ such that $z(k) \neq 0$ for all $k \in \mathbb{Z}$. The null sequence, also called the *trivial sequence*, is denoted by $\mathtt{0}$.

Given $z \in \ell(\mathbb{Z})$ and $p \in \mathbb{N}^*$, for any $m \in \mathbb{Z}$ we denote by $z_{p,m} \in \ell(\mathbb{Z})$ the subsequence of z defined as

$$z_{p,m}(k) = z(kp + m), \quad k \in \mathbb{Z}.$$

Clearly, any sequence $z \in \ell(\mathbb{Z})$ is completely determined by the values of the sequences $z_{p,j}$, for $0 \le j \le p-1$. In particular, $z_{1,0} = z$, whereas $z_{2,0}$ and $z_{2,1}$ are the subsequences of z formed by the even or odd indexes, respectively. Moreover, the sequences $z_{1,m}$ are the *shift subsequences* of z, since $z_{1,m}(k) = z(k+m)$ for any $k \in \mathbb{Z}$. Notice that if we also allow $p = -1$, then $z_{-1,m}$ are the *flipped shift subsequences* of z, since $z_{-1,m}(k) = z(m-k)$ for any $k \in \mathbb{Z}$.

The sequence $z \in \ell(\mathbb{Z})$ is called *periodic with period* $p \in \mathbb{N}^*$ if it satisfies that

$$z(p + k) = z(k), \quad k \in \mathbb{Z},$$

which also implies that $z(kp + m) = z(m)$ for any $k, m \in \mathbb{Z}$.

The set of periodic sequences with period p is denoted by $\ell(\mathbb{Z}; p)$ and we define $\ell^*(\mathbb{Z}; p) = \ell(\mathbb{Z}; p) \cap \ell^*(\mathbb{Z})$. In particular $\ell(\mathbb{Z}; 1)$ consists of all constant sequences and then, it is identified with $\mathbb{R}$.

Lemma 2.1. *Given $z \in \ell(\mathbb{Z})$, then z is bounded iff there exists $p \in \mathbb{N}^*$ such that $z_{p,j}$ is bounded, for $0 \le j \le p-1$ and then, $z_{r,m}$ is bounded for any $r \in \mathbb{N}^*$ and any $m \in \mathbb{Z}$.*

Given $p \in \mathbb{N}^*$, then $z \in \ell(\mathbb{Z}; p)$ iff $z_{p,m} \in \ell(\mathbb{Z}; 1)$ for any $m \in \mathbb{Z}$. Moreover, all periodic sequence is also bounded.

Given $p \in \mathbb{N}^*$, $a \in \ell(\mathbb{Z}; p)$ and $c \in \ell^*(\mathbb{Z}; p)$, consider the associated self–adjoint operator $\Delta_{a,c}\colon \ell(\mathbb{Z}) \longrightarrow \ell(\mathbb{Z})$, defined as

$$\Delta_{a,c}(z)(k) = c(k)z(k+1) + c(k-1)z(k-1) - a(k)z(k), \quad k \in \mathbb{Z} \tag{1}$$

and the corresponding (irreductible) homogeneous equation

$$\Delta_{a,c}(z) = \mathtt{0}. \tag{2}$$

The sequences a and c are called the *coefficients of the Equation* (2) and any sequence $z \in \ell(\mathbb{Z})$ satisfying the Identity (2) is called a *solution of the equation*. It is well-known that for any $z_0, z_1 \in \mathbb{R}$ and any $m \in \mathbb{Z}$, there exists a unique solution of Equation (2) satisfying $z(m) = z_0$ and $z(m+1) = z_1$.

The problem we are interested in, can be formulated as follows:

For which coefficients $a, c \in \ell(\mathbb{Z}; p)$ has the equation $\Delta_{a,c}(z) = \mathtt{0}$ bounded solutions, other than the trivial one?

Operator (1), and hence Equation (2), encompasses many specific examples that have been widely considered in the literature. For instance, when $c(k) > 0$ for any $k \in \mathbb{Z}$, then $\Delta_{a,c} = -\mathcal{L}_q$, the Schrödinger operator on the infinite path with conductance c and potencial $q(k) = a(k) - c(k) - c(k-1)$; that is,

$$\mathcal{L}_q(z)(k) = c(k)\big(z(k) - z(k+1)\big) + c(k-1)\big(z(k) - z(k-1)\big) + q(k)z(k).$$

In particular, when $c(k) = 1$, for any $k \in \mathbb{Z}$, then $\Delta_{a,c}$ is known as the *Harper operator* and denoted by $\mathcal{H}_a$. More specifically when, in addition, the coefficient a is given by $a(k) = E - \lambda \cos(2\pi\omega k + \theta)$, $k \in \mathbb{Z}$, then the operator $\mathcal{H}_a$ is called *Mathieu operator* and the parameters $E, \lambda \in \mathbb{R}$, $\omega \in \mathbb{Q}$, $\theta \in [0, 2\pi)$, are called the *energy, coupling, frequency, and phase*, respectively. In this case the operator $\mathcal{H}_a$ is usually represented as $\mathcal{H}_{E,\lambda,\omega,\theta}$. If we permit the frequency not to be a rational number; that is, $\omega \in \mathbb{R}$, then $\mathcal{H}_{E,\lambda,\omega,\theta}$ is called *almost Mathieu operator*, but it does not have periodic coefficients. Therefore, in this work we are only interested in Mathieu operators; that is, in rational frequencies. We must bear in mind that when $\omega = \frac{m}{p}$, where $m \in \mathbb{Z}$ and $p \in \mathbb{N}^*$ are relative primes, then $a \in \ell(\mathbb{Z}; p)$.

Observe that the equation $\mathcal{H}_{E,\lambda,\omega,\theta}(u) = 0$ is equivalent to the equation $\mathcal{H}_{\lambda,\omega,\theta}(u) = Eu$; where $\mathcal{H}_{\lambda,\omega,\theta}$ denotes the operator $\mathcal{H}_{E,\lambda,\omega,\theta}$ when $E = 0$. Therefore, the energy E is an eigenvalue and u a corresponding eigenfunction of the operator $\mathcal{H}_{\lambda,\omega,\theta}$.

The interested reader can find the physics meaning of these parameters and the physics background of these kind of operators in [1–3] and also in [7].

The paper [5] was devoted to the Floquet Theory for the equation $\Delta_{a,c} = 0$; that is, to the condition under which the above equation has periodic solutions. Since any periodic solution is bounded, this characterization gives us only a partial answer to the main question. However, we can follow the same techniques as in [5] to completely solve the question.

We end this preliminary section by remarking that when only a finite interval in $\mathbb{Z}$ is considered, namely when $k = 0, 1, \dots, n$ for some $n \in \mathbb{N}$, then Equation (2) must be supplied with some *boundary conditions* and it is related with the inversion of finite and symmetric Jacobi matrices, see for instance [8]. Another interesting application of these boundary value problems falls in the ambit of Organic Chemistry, see Examples 1 and 2 in page 364 of [5]. In this case, all the eigenfuncions are bounded, so the main problem is nothing else that the consideration of the eigenvalue problem. For the Mathieu equation with null frequency, this analysis in the finite interval case can be found in [9].

3 The easiest case

The most simple case of the proposed problem corresponds to $a, c \in \ell(\mathbb{Z}; 1)$; that is, when the coefficients of $\Delta_{a,c}$ are constant; *i.e.* $a \in \mathbb{R}$ and $c \in \mathbb{R}^*$. Self–adjoint linear difference equations with constant coefficients can be characterized as those satisfying that $z \in \ell(\mathbb{Z})$ is a solution iff any shift and any flipped shift of z is also a solution. Moreover, in this case, Equation (2) is equivalent, in the sense that both have the same solutions, to the *Chebyshev equation with parameter q*

$$z(k+1) - 2qz(k) + z(k-1) = 0, \quad k \in \mathbb{Z}, \tag{3}$$

where $q = \dfrac{a}{2c}$. So, we can say that the most simple case to analyze corresponds to both the uncoupled Harper equation and the coupled Harper equation with null frequency. Moreover, these two kinds of equations can be viewed in an unified manner as Chebyshev equations. Any solution of a Chebyshev equation with parameter q is called *Chebyshev sequence with parameter q*.

Recall that a polynomial sequence $\{P_k(x)\}_{k\in\mathbb{Z}} \subset \mathbb{R}[x]$ is a *sequence of Chebyshev polynomials* if it satisfies the following three-term recurrence, see [10],

$$P_{k+1}(x) = 2xP_k(x) - P_{k-1}(x), \quad k \in \mathbb{Z}. \tag{4}$$

Therefore, any Chebyshev sequence with parameter q is of the form $\{P_k(q)\}_{k\in\mathbb{Z}}$, where $\{P_k(x)\}_{k\in\mathbb{Z}}$ is a sequence of Chebyshev polynomials. So, many properties of Chebyshev sequences are the consequence of properties of Chebyshev polynomials and conversely.

As a by–product of the Proposition 2.1 in [5], we have the following basic result about periodic and bounded Chebyshev sequences.

Proposition 3.1. *Given $a \in \mathbb{R}$, $c \neq 0$ we have the following results:*

(i) *The equation $\Delta_{a,c}(z) = 0$ has bounded solutions iff $|a| \le 2|c|$. Moreover, when $|a| < 2|c|$ all the solutions are bounded.*

(ii) *When $a = 2c$ the unique bounded solutions are the constant ones, whereas when $a = -2c$ the bounded solutions are all multiple of $z(k) = (-1)^k$.*

(iii) *The equation $\Delta_{a,c}(z) = 0$ has periodic solutions with period p iff $a = 2c\cos\left(\frac{2\pi j}{p}\right)$, $j = 0, \dots, \lceil\frac{p-1}{2}\rceil$.*

(iv) *The equation $\Delta_{a,c}(z) = 0$ has constant solutions iff $a = 2c$*

The main result about the boundedness of solutions for difference equations with constant coefficients has the following particularization for Mathieu operators.

Corollary 3.2. *Given $E, \lambda \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ the Mathieu equation with null frequency*

$$z(k+1) + z(k-1) - \lambda\cos(\theta)z(k) = Ez(k), \quad k \in \mathbb{Z}$$

has bounded solutions iff $|E - \lambda\cos(\theta)| \le 2$ and all its solutions are bounded when the inequality is strict.

The equation has constant solutions iff $E = 2 + \lambda\cos(\theta)$ in which case the constant ones are the unique bounded solutions, whereas when $E = \lambda\cos(\theta) - 2$ the bounded solutions are multiple of $z(k) = (-1)^k$. Finally, the equation has has periodic solutions with period p iff $E = \lambda\cos(\theta) + 2\cos\left(\frac{2\pi j}{p}\right)$, $j = 0, \dots, \lceil\frac{p-1}{2}\rceil$.

4 The general case

Back to the general case, consider $p \in \mathbb{N}^*$, $a \in \ell(\mathbb{Z}; p)$, $c \in \ell^*(\mathbb{Z}; p)$ and the associate self–adjoint operator $\Delta_{a,c}$. Although this scenario seems to be far away from the easiest one analyzed in the previous section, we will show that in fact Chebyshev equations contain all the information needed to conclude the existence of bounded solutions for the difference equation $\Delta_{a,c}(z) = 0$. This is true because the main result in [5] establishes that (irreductible) second order difference equations (not necessarily self-adjoint) with periodic coefficients are basically equivalent to some Chebyshev equation. For the setting concerning to this paper we have the following facts.

Lemma 4.1 ([5, Theorem 3.3]). *Given $p \in \mathbb{N}^*$, $a \in \ell(\mathbb{Z}; p)$ and $c \in \ell^*(\mathbb{Z}; p)$; there exists $q(a, c; p) \in \mathbb{R}$, depending only on the coefficients a and c and on the period p, such that $z \in \ell(\mathbb{Z})$ is a solution of the equation $\Delta_{a,c}(z) = 0$ iff for any $m \in \mathbb{Z}$, $z_{p,m}$ is a solution of the Chebyshev equation with parameter $q(a, c; p)$; that is*

$$v(k+1) - 2q(a, c; p)v(k) + v(k-1) = 0, \quad k \in \mathbb{Z}.$$

As the boundedness of z is equivalent to the boundedness of the sequences $z_{p,m}$, $m = 0, \dots, p-1$, we can conclude that existence of bounded solutions for the equation $\Delta_{a,c}(z) = 0$, depends only on the knowledge of the specific value $q(a, c; p)$. Since in [5, Theorem 3.3] the existence of this parameter was proved by induction the above result is not useful in practice. For this reason, most of the above mentioned paper was devoted to the explicit computation of the so–called *Floquet function*; that is, the function assigning the value $q(a, c; p)$ to any $a \in \ell(\mathbb{Z}; p)$ and $c \in \ell^*(\mathbb{Z}; p)$. Notice that, in fact, the value $q(a, c; p)$ only depends on $a(j), c(j)$, $j = 0, \dots, p-1$. Once this function was obtained, the characterization of the existence of periodic solutions for the equation $\Delta_{a,c}(z) = 0$ appears as a simple by–product, since from Lemma 2.1, they are characterized as being constant the sequences $z_{p,m}$, $0 \le m \le p-1$, see [5, Corollary 4.8]. So, the main novelty of this paper is to derive the characterization of the existence of bounded solutions for the equation $\Delta_{a,c}(z) = 0$, from the value $q(a, c; p)$. To do this, we need to introduce some notations and concepts.

A *binary multi-index of order p* is a p-tuple $\alpha = (\alpha_0, \dots, \alpha_{p-1}) \in \{0, 1\}^p$ and its *length* is defined as $|\alpha| = \sum_{j=0}^{p-1} \alpha_j \le p$. So $|\alpha| = m$ iff exactly m components of α are equal to 1 and exactly $p - m$ components of α are equal to 0.

Given a binary multi-index of order p, $\alpha \in \{0, 1\}^p$ such that $|\alpha| = m \ge 1$, we consider $0 \le i_1 < \dots < i_m \le p-1$ such that $\alpha_{i_1} = \dots = \alpha_{i_m} = 1$. Given $p \in \mathbb{N}^*$, we define the following subsets of the set $\{0, 1\}^p$ of binary multi-indexes of order p:

(i) $\Lambda_p^0 = \{(0, \dots, 0)\}$, for $p \ge 1$.
(ii) $\Lambda_p^1 = \{\alpha : |\alpha| = 1\}$, for $p \ge 2$.
(iii) $\Lambda_p^m = \{\alpha : |\alpha| = m, \ i_{j+1} - i_j \ge 2, \ 1 \le j \le m-1 \text{ and } i_m \le p-2 \text{ if } i_1 = 0\}$ for $p \ge 4$, and $m = 2, \dots, \lfloor\frac{p}{2}\rfloor$, where $0 \le i_1 < \dots < i_m \le p-1$ are the indexes such that $\alpha_{i_1} = \dots = \alpha_{i_m} = 1$.

In addition, if $p \geq 2$, $m = 1, \dots, \lfloor \frac{p}{2} \rfloor$ and $\alpha \in \Lambda_p^m$, let $0 \leq i_1 < \cdots < i_m \leq p-1$ be the indexes such that $\alpha_{i_1} = \cdots = \alpha_{i_m} = 1$. Then, we define the binary multi-index $\bar{\alpha}$ of order p as

$$\bar{\alpha}_{i_j} = \bar{\alpha}_{i_j+1} = 0, \ \ j = 1, \dots, m, \quad \text{and} \quad \bar{\alpha}_i = 1 \quad \text{otherwise},$$

where if $i_m = p-1$, then $\bar{\alpha}_{p-1} = \bar{\alpha}_0 = 0$. Moreover, if $\alpha = (0, \dots, 0)$; that is, if $\alpha \in \Lambda_p^0$, then we define $\bar{\alpha} = (1, \dots, 1)$. It is clear that, in any case, $|\bar{\alpha}| = p - 2m$.

We are now ready to show the expression for the value of $q(a, c; p)$. In the sequel, we always assume that $0^0 = 1$ and also the usual convention that empty sums and empty products are defined as 0 and 1, respectively.

Lemma 4.2 ([5, Theorem 4.4]). *Given $p \in \mathbb{N}^*$, $a \in \ell(\mathbb{Z}; p)$ and $c \in \ell^*(\mathbb{Z}; p)$, then*

$$q(a, c; p) = \frac{1}{2}\Big(\prod_{i=0}^{p-1} c(i)\Big)^{-1} \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \sum_{\alpha \in \Lambda_p^j} \prod_{i=0}^{p-1} c(i)^{2\alpha_i} a(i)^{\bar{\alpha}_i}.$$

Observe that when $p = 1$, the above identity becomes $q(a, c; 1) = \dfrac{a}{2c}$; that is, the value corresponding to the case in which the coefficients a and c are constant; or equivalent both have period $p = 1$.

Our main result appears now as a consequence of the Proposition 3.1 together with Lemma 2.1 and also the above Lemma.

Theorem 4.3. *Given $p \in \mathbb{N}^*$, $a \in \ell(\mathbb{Z}; p)$ and $c \in \ell^*(\mathbb{Z}; p)$, then the equation*

$$c(k)z(k+1) + c(k-1)z(k-1) - a(k)z(k) = 0, \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\Big| \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \sum_{\alpha \in \Lambda_p^j} \prod_{i=0}^{p-1} c(i)^{2\alpha_i} a(i)^{\bar{\alpha}_i} \Big| \leq 2 \prod_{i=0}^{p-1} |c(i)|$$

and when the inequality is strict, all the solutions are bounded. Moreover, if

$$\sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \sum_{\alpha \in \Lambda_p^j} \prod_{i=0}^{p-1} c(i)^{2\alpha_i} a(i)^{\bar{\alpha}_i} = 2 \prod_{i=0}^{p-1} c(i)$$

then the equation has periodic solutions with period p and these are the unique bounded solutions.

Corollary 4.4. *Given $E, \lambda \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and $\omega = \dfrac{m}{p}$, where $p \in \mathbb{N}^*$, $m \in \mathbb{Z}$ and $(p, m) = 1$, then the Mathieu equation*

$$z(k+1) + z(k-1) + \lambda \cos(2\pi\omega k + \theta) z(k) = Ez(k), \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\Big| \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \sum_{\alpha \in \Lambda_p^j} \prod_{i=0}^{p-1} \big(E - \lambda \cos(2\pi\omega i + \theta)\big)^{\bar{\alpha}_i} \Big| \leq 2$$

and when the inequality is strict, all the solutions are bounded. Moreover, if

$$\sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \sum_{\alpha \in \Lambda_p^j} \prod_{i=0}^{p-1} \big(E - \lambda \cos(2\pi\omega i + \theta)\big)^{\bar{\alpha}_i} = 2$$

then the Mathieu equation has periodic solutions with period p and these are the unique bounded solutions.

Clearly, the main difficulty to apply the above characterizations is to obtain the binary multi–indexes involved in them. In general, this is a difficult task and, in fact, the number of multi–indexes in Λ_p^j, $0 \leq j \leq \lfloor \frac{p}{2} \rfloor$, grows dramatically with p. Specifically, we have $|\Lambda_p^j| = \frac{p}{p-j}\binom{p-j}{j}$ for any $0 \le j \le \lfloor \frac{p}{2} \rfloor$, which for any $m \in \mathbb{N}^*$ implies that $\sum_{j=0}^{m} |\Lambda_{2m}^j| = 2T_m\left(\frac{3}{2}\right)$ and that $\sum_{j=0}^{m} |\Lambda_{2m+1}^j| = W_m\left(\frac{3}{2}\right)$, see [5, Proposition 4.2].

We end this paper with some specific examples using the given characterization for the existence of bounded solutions for difference equations with periodic coefficients with period up to 4. Remember that the case $p = 1$, the easiest case, has been analyzed in the previous sections.

Corollary 4.5 (Period $p = 2$). *Given $a \in \ell(\mathbb{Z}; 2)$ and $c \in \ell^*(\mathbb{Z}; 2)$, then the equation*

$$c(k)z(k+1) + c(k-1)z(k-1) - a(k)z(k) = 0, \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\big(|c(0)| - |c(1)|\big)^2 \le a(0)a(1) \le \big(|c(0)| + |c(1)|\big)^2$$

and when both inequalities are strict, then all the solutions are bounded. In particular, given $E, \lambda \in \mathbb{R}, \theta \in [0, 2\pi)$, then the Mathieu equation

$$z(k+1) + z(k-1) + \lambda \cos(\pi k + \theta)z(k) = Ez(k), \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$0 \le E^2 - \lambda^2 \cos^2(\theta) \le 4.$$

Proof. In this case we have $\Lambda_2^0 = \{(0,0)\}$, $\Lambda_2^1 = \{(1,0),(0,1)\}$, $\bar{\Lambda}_2^0 = \{(1,1)\}$ and $\bar{\Lambda}_2^1 = \{(0,0),(0,0)\}$, which implies that $q(a,c;2) = \dfrac{a(0)a(1) - c(0)^2 - c(1)^2}{2c(0)c(1)}$.

In particular, for the Mathieu which coefficient has period 2, the frequency is $\omega = \dfrac{m}{2}$, where $m \in \mathbb{Z}$ is odd and hence $\omega = n + \frac{1}{2}$ with $n \in \mathbb{Z}$. Therefore, the coefficient is $a(k) = E - \lambda \cos(\pi k + \theta)$, which implies that $a(0) = E - \lambda \cos(\theta)$, whereas $a(1) = E - \lambda \cos(\pi + \theta) = E + \lambda \cos(\theta)$. □

Corollary 4.6 (Period $p = 3$). *Given $a \in \ell(\mathbb{Z}; 3)$ and $c \in \ell^*(\mathbb{Z}; 3)$, then the equation*

$$c(k)z(k+1) + c(k-1)z(k-1) - a(k)z(k) = 0, \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\Big|a(0)a(1)a(2) - c(0)^2a(2) - c(1)^2a(0) - c(2)^2a(1)\Big| \le 2|c(0)c(1)c(2)|$$

and when the inequality is strict, all the solutions are bounded. In particular, given $E, \lambda \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and $\omega = \dfrac{m}{3}$, where $m \in \mathbb{Z}$ and $(3, m) = 1$, then the Mathieu equation

$$z(k+1) + z(k-1) + \lambda \cos(2\pi\omega k + \theta)z(k) = Ez(k), \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\Big|\Big(E - \lambda \cos(\theta)\Big)\Big(E^2 - \frac{3}{4}\lambda^2 + \lambda^2\cos^2(\theta) + E\lambda\cos(\theta)\Big) - 3E\Big| \le 2.$$

Proof. In this case we have $\Lambda_3^0 = \{(0,0,0)\}$ and $\Lambda_3^1 = \{(1,0,0),(0,1,0),(0,0,1)\}$, which implies that $\bar{\Lambda}_3^0 = \{(1,1,1)\}$ and $\bar{\Lambda}_3^1 = \{(0,0,1),(1,0,0),(0,1,0)\}$ and hence,

$$q(a,c;3) = \frac{a(0)a(1)a(2) - c(0)^2a(2) - c(1)^2a(0) - c(2)^2a(1)}{2c(0)c(1)c(2)}.$$

In particular, for the Mathieu equation the condition for the existence of bounded solutions becomes

$$|a(0)a(1)a(2) - a(0) - a(1) - a(2)| \le 2.$$

On the other hand, the frequency is $\omega = \frac{m}{3}$ where $(m, 3) = 1$ which implies that $\omega = n + \frac{r}{3}$, where $n \in \mathbb{Z}$ and $r = 1, 2$. Therefore, the coefficient is given by $a_r(k) = E - \lambda \cos\left(\pi \frac{2r}{3} k + \theta\right)$, and hence

$$\begin{aligned} a_1(0) &= a_2(0) = E - \lambda \cos(\theta), \\ a_1(1) &= a_2(2) = E - \lambda \cos\left(\pi\tfrac{2}{3} + \theta\right) = E + \frac{\lambda}{2}\left[\cos(\theta) + \sqrt{3}\sin(\theta)\right], \\ a_1(2) &= a_2(1) = E - \lambda \cos\left(\pi\tfrac{4}{3} + \theta\right) = E + \frac{\lambda}{2}\left[\cos(\theta) - \sqrt{3}\sin(\theta)\right] \end{aligned}.$$

□

Corollary 4.7 (Period $p = 4$). *Given $a \in \ell(\mathbb{Z}; 4)$ and $c \in \ell^*(\mathbb{Z}; 4)$, then the equation*

$$c(k)z(k+1) + c(k-1)z(k-1) - a(k)z(k) = 0, \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$\Big| a(0)a(1)a(2)a(3) - c(0)^2a(2)a(3) - c(1)^2a(0)a(3) - c(2)^2a(0)a(1) \\ -c(3)^2a(1)a(2) + c(0)^2c(2)^2 + c(1)^2c(3)^2\Big| \le 2|c(0)c(1)c(2)c(3)|$$

and when the inequality is strict, all the solutions are bounded. In particular, given $E, \lambda \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and $\omega = \frac{m}{4}$, where $m \in \mathbb{Z}$ and $(4, m) = 1$, then the Mathieu equation

$$z(k+1) + z(k-1) + \lambda\cos(2\pi\omega k + \theta)z(k) = Ez(k), \quad k \in \mathbb{Z}$$

has bounded solutions iff

$$4(E^2 - 1) \le \left(E^2 - \lambda^2\cos^2(\theta)\right)\left(E^2 - \lambda^2\sin^2(\theta)\right) \le 4E^2.$$

Proof. In this case we have $\Lambda_4^0 = \{(0,0,0,0)\}$,

$$\Lambda_4^1 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}, \quad \Lambda_4^2 = \{(1,0,1,0), (0,1,0,1)\}$$

which implies that $\bar{\Lambda}_4^0 = \{(1,1,1,1)\}$ and

$$\bar{\Lambda}_4^1 = \{(0,0,1,1), (1,0,0,1), (1,1,0,0), (0,1,1,0)\}, \quad \bar{\Lambda}_4^2 = \{(0,0,0,0), (0,0,0,0)\}$$

and hence,

$$\begin{aligned} q(a, c; 4) = \frac{1}{2c(0)c(1)c(2)c(3)}\Big[&a(0)a(1)a(2)a(3) - c(0)^2a(2)a(3) - c(1)^2a(0)a(3) \\ &-c(2)^2a(0)a(1) - c(3)^2a(1)a(2) + c(0)^2c(2)^2 + c(1)^2c(3)^2\Big] \end{aligned}$$

In particular, for the Mathieu equation the condition for the existence of bounded solutions becomes

$$-4 \le a(0)a(1)a(2)a(3) - a(2)a(3) - a(0)a(3) - a(0)a(1) - a(1)a(2) \le 0.$$

On the other hand, the frequency is $\omega = \frac{m}{4}$ where $(m, 4) = 1$ which implies that $\omega = n + \frac{r}{4}$, where $n \in \mathbb{Z}$ and $r = 1, 3$. Therefore, the coefficient is given by $a_r(k) = E - \lambda\cos\left(\pi\frac{r}{2} k + \theta\right)$, and hence

$$\begin{aligned} a_1(0) &= a_3(0) = E - \lambda\cos(\theta), \\ a_1(1) &= a_3(3) = E + \lambda\sin(\theta), \\ a_1(2) &= a_3(2) = E + \lambda\cos(\theta), \\ a_1(3) &= a_3(1) = E - \lambda\sin(\theta) \end{aligned}.$$

□

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