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Singular Cauchy problem for the general Euler-Poisson-Darboux equation

General Euler-Poisson-Darboux equation

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Abstract: In this paper we obtain the solution of the singular Cauchy problem for the Euler-Poisson-Darboux equation when differential Bessel operator acts by each variable.

Keywords: Bessel operator, Euler-Poisson-Darboux equation, Singular Cauchy problem

MSC: 26A33, 44A15

1 Introduction

The classical Euler-Poisson-Darboux equation has the form

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},\quad u=u(x,t),\quad x\in\mathbb{R}^n,\quad t>0,\quad -\infty<k<\infty. \tag{1}$$

The operator acting by t in (1) is called the **Bessel operator**. For the Bessel operator we use the notation (see. [1], p. 3)

$$(B_k)_t=\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}.$$

The Euler-Poisson-Darboux equation for $n = 1$ appears in Euler's work (see [2], p. 227). Further Euler's case of (1) was studied by Poisson in [3], Riemann in [4] and Darboux in [5] (for the history of this issue see also in [6], p. 532 and [7], p. 527). The generalization of it was studied in [8]. When $n \geq 1$ the equation (1) was considered, for example, in [9, 10]. The Euler-Poisson-Darboux equation appears in different physics and mechanics problems (see [11–15]). In [16] (see also [17], p. 243) and in [18] there were different approaches to the solution of the Cauchy problem for the general Euler-Poisson-Darboux equation

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},\qquad 0<\gamma_i,\qquad i=1,\dots,n,\qquad k>0 \tag{2}$$

with the initials conditions

$$u(x,0)=f(x),\qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0. \tag{3}$$

The Cauchy problem with the nonequal to zero first derivative by t of u for the (2) (and for (1)) is incorrect. However, if we use the special type of the initial conditions containing the nonequal to zero first derivative by t of u then such Cauchy problem for the (2) will by solvable. Following [17] and [19] we will use the term

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singular Cauchy problem in this case. The abstract Euler-Poisson-Darboux equation (when in the left hand of (2) an arbitrary closed linear operator is presented) was studied in [20–22].

In this article we consider the solution of the problem (2)-(3) when $-\infty < k < +\infty$ and its properties. Besides this, we get the formula for the connection of solution of the problem (2)-(3) and solution of a simpler problem. Also using the solution of the problem (2)-(3) we obtain solution of the singular Cauchy problem for the equation (2) when $k < 1$ with the conditions

$$u(x,0)=0, \qquad \lim_{t\to 0} t^k \frac{\partial u}{\partial t} = \varphi(x). \tag{4}$$

2 Property of general Euler-Poisson-Darboux equations' solutions

In this section we give some necessary definitions and obtain two fundamental recursion formulas for solution of (2).

Let

$$\mathbb{R}^n_+=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n,\ \ x_1>0,\dots,x_n>0\}$$

and Ω is open set in $\mathbb{R}_n$ which is symmetric correspondingly to each hyperplane x_i=0, i=1, ..., n, $\Omega_+ = \Omega\cap\mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega\cap\overline{\mathbb{R}}^n_+$ where

$$\overline{\mathbb{R}}^n_+=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n,\ \ x_1\geq 0,\dots,x_n\geq 0\}.$$

We have $\Omega_+ \subseteq \mathbb{R}^n_+$ and $\overline{\Omega}_+ \subseteq \overline{\mathbb{R}}^n_+$. Consider the set $C^m(\Omega_+)$, $m \geq 1$, consisting of differentiable functions on Ω_+ by order m. Let $C^m(\overline{\Omega}_+)$ be the set of functions from $C^m(\Omega_+)$ such that all their derivatives by x_i for all $i = 1, \dots, n$ are continuous up to the x_i=0. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$ for all non-negative integers $k \leq \frac{m-1}{2}$ and all x_i, $i = 1, \dots, n$ (see [1], p. 21). A multi-index $\gamma=(\gamma_1,\dots,\gamma_n)$ consists of fixed positive numbers $\gamma_i > 0$, i=1, ..., n and $|\gamma|=\gamma_1+\dots+\gamma_n$.

We consider the multidimensional Euler-Poisson-Darboux equation wherein the Bessel operator acts in each of the variables:

$$(\triangle_\gamma)_x u = (B_k)_t\, u, \quad -\infty < k < \infty, \quad u = u^k(x,t), \quad x\in\mathbb{R}^n_+, \quad t>0, \tag{5}$$

where

$$(\triangle_\gamma)_x = \triangle_\gamma = \sum_{i=1}^n (B_{\gamma_1})_{x_i} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x}, \tag{6}$$

$$(B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t}, \qquad k\in\mathbb{R}.$$

Equation (5) we will call **the general Euler-Poisson-Darboux equation**.

Statement 2.1. *Let $u^k = u^k(x,t)$ denote the solution of* (5) *when the next two fundamental recursion formulas hold*

$$u^k = t^{1-k}u^{2-k}, \tag{7}$$

$$u^k_t = t u^{k+2}. \tag{8}$$

Proof. Following [23] we prove (7). Putting $w = t^{k-1}v$, $v = u^k$ we have

$$w_t = (k-1)t^{k-2}v + t^{k-1}v_t = \frac{k-1}{t}w + t^{k-1}v_t,$$

$$w_{tt} = (k-1)(k-2)t^{k-3}v + (k-1)t^{k-2}v_t + (k-1)t^{k-2}v_t + t^{k-1}v_{tt} =$$
$$= \frac{(k-1)(k-2)}{t^2}w + 2(k-1)t^{k-2}v_t + t^{k-1}v_{tt},$$

$$\frac{2-k}{t}w_t = -\frac{(k-1)(k-2)}{t^2}w + (2-k)t^{k-2}v_t,$$

$$w_{tt} + \frac{2-k}{t}w_t = 2(k-1)t^{k-2}v_t + t^{k-1}v_{tt} + (2-k)t^{k-2}v_t = t^{k-1}\left(v_{tt} + \frac{k}{t}v_t\right)$$

or

$$w_{tt} + \frac{2-k}{t}w_t = t^{k-1}\left(v_{tt} + \frac{k}{t}v_t\right). \tag{9}$$

If $w = t^{k-1}v$ satisfies the equation

$$\Delta_\gamma w = w_{tt} + \frac{2-k}{t}w_t,$$

then using (9) we get

$$t^{k-1}\,\Delta_\gamma v = t^{k-1}\left(v_{tt} + \frac{k}{t}v_t\right)$$

which means that v satisfies the equation

$$\Delta_\gamma v = v_{tt} + \frac{k}{t}v_t.$$

Denoting $w = u^{2-k}$ we obtain (7).

Now we prove the (8). Let $tw = v_t$, $v = u^k$. We obtain

$$w_t = -\frac{1}{t^2}v_t + \frac{1}{t}v_{tt},$$

$$w_{tt} = \frac{2}{t^3}v_t - \frac{2}{t^2}v_{tt} + \frac{1}{t}v_{ttt}.$$

We find now $\frac{k+2}{t}w_t$:

$$\frac{k+2}{t}w_t = -\frac{k+2}{t^3}v_t + \frac{k+2}{t^2}v_{tt}.$$

Then we get

$$w_{tt} + \frac{k+2}{t}w_t = \frac{2}{t^3}v_t - \frac{2}{t^2}v_{tt} + \frac{1}{t}v_{ttt} - \frac{k+2}{t^3}v_t + \frac{k+2}{t^2}v_{tt} =$$

$$= \frac{1}{t}v_{ttt} - \frac{k}{t^3}v_t + \frac{k}{t^2}v_{tt} = \frac{1}{t}\left(v_{ttt} - \frac{k}{t^2}v_t + \frac{k}{t}v_{tt}\right) = \frac{1}{t}\frac{\partial}{\partial t}\left(v_{tt} + \frac{k}{t}v_t\right)$$

or

$$w_{tt} + \frac{k+2}{t}w_t = \frac{1}{t}\frac{\partial}{\partial t}\left(v_{tt} + \frac{k}{t}v_t\right). \tag{10}$$

□

Recursion formulas (7) and (8) allow us to obtain, from a solution u_k of equation (5), the solutions of the same equation with the parameter $k+2$ and $2-k$, respectively. Both formulas are proved for Euler-Poisson-Darboux equation $\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} - \triangle u = 0$.

3 Weighted spherical mean and the first Cauchy problem for the general Euler-Poisson-Darboux equation

Here we present the solutions of the problem (2)-(3) for different values of k for which we obtain solution of (2)-(4) in the next section, and get formula for the connection of solution of problem (2)-(3) and solution of simpler problem when $k = 0$ in (2).

In $\mathbb{R}^n_+$ we will use multidimensional generalized translation corresponding to multi-index γ:

$${}^\gamma T^t = {}^{\gamma_1}T^{t_1}_{x_1} \ldots {}^{\gamma_n}T^{t_n}_{x_n},$$

where each ${}^{\gamma_i}T^{\tau_i}_{x_i}$ is defined by the formula (see [24])

$$ {}^{\gamma_i}T^{\tau_i}_{x_i}f(x)=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^{\pi} f(x_1,\ldots,x_{i-1},\sqrt{x_i^2+\tau_i^2-2x_i\tau_i\cos\alpha_i},x_{i+1},\ldots,x_n)\ \sin^{\gamma_i-1}\alpha_i\,d\alpha_i\,. $$

The below-considered weighted spherical mean generated by a multidimensional generalized translation ${}^{\gamma}T^{t}$ has the form (see [25])

$$ M_f^{\gamma}(x;r)=\frac{1}{|S_1^+(n)|_{\gamma}}\int\limits_{S_1^+(n)} {}^{\gamma}T_x^{r\theta}f(x)\theta^{\gamma}\,dS, \tag{11} $$

where $\theta^{\gamma}=\prod\limits_{i=1}^{n}\theta_i^{\gamma_i}$, $S_1^+(n)=\{\theta{:}|\theta|=1,\,\theta\in\mathbb{R}^n_+\}$ and the coefficient $|S_1^+(n)|_{\gamma}$ is computed by the formula

$$ |S_1^+(n)|_{\gamma}=\int\limits_{S_1^+(n)}\prod_{i=1}^{n}x_i^{\gamma_i}\,dS(y)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} \tag{12} $$

(see [26], p. 20, formula (1.2.5) in which we should put $N{=}n$). Construction of a multidimensional generalized translation and the weighted spherical mean are transmutation operators (see [27]).

Theorems 3.1-3.4 have been proved in [28]. We give formulations of these theorems here because they will be needed in the next section.

Theorem 3.1. *The weighted spherical mean of $f\in C^2_{ev}$ satisfies the general equation Euler–Poisson–Darboux equation*

$$ (\Delta_{\gamma})_x M_f^{\gamma}(x;t)=(B_k)_t M_f^{\gamma}(x;t),\quad k=n+|\gamma|-1 \tag{13} $$

and the conditions

$$ M_f^{\gamma}(x;0)=f,\quad (M_f^{\gamma})'_t(x;0)=0. \tag{14} $$

This theorem has been proved in [25]).

We give theorems on the solution of the Cauchy problem for the general Euler–Poisson–Darboux equation for the remaining values of k.

$$ (\Delta_{\gamma})_x u=(B_k)_t u,\quad u=u^k(x,t),\qquad x\in\mathbb{R}^n_+,\quad t>0, \tag{15} $$

$$ u^k(x,0)=f(x),\qquad u^k_t(x,0)=0. \tag{16} $$

Theorem 3.2. *Let $f\in C^2_{ev}$. Then for the case $k>n+|\gamma|-1$ the solution of* (15)–(16) *is unique and given by*

$$ u^k(x,t)=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\int\limits_{B_1^+(n)}[{}^{\gamma}T^{ty}f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^{\gamma}\,dy. \tag{17} $$

Using weighted spherical mean we can write

$$ u^k(x,t)=\frac{2t^{1-k}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^t(t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}M_f^{\gamma}(x;r)\,dr. \tag{18} $$

Theorem 3.3. *If $f\in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$ then the solution of* (15)–(16) *for $k<n+|\gamma|-1$, $k\neq-1,-3,-5,\ldots$*

$$ u^k(x,t)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m\left(t^{k+2m-1}u^{k+2m}(x,t)\right), \tag{19} $$

where m is a minimum integer such that $m\geq\frac{n+|\gamma|-k-1}{2}$ and $u^{k+2m}(x,t)$ is the solution of the Cauchy problem

$$ (B_{k+2m})_t u^{k+2m}(x,t)=(\Delta_{\gamma})_x u^{k+2m}(x,t), \tag{20} $$

$$u^{k+2m}(x,0)=\frac{f(x)}{(k+1)(k+3)\dots(k+2m-1)},\qquad u_t^{k+2m}(x,0)=0. \tag{21}$$

The solution of (15)–(16) *is unique for* $k\geq 0$ *and not unique for negative* k.

Theorem 3.4. *If* $f\in C_{ev}^{1-k}$ *is B–polyharmonic of order* $\frac{1-k}{2}$ *then one of the solutions of the Cauchy problem* (20)–(21) *for the* $k=-1,-3,-5,\dots$ *is given by*

$$u^{-1}(x,t)=f(x), \tag{22}$$

$$u^{k}(x,t)=f(x)+\sum_{h=1}^{-\frac{k+1}{2}}\frac{\Delta_\gamma^h f}{(k+1)\dots(k+2h-1)}\,\frac{t^{2h}}{2\cdot 4\cdot\ldots\cdot 2h},\qquad k=-3,-5,\dots \tag{23}$$

The solution of (15)–(16) *is not unique for negative* k.

The theorem 3.5 contains the explicit form of the transmutation operator for the solution. Definition, methods of construction and applications of the transmutation operators can be found in [27, 29, 30].

Theorem 3.5. *Let* $k>0$. *The twice continuously differentiable on* $\mathbb{R}_+^{n+1}$ *solution* $u=u^k(x,t)$ *of the Cauchy problem*

$$(\Delta_\gamma)_x u=(B_k)_t u,\quad u=u^k(x,t),\qquad x\in\mathbb{R}_+^n,\quad t>0, \tag{24}$$

$$u^k(x,0)=f(x),\qquad u_t^k(x,0)=0 \tag{25}$$

such that $u_{x_i}^k(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n,t)=0$, $i=1,\dots,n$ *is connected with the twice continuously differentiable on* $\mathbb{R}_+^n\times\mathbb{R}$ *solution* $w=w(x,t)$ *of the Cauchy problem*

$$(\Delta_\gamma)_x w=w_{tt},\quad w=w(x,t),\qquad x\in\mathbb{R}_+^n,\quad t\in\mathbb{R}, \tag{26}$$

$$w(x,0)=f(x),\qquad w_t(x,0)=0 \tag{27}$$

such that $w_{x_i}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n,t)=0$, $i=1,\dots,n$ *by formula*

$$u^k(x,t)=(\mathcal{P}_1^{\frac{k-1}{2}})_\alpha w(x,\alpha t), \tag{28}$$

where $(\mathcal{P}_\tau^\lambda)_\alpha$ *is transmutation Poisson operator (see [24]) acting by* α

$$(\mathcal{P}_\tau^\lambda)_\alpha g(\alpha)=\frac{2\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}\frac{1}{\tau^{2\lambda}}\int\limits_0^\tau g(\alpha)[\tau^2-\alpha^2]^{\lambda-\frac{1}{2}}\,d\alpha.$$

Proof. The fact that the function u^k defined by the equality (28) satisfies the conditions (31) is obvious. Let us show that u^k defined by (28) satisfies (24)

$$(\Delta_\gamma)_x u=(\mathcal{P}_1^{\frac{k-1}{2}})_\alpha(\Delta_\gamma)_x w(x,\alpha t)=(\mathcal{P}_1^{\frac{k-1}{2}})_\alpha w_{\xi\xi}(x,\alpha t)==\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)}\int\limits_0^1(\Delta_\gamma)_x w(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha,$$

where $\xi=\alpha t$. Further integrating by parts we obtain

$$\frac{\partial u^k}{\partial t}=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)}\int\limits_0^1\alpha\, w_\xi(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha=$$

$$=\left\{u=w_\xi(x,\alpha t),\,dv=\alpha[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha,\,du=tw_{\xi\xi}(x,\alpha t)d\alpha,\,v=-\frac{1}{k}[1-\alpha^2]^{\frac{k}{2}}\right\}=$$

$$=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\frac{t}{k}\int\limits_0^1 w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}}\,d\alpha=$$

$$=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\frac{t}{k}\int\limits_0^1 w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}}\,d\alpha.$$

For $\frac{\partial^2 u^k}{\partial t^2}$ we have

$$\frac{\partial^2 u^k}{\partial t^2}=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\int\limits_0^1 \alpha^2\, w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha=$$

$$=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\int\limits_0^1 (\Delta_\gamma)_x w(x,\alpha t)\alpha^2\,[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha.$$

Finally,

$$\frac{\partial^2 u^k}{\partial t^2}+\frac{k}{t}\frac{\partial u^k}{\partial t}=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\left[\int\limits_0^1 (\Delta_\gamma)_x w(x,\alpha t)\alpha^2\,[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha+\int\limits_0^1 (\Delta_\gamma)_x w(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}}\,d\alpha\right]=$$

$$=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\int\limits_0^1 (\Delta_\gamma)_x w(x,\alpha t)[1-\alpha^2]^{\frac{k}{2}-1}\,d\alpha=(\Delta_\gamma)_x u^k.$$

Thus the function u^k defined by equality (28) satisfies the problem (24)–(31).

Let us prove that from the relation (28) we can uniquely obtain a solution of the problem (26)–(27). By introducing new variables $\alpha t=\sqrt{\tau}$, $t=\sqrt{y}$, we get

$$y^{\frac{k-1}{2}}\,u^k(x,\sqrt{y})=\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\int\limits_0^y \frac{w(x,\sqrt{\tau})}{\sqrt{\tau}}\,(y-\tau)^{\frac{k}{2}-1}d\tau.$$

Let $k>0$ then $y^{\frac{k-1}{2}}\,u^k(x,\sqrt{y})$ is the Riemann-Liouville left-sided fractional integral of the order $\frac{k}{2}$ (see [31], p. 33):

$$y^{\frac{k-1}{2}}\,u^k(x,\sqrt{y})=\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}}\left(I^{\frac{k}{2}}_{0+}\frac{w(x,\sqrt{\tau})}{\sqrt{\tau}}\right)(y).$$

Thus we have unique representation of $w(x,\sqrt{\tau})$ (see [31], p. 44, theorem 24)

$$w(x,\sqrt{\tau})=\frac{\sqrt{\tau}\sqrt{\pi}}{\Gamma\left(\frac{k+1}{2}\right)}\left(D^{\frac{k}{2}}_{0+}y^{\frac{k-1}{2}}\,u^k(x,\sqrt{y})\right)(\tau)$$

or

$$w(x,t)=\frac{2}{\Gamma\left(n-\frac{k}{2}\right)}\left(\frac{d}{2tdt}\right)^n\int\limits_0^t\frac{u^k(x,z)z^k}{(t^2-z^2)^{\frac{k}{2}-n+1}}dz.$$ □

4 The second Cauchy problem for the general Euler-Poisson-Darboux equation

In this section we obtain solution of (2)-(4).

Theorem 4.1. *If* $\varphi\in C_{ev}^{\left[\frac{n+|\gamma|+k-1}{2}\right]}$ *then the solution* $v=v^k(x,t)$ *of*

$$(\triangle_\gamma)_x v=(B_k)_t v,\quad 0<\gamma_i,\quad i=1,\dots,n,\quad k<1,\quad x\in\mathbb{R}^n_+,\quad t>0,\tag{29}$$

$$v^k(x,0)=0, \qquad \lim_{t\to 0} t^k \frac{\partial v}{\partial t}=\varphi(x) \tag{30}$$

is given by

$$v^k(x,t)=\frac{\Gamma\left(\frac{3-k}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{2-k+2q-n-|\gamma|+1}{2}\right)}{2^{n+q}(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{2-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\times$$
$$\times\left(t^{1-k+2q}\int\limits_{B_1^+(n)}[{}^\gamma T^{ty}\varphi(x)](1-|y|^2)^{\frac{2-k+2q-n-|\gamma|-1}{2}}y^\gamma dy\right)$$

if $n+|\gamma|+k$ is not an odd integer and

$$v^k(x,t)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(t^{n+|\gamma|-2}M_\varphi^\gamma(x;t)\right).$$

if $n+|\gamma|+k$ is an odd integer, where $q\geq 0$ is the smallest positive integer number such that $2-k+2q\geq n+|\gamma|-1$.

Proof. Let $q\geq 0$ be the smallest positive integer number such that $2-k+2q\geq n+|\gamma|-1$ i.e. $q=\left[\frac{n+|\gamma|+k-1}{2}\right]$ and let $v^{2-k+2q}(x,t)$ be a solution of (29) when we take $2-k+2q$ instead of k such that

$$v^{2-k+2q}(x,0)=\varphi(x), \qquad v_t^{2-k+2q}(x,0)=0. \tag{31}$$

Then by property (7) we obtain that

$$v^{k-2q}=t^{1-k+2q}v^{2-k+2q}$$

is a solution of the equation

$$(\triangle_\gamma)_x v=\frac{\partial^2 v}{\partial t^2}+\frac{k-2q}{t}\frac{\partial v}{\partial t}.$$

Further, applying q-times the formula (8) we obtain that

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q v^{k-2q}=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q(t^{1-k+2q}v^{2-k+2q})$$

is a solution of the (29).

Let's consider

$$v^k(x,t)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q(t^{1-k+2q}v^{2-k+2q}). \tag{32}$$

We have shown that (32) satisfies the equation (29).

Now we will prove that v^k satisfies the conditions (31). For $v^k\in C_{ev}^q(\Omega_+)$ we have the formula (see [19], p.9)

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q(t^{1-k+2q}v^{2-k+2q})=\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s v^{2-k+2q}. \tag{33}$$

Taking into account formula (33) we obtain $v^k(x,0)=0$ and

$$\lim_{t\to 0}t^k v_t^k(x,t)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\lim_{t\to 0}t^k\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q(t^{1-k+2q}v^{2-k+2q})=$$

$$=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\lim_{t\to 0}t^k\frac{\partial}{\partial t}\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s v^{2-k+2q}=$$

$$=\frac{1}{1-k}\lim_{t\to 0}t^k\frac{\partial}{\partial t}\left(t^{1-k}v^{2-k+2q}\right)=\frac{1}{1-k}\lim_{t\to 0}t^k\left((1-k)t^{-k}v^{2-k+2q}+t^{1-k}v_t^{2-k+2q}\right)=$$

$$=\frac{1}{1-k}\lim_{t\to 0}\left((1-k)v^{2-k+2q}+tv_t^{2-k+2q}\right)=\varphi(x).$$

Now we obtain the representation of v^k through the integral. Using formula (18) we get

$$v^{2-k+2q}=\frac{2\Gamma\left(\frac{3-k+2q}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^1(1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}}r^{n+|\gamma|-1}M_\varphi^\gamma(x;rt)dr.$$

If $2-k+2q>n+|\gamma|-1$ then by applying (32) and (33) we write

$$v^k=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{3-k}{2}+s\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s v^{2-k+2q}=$$

$$=\frac{\Gamma\left(\frac{3-k}{2}\right)}{1-k}\sum_{s=0}^{q}\frac{C_q^s\,t^{1-k+2s}}{2^s\Gamma\left(\frac{3-k}{2}+s\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s v^{2-k+2q}=$$

$$=\frac{\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\sum_{s=0}^{q}\frac{C_q^s\,t^{1-k+2s}}{2^s\Gamma\left(\frac{3-k}{2}+s\right)}\times$$

$$\times\int\limits_0^1(1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}}r^{n+|\gamma|-1}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_\varphi^\gamma(x;rt)dr.$$

If $2-k+2q=n+|\gamma|-1$ then $v^{2-k+2q}=M_\varphi^\gamma(x;t)$ and

$$v^k=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(t^{n+|\gamma|-2}M_f^\gamma(x;t)\right)=$$

$$=\frac{2^{-1-q}\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q}{2}\right)}\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{3-k}{2}+q\right)}{\Gamma\left(\frac{3-k}{2}+s\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_f^\gamma(x;t)=$$

$$=\sum_{s=0}^{q}\frac{C_q^s\,\Gamma\left(\frac{1-k}{2}\right)}{2^{s+1}\Gamma\left(\frac{3-k}{2}+s\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_f^\gamma(x;t).$$

□

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