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A. Zuevsky*

Algebraic proofs for shallow water bi–Hamiltonian systems for three cocycle of the semi-direct product of Kac–Moody and Virasoro Lie algebras

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Abstract: We prove new theorems related to the construction of the shallow water bi-Hamiltonian systems associated to the semi-direct product of Virasoro and affine Kac–Moody Lie algebras. We discuss associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

Keywords: Affine Kac–Moody Lie algebras, Bi-Hamiltonian systems, Verma modules, Coadjoint orbits

MSC: 17B69, 17B08, 70G60, 82C23

1 Introduction: The semi-direct product of Virasoro algebra with the Kac–Moody algebra

This paper is a continuation of the paper [1] where we studied bi-Hamiltonian systems associated to the three-cocycle extension of the algebra of diffeomorphisms on a circle. In this note we show that certain natural problems (classification of Verma modules, classification of coadjoint orbits, determination of Casimir functions) [2–5] for the central extensions of the Lie algebra $Vect(S^1) \ltimes \mathcal{LG}$ reduce to the equivalent problems for Virasoro and affine Kac–Moody algebras (which are central extensions of $Vect(S^1)$ and $\mathcal{LG}$ respectively). Let G be a Lie group and $\mathcal{G}$ its Lie algebra. The group $Diff(S^1)$ of diffeomorphisms of the circle is included in the group of automorphisms of the Loop group LG of smooth maps from S^1 to G. For any pairs $(\phi, \psi) \in Diff(S^1)^2$ and $(g, h) \in LG^2$ the composition law of the group $Diff(S^1) \ltimes \mathcal{LG}$ is

$$(\phi, a) \cdot (\psi, b) = (\phi \circ \psi, a.b \circ \phi^{-1}).$$

The Lie algebra of $Diff(S^1) \ltimes LG$ is the semi-direct product $Vect(S^1) \ltimes \mathcal{LG}$ of the Lie algebras $Vect(S^1)$ and $\mathcal{LG}$.

Let $\mathcal{G}$ be a Lie algebra and $\langle ., . \rangle$ a non-degenerated invariant bilinear form. $Vect(S^1)$ is the Lie algebra of vector fields on the circle and $\mathcal{LG}$ the loop algebra (i.e., the Lie algebra of smooth maps from S^1 to $\mathcal{G}$), $Vect(S^1)_{\mathbb{C}}$ is the Lie algebra over $\mathbb{C}$ generated by the elements L_n, $n \in \mathbb{Z}$ with the relations

$$[L_m, L_n] = (n - m)L_{n+m}.$$

We denote by $\mathcal{LG}_{\mathbb{C}}$ the Lie algebra over $\mathbb{C}$ generated by the elements g_n, $n \in \mathbb{Z}$, $g \in \mathcal{G}$ where $(\lambda g + \mu h)_n$ is identified with $\lambda g_n + \mu h_n$ with the relations

$$[g_n, h_m] = [g, h]_{n+m}.$$

***Corresponding Author: A. Zuevsky:** Institute of Mathemtics, Czech Academy of Sciences, Prague, Czech Republic,
E-mail: zuevsky@yahoo.com

The semi-direct product of $Vect(S^1)$ with $\mathcal{LG}$ is as a vector space isomorphic to $C^\infty(S^1,\mathbb{R}) \oplus C^\infty(S^1,\mathcal{G})$ [6]. The Lie bracket of $\mathcal{SU}(\mathcal{G})$ has the form

$$[(u,a),(v,b)] = \left([.,\partial_t.].u \otimes v, va' - ub' + [a,b]\right),$$

for any $(u,v) \in C^\infty(S^1,\mathbb{R})^2$ and any $(a,b) \in C^\infty(S^1,\mathcal{G})^2$, where prime denote derivative with respect to a coordinate on S^1. The Lie algebra $Vect(S^1) \ltimes \mathcal{LG}$ can be extended with a universal central extension $\mathcal{SU}(\mathcal{G})$ by a two-dimensional vector space. Let us denote by $\mathcal{J}(u) = \int_{S^1} u$. Two independent cocycles are given by

$$\omega_{Vir}\big((u,a),(v,b)\big) = \mathcal{J}(u'''v), \qquad \omega_{K-M}\big((u,a),(v,b)\big) = \mathcal{J}(\langle a',b\rangle).$$

We denote by (u,a,χ,α) the elements of $\mathcal{SU}(\mathcal{G})$ with $u \in C^\infty(S^1,\mathbb{R})$, $a \in C^\infty(S^1,\mathcal{G})$ and $(\chi,\alpha) \in \mathbb{R}^2$. The algebra $\mathcal{SU}(\mathcal{G})$ can be also represented as the semi-direct product of Virasoro algebra on the affine Kac–Moody algebra. We denote by c_{Vir} and c_{K-M} the elements $(0,0,1,0)$ and $(0,0,0,1)$ respectively. If $\mathcal{G} = \mathbb{R}$, then the Lie algebra $Vect(S^1) \ltimes \mathcal{L}\mathbb{R}$ has a universal central extension $\widetilde{\mathcal{SU}}(\mathbb{R})$ by a three-dimensional vector space. The third independent cocycle is given by

$$\omega_{sp}\left((u,a),(v,b)\right) = \mathcal{J}\left(ub'' - va''\right).$$

We denote by $(u,a,\chi,\alpha,\gamma,\delta)$ elements of $\widetilde{\mathcal{SU}}(\mathbb{R})$ with $u \in C^\infty(S^1,\mathbb{R})$, $a \in C^\infty(S^1,\mathcal{G})$, and $(\chi,\alpha,\gamma) \in \mathbb{R}^3$. The Lie bracket of $\widetilde{\mathcal{SU}}(\mathbb{R})$ is given by

$$[(u,a,\phi,\alpha,\gamma),(v,b,\xi,\beta,\delta)] = \left(uv' - u'v, [a,b] - ub' + va', \mathcal{J}(u'''v), \mathcal{J}(\langle a',b\rangle), \mathcal{J}(ub'' - va'')\right).$$

In this paper we discuss a few questions. Let us mention the main results. First, in Section 2 we consider Kirillov-Kostant Poisson brackets [7] of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac–Moody Lie algebra. Let us denote by $\mathcal{SU}(\mathcal{G})'$ the subset of $\mathcal{SU}(\mathcal{G})$ of elements (u,a,ξ,β) with non-vanishing β. We denote by $(\widetilde{Vect(S^1)} \oplus \widetilde{\mathcal{LG}})'$ the subset of $\widetilde{Vect(S^1)} \oplus \widetilde{\mathcal{LG}}$ composed of elements (u,a,ξ,β) with $\beta \neq 0$. Then introduce two new maps $\mathcal{I}(u,a,\xi,\beta)$ from $\mathcal{SU}(\mathcal{G})'$ to $(\widetilde{Vect(S^1)} \oplus \widetilde{\mathcal{LG}})'$, and $\widetilde{\mathcal{I}}(u,a,\xi,\beta,\gamma)$ from $\mathcal{SU}(\mathcal{G})$ to $\widetilde{Vect(S^1)} \oplus \widetilde{\mathbb{L}\mathcal{R}}$. We prove that $\mathcal{I}(u,a,\xi,\beta)$ and $\widetilde{\mathcal{I}}(u,a,\xi,\beta,\gamma)$ are Poisson maps. In Section 3 we discuss coadjoint orbits and Casemir functions for $\mathcal{SU}(\mathcal{G})$. Let $\widetilde{\mathcal{H}}$ be a central extension of a Lie algebra $\mathcal{H}$ and H be a Lie group with Lie algebra is $\mathcal{H}$. We find explicit form for the the coadjoint actions of the groups $Diff(S^1) \ltimes LG$ and $Diff(S^1) \ltimes L\mathbb{R}^*_+$. As a result we obtain the following new theorem. We prove that a coadjoint orbit of $\mathcal{SU}(\mathcal{G})$ is mapped by $\mathcal{I}$ to a coadjoint orbit of $\widetilde{Vect(S^1)} \otimes \widetilde{\mathcal{LG}}$ to a coadjoint orbits of $\widetilde{Vect(S^1)}$. We prove that the map $\widetilde{\mathcal{I}}$ sends the coadjoint orbits of $\widetilde{\mathcal{SU}(\mathcal{G})}$ to coadjoint orbits of $\widetilde{Vect(S^1)} \otimes \widetilde{\mathcal{LG}}$. Previously, we determined Casemir functions on $\widetilde{\mathcal{SU}(\mathcal{G})}'$ and $\widetilde{\mathcal{SU}(\mathbb{R})}$. We then prove new propositions concerning the exlicit form of Casemir functions on $\widetilde{\mathcal{SU}(\mathcal{G})}'$, and in particular on on $\widetilde{\mathcal{SU}(\mathbb{R})}'$. This paper was partially inspired by the construction of bi-Hamiltonian systems as natural generalization of the classical Korteweg-de Vries equation. [1, 8–11]. It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In Section 4 some results of [1] are obtained from another point of view. We prove new proposition for pairwise commuting functions under certain brackets. In section 5 we discuss properties of the universal enveloping algebra of $\mathcal{SU}(\mathcal{G})$. In subsection 5.1 we consider a decomposition of the enveloping algebra of a semi-direct product. We introduce the notion of realizability of the action of $\mathcal{K}$ on $\mathcal{H}$ in $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$. Then we show (Theorem 5.1) that the realizability of the action of $\mathcal{K}$ in $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$ leads to the isomorphism

$$\mathcal{U}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H}) \simeq \mathcal{U}_{\omega_{\mathcal{K}}-\alpha}(\mathcal{K}) \otimes \mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H}).$$

In subsection 5.2 the case of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ is considered. In subsection 5.3 we discuss representations of $\mathcal{SU}(\mathcal{G})$. We prove that positive energy representation V of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ with non-vanishing βId-action of the cocyle c_{K-M} delivers a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras. This proposition determines whether a $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ Verma module is a sub-module of another Verma module of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$. We also prove a proposition regarding a linear form over $\mathfrak{h}$ with non-vanishing $\lambda(c_{K-M})$. In this paper we present proofs for corresponding theorems and lemmas.

2 The Kirillov-Kostant structure of $\mathcal{SU}(\mathcal{G})$

Now we consider Kirillov-Kostant Poisson brackets of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac–Moody Lie algebra. Let $\mathcal{K}$ be a Lie algebra with a non-degenerated bilinear form $\langle.,.\rangle$. A function $f:\mathcal{K}\to\mathbb{R}$ is called regular at $x\in\mathcal{K}$ if there exists an element $\nabla f(x)$ such that

$$f(x+\epsilon a)=f(x)+\epsilon\langle\nabla f(x),a\rangle+o(\epsilon),$$

for any $a\in\mathcal{K}$. For two regular functions $f,g:\mathcal{K}\longrightarrow\mathbb{R}$, we define the Kirillov-Kostant structure as a Poisson structure on $\mathcal{K}$ with

$$\{f,g\}(x)=\langle x,[\nabla f(x),\nabla g(x)]\rangle.$$

Then for any $e\in\mathcal{G}$, the second Poisson structure $\{f,g\}_e(x)$ compatible with the Kirillov-Kostant Poisson structure is defined by

$$\{f,g\}_e(x)=\langle e,[\nabla f(x),\nabla g(x)]\rangle.$$

A non-degenerated bilinear form on $\mathcal{SU}(\mathcal{G})$ and $\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}}$ is defined by

$$\langle(u_1,a_1,\beta_1,\xi_1),(u_2,a_2,\beta_2,\xi_2)\rangle=\int_{S^1}u_1u_2+\int_{S^1}\langle a_1,a_2\rangle+\xi_1\xi_2+\beta_1\beta_2.$$

We denote by $\mathcal{SU}(\mathcal{G})'$ the subset of $\mathcal{SU}(\mathcal{G})$ of elements (u,a,ξ,β) with non-vanishing β. Let $u'=u-\frac{\|a\|^2}{2\beta}$ We denote by $(\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}})'$ the subset of $\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}}$ composed of elements (u,a,ξ,β) with $\beta\neq0$. Let us introduce a new map $\mathcal{I}(u,a,\xi,\beta)=(u',a,\xi,\beta)$ from $\mathcal{SU}(\mathcal{G})'$ to $(\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}})'$. Then for non-vanishing β, let us introduce another new map $\widetilde{\mathcal{I}}(u,a,\xi,\beta,\gamma)=\left(u'-\frac{\gamma}{\beta}a',a,\xi-\frac{\gamma^2}{\beta},\beta\right)$ from $\mathcal{SU}(\mathcal{G})$ to $\widetilde{Vect(S^1)}\oplus\widetilde{\mathbb{L}\mathcal{R}}$. Here we give a proof for the following new theorem:

Theorem 2.1. *$\mathcal{I}$ and $\widetilde{\mathcal{I}}$ are Poisson maps.*

Proof. For any regular function $f(u,a,\xi,\beta)$ from $\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}}$ to $\mathbb{R}$ let us define a regular function $\widehat{f}$ from $\mathcal{SU}(\mathcal{G})'$ to $\mathbb{R}$ by $\widehat{f}(u,a,\xi,\beta)=f\left(u',a,\xi,\beta\right)$. For $f(u,a,\xi,\beta)$ a function on $\mathcal{SU}(\mathcal{G})$ or $(\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}})$, let us denote by f_u the function of the variables a and β that we get when we fix u and ξ. Let us denote f_a the function of the variables u and ξ that we get when we fix a and β. With the previous notations, one has for $\beta\neq0$ for the bracket $\{.,.\}^U=\{.,.\}^{\mathcal{SU}(\mathcal{G})}$

$$\{f,\widehat{g}\}^U(u,a,\xi,\beta)=\left[\{f_u,g_u\}^U+\{f_u,g_a\}^U+\{f_a,g_u\}^U+\{f_a,g_a\}^U\right](u,a,\xi,\beta),$$

and for the bracket $\{.,.\}^{V\mathcal{LG}}=\{.,.\}^{\widetilde{Vect(S^1)}\oplus\widetilde{\mathcal{LG}}}$ we have

$$\{f,g\}^{V\mathcal{LG}}(u,a,\xi,\beta)=\{f_u,g_u\}^{V\mathcal{LG}}+\{f_a,g_a\}^{V\mathcal{LG}}.$$

Then the map π_1 from $\mathcal{SU}(\mathcal{G})$ onto $\widetilde{Vect(S^1)}$ which sends (u,a,ξ,β) onto (u',ξ) is a Poisson morphism. The map π_2 from $\mathcal{SU}(\mathcal{G})$ onto $\widetilde{\mathcal{LG}}$ which sends (u,a,ξ,β) to (a,β) is a Poisson morphism. For any regular function f on $\widetilde{Vect(S^1)}$ and any regular function g on $\widetilde{\mathcal{LG}}$ we have

$$\{\pi_1^*f,\pi_2^*g\}_U=0.$$

Indeed, for $i=1,2$, $\left(\delta_a-\frac{a}{\beta}\delta_u\right)f_i(\widehat{u},\xi)=0$. We have:

$$\begin{aligned}\{f_1(\widehat{u},\xi),f_2(\widehat{u},\xi)\}^U_{\xi,\beta}(u,a,\xi,\beta) &= \mathcal{J}([\xi(\delta f_{1,u}(\widehat{u}),\xi)_{xxx}\,\delta f_{2,u}(\widehat{u},\xi)+2(\delta f_{1,u}(\widehat{u},\xi))_x\,\delta f_{2,u}(\widehat{u},\xi)\,u\\ &+\delta f_{1,u}(\widehat{u},\xi)\,u_x\,\delta f_{2,u}(\widehat{u},\xi)-\beta^{-1}(\delta f_{1,u}(\widehat{u},\xi)_x)\parallel a\parallel^2\delta f_{2,u}(\widehat{u},\xi)\\ &-\langle(\delta f_{1,u}(\widehat{u})a/\beta,\xi)_x,\delta f_{2,u}(\widehat{u},\xi)\rangle+\beta^{-1}\langle(\delta f_{1,u}(\widehat{u})a,\xi)_x,\delta f_{2,u}(\widehat{u},\xi)a\rangle\Big]).\end{aligned}$$

This gives

$$\{f_1(\widehat{u},\xi),f_2(\widehat{u},\xi)\}^U_{\xi,\beta}(u,a)=\mathcal{J}(\xi\left[(\delta f_{1,u}(\widehat{u},\xi))_{xxx}\delta f_{2,u}(\widehat{u},\xi)\right.$$

$$+2(\delta f_{1,u}(\widehat{u},\xi))_x \delta f_{2,u}(\widehat{u},\xi)(\widehat{u},\xi) + \delta f_{1,u}(\widehat{u},\xi)(\widehat{u})_x \delta f_{2,u}(\widehat{u},\xi)]),$$

and

$$\{f_1(\widehat{u},\xi), f_2(\widehat{u},\xi)\}^U_{\xi,\beta}(u,a) = \{f_1,f_2\}^{Vir}_{\xi}(\widehat{u},\xi).$$

Let $g_i(a,\beta)$, $i = 1,2$ be two regular functions on the affine Kac–Moody algebra. One notes that $\delta g_{1,u} = \delta g_{2,u} = 0$. Therefore,

$$\{g_1,g_2\}^U_{\xi,\beta}(u,a) = \beta \mathcal{J}(\langle dx(\delta g_{1,a}(a,\beta)), \delta g_{2,a}(a,\beta)\rangle + \langle [a, \delta g_{1,a}(a,\beta)], \delta g_{2,a}(a,\beta)\rangle).$$

Then,

$$\{g_1,g_2\}^U(u,a,\xi,\beta) = \{f,g\}^{\widetilde{\mathcal{LG}}}(a,\beta).$$

We have:

$$\begin{aligned}\{f(\widehat{u},\xi), g(a,\beta)\}^U &= \mathcal{J}(\langle(\delta f_u(\widehat{u})_x\, a,\xi), \delta g, a(a,\beta)\rangle \\ &\quad -\beta dx(\delta f_u(\widehat{u},\xi)a), \delta g_a(a,\beta)\rangle + [a, \delta f_u a], \delta g_a\rangle).\end{aligned}$$

The sum of the first two terms is equal to 0. The last term is $\mathcal{J}(\delta f_u\langle [a,a], \delta g_a\rangle)$, and is equal to zero. One can proceed similarly for $\widetilde{\mathcal{I}}$. □

3 Coadjoint orbits Casimir functions and for $\mathcal{SU}(\mathcal{G})$

Let $\widetilde{\mathcal{H}}$ be a central extension of a Lie algebra $\mathcal{H}$, and H be a Lie group with Lie algebra is $\mathcal{H}$. Then H acts on $\widetilde{\mathcal{H}}^\star$ by the coadjoint action along coadjoint orbits.

Proposition 3.1. *The coadjoint actions of the groups $Diff(S^1) \ltimes LG$ and $Diff(S^1) \ltimes L\mathbb{R}^*_+$ are given by*

$$\begin{aligned}Ad^*(\phi,g)^{-1}(u,a,\xi,\beta) &= \Big((u\circ\phi)\phi'^2 + \xi S(\phi) + \langle g^{-1}g', a\rangle\phi'^2 \\ &+ \quad \frac{1}{2}\beta \parallel g^{-1}g' \parallel^2, \phi' Ad(g^{-1})a\circ\phi + \beta g^{-1}g', \xi, \beta\Big), \\ &\Big((u\circ\phi)\phi'^2 + \xi S(\phi) + \langle g'g^{-1}, a\rangle\phi'^2 + \frac{1}{2}\beta(g'g^{-1})^2 + \gamma g''g^{-1}, \\ &\quad \phi' Ad(g^{-1})a\circ\phi + \beta g^{-1}g' - \gamma g''g^{-1}, \xi, \beta, \gamma\Big).\end{aligned}$$

The classification of coadjoint orbits of $Vect(S^1) \ltimes \mathcal{LG}$ can be known from the classification of coadjoint orbits of the Virasoro and affine Kac-moody algebra. Here we obtain the following new

Theorem 3.2. *A coadjoint orbit of $\mathcal{SU}(\mathcal{G})$ is mapped by $\mathcal{I}$ to a coadjoint orbit of $\widetilde{Vect(S^1)} \otimes \widetilde{\mathcal{LG}}$ to a coadjoint orbits of $\widetilde{Vect(S^1)}$.*

In other words, this means that if $\beta_1 \neq 0$, the elements $(u_1, a_1, \xi_1, \beta_1)$ and $(u_1, a_1, \xi_2, \beta_2)$ are in the same coadjoint orbit if and only if: $\xi_1 = \xi_2$, $\beta_1 = \beta_2$, (a_1, β_1) and (a_2, β_2) are on the same coadjoint orbit of $\widetilde{\mathcal{LG}}$, $(u_1 - \frac{\|a_1\|}{2\beta_1}), \xi_1)$ and $(u_2 - \frac{\|a_2\|}{2\beta_2}), \xi_2)$ are elements of the same coadjoint orbit of $\widetilde{Vect(S^1)}$.

Proof. For any $\phi \in Diff(S^1)$, there exists $h \in LG$ such that

$$hah^{-1} + \beta\frac{\partial h(x)}{\partial x}.h^{-1} = a\circ\phi.\phi'.$$

By direct computation we check that

$$\mathcal{I}(Ad^*(\phi,g)(u,a,\xi,\beta) = (Ad^*(\phi,g.h)\mathcal{I}(u,a,\xi,\beta).$$

This implies Theorem 3.2. □

Proposition 3.3. *The map $\widetilde{\mathcal{I}}$ sends the coadjoint orbits of $\widetilde{\mathcal{SU}(\mathcal{G})}$ to coadjoint orbits of $\widetilde{Vect(S^1)} \otimes \widetilde{\mathcal{LG}}$.*

In other words, this means that if $\beta_1 \neq 0$ the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2, \gamma_2)$ are in the same coadjoint orbit if and only if $\gamma_1 = \gamma_2$, $\xi_1 = \xi_2$, $\beta_1 = \beta_2$, (a_1, β_1) and (a_2, β_2) are on the same coadjoint orbit of $\widetilde{\mathcal{LG}}$, $(u_1 - \frac{a_1^2}{2\beta}, \xi_1 - \frac{\gamma_1^2}{\beta_1})$ and $(u_2 - \frac{a_2^2}{2\beta}, \xi_2 - \frac{\gamma_2^2}{\beta_2})$ are elements of the same coadjoint orbit of $\widetilde{Vect(S^1)}$. In a particular case, if $\beta_1 = \beta_2 = 0$, then:

Proposition 3.4. *If the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2\gamma_2)$ are in the same coadjoint orbit then $\gamma_1 = \gamma_2$, $(a_1^2 + \gamma_1 a_1', \gamma_1)$ and $(a_2^2 + \gamma_2 a_2', \gamma_2)$ are in the same coadjoint orbit of the Virasoro Lie algebra.*

Proof. We have: $Ad(\phi, g)(a_1^2 + \gamma_1 a_1') = (a_1^2 + \gamma_1 a_1') \circ \phi + \gamma_1 S(\phi)$. □

Previously, we determined Casemir functions on $\widetilde{\mathcal{SU}(\mathcal{G})}'$ and $\widetilde{\mathcal{SU}(\mathbb{R})}$. We gave the following proposition:

Proposition 3.5. *Let $\mathcal{C}_{Vir}$, $\mathcal{C}_{K-M}$ $\mathcal{C}_{\mathcal{A}}$ be Casimir functions for Virasoro, affine Kac–Moody, and the Heisenberg Lie algebras $\mathcal{A}$ correspondingly. Let $\mathcal{S}_P\mathcal{U}(\mathcal{G})$, $\widetilde{\mathcal{S}_P\mathcal{U}(\mathbb{R})}$ be Poisson submanifolds of $\mathcal{SU}(\mathcal{G})$ and $\widetilde{\mathcal{SU}(\mathbb{R})}$ defined by $\xi = 0$. Then the functions $\mathcal{C}_{Vir}(u', \xi)$, $c(u, a, \beta, \xi) = \mathcal{C}_{K-M}(a, \beta)$, and $\int_{S^1} |u'|^{1/2}$, are Casimir functions on $\widetilde{\mathcal{SU}(\mathcal{G})}'$. In particular, the functions $c_{\mathcal{A}}(u, a, \beta, \xi) = C_{\mathcal{A}}(a, \beta)$, $\mathcal{C}_{Vir}(u' - \frac{\gamma^2}{\beta} a', \xi)$, and $\int_{S^1} |u' - \frac{\gamma^2}{\beta} a'|^{1/2}$, are Casimir functions on $\widetilde{\mathcal{SU}(\mathbb{R})}'$.*

4 Bi-hamiltonian dispersive water waves systems associated to $\mathcal{SU}(\mathcal{G})$

It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In this section some results of [1] are obtained from another point of view. We obtain new

Proposition 4.1. *The functions $\left\{\phi_1\left(A(u + B\frac{da}{dx} + C)\right) | \lambda \in \mathbb{R}\right\}$ commute pairwise for the Sugawara $\{.,.\}'_{Sug}$ and e-braket $\{.,.\}_e$ with $e = (1, 0, 0, 2, 0)$, and $A = \left(\xi - \frac{\gamma}{\beta - 2\lambda}\right)^{-2}$, $B = -\frac{\gamma}{\beta - 2\lambda}$, $C = -\frac{\|a\|^2}{2\beta - 4\lambda} - \lambda$.* □

The function $\lambda \mapsto \phi_1\left(A(u + B\frac{da}{dx} + C)\right)$ has an asymptotic development. The coefficients of this development form a hierarchy. The first term of this development is $\int_{S^1} u$, and the second one is $\int_{S^1} (u^2 + \gamma u + \| a \|^2)$. A linear combination of these two terms gives the Hamiltonian of equations $H(u, a) = \int_{S^1} (u^2 + \| a \|^2)$.

Let $\{\phi_i, i \in I\}$ be a set of Casimir functions and $e \in \mathcal{G}$. Define $x_\chi = x - \chi e$, for some $\chi \in \mathbb{R}$.

Lemma 4.2. *For any $(i, j) \in I^2$ and any $(\lambda, \mu) \in \mathbb{R}^2$ we have $\{\phi_i(x_\lambda), \phi_j(x_\mu)\} = \{\phi_i(x_\lambda), \phi_j(x_\mu)\}_e = 0$.*

Lemma 4.3. *Suppose $\phi_i(x_\lambda)$ can be expanded in terms of inverse powers of λ with some extra function $f(\lambda)$, and modes $F_{i,k}(x)$, i.e.,*

$$\phi_i(x_\lambda) = f(\lambda) \sum_{k \in \mathbb{R}} \lambda^{-k} F_{i,k}(x),$$

then $\{F_{i,k+1}, f\}_e = \{F_{i,k}, f\}_0$. We can choose e so that the Hamiltonian $H(x) = \frac{1}{2}\langle x, x\rangle$ commute with these functions.

Lemma 4.4. *If an element $e \in \mathcal{G}$ satisfies two conditions: (i) $ad^*(e)e = 0$; (ii) for any $u \in \mathcal{G}$, $ad^*(u)e$ belongs to the tangent space to the coadjoint orbit of u (i.e., for any $u \in \mathcal{G}$ there exists $v \in \mathcal{G}$ such that $ad^*(u)e = ad^*(v)u$). then the functions $\phi(a - \lambda e)$ commute with the Hamiltonian of the geodesics $H(a) = \frac{1}{2} \| a \|^2$ with respect to the brackets $\{.,.\}_0$ and $\{.,.\}_e$.*

5 The universal enveloping algebra of $\mathcal{SU}(\mathcal{G})$

When $\mathcal{H} = \sum_{k\in\mathbb{Z}} \mathcal{H}_k$ has a structure of graded algebra, its universal enveloping algebra $\mathcal{UH}$ is also naturally endowed with a structure of a graded Lie algebra. Indeed, the weight of a product $h_1, \dots, h_n \in \mathcal{UH}$ of homogeneous elements is defined to be the sum of the weights of the elements h_i, $i = 1, \dots, n$. The universal enveloping algebra $\mathcal{UH}$ admits a filtration $\mathcal{UH} = \cup_{i=0}^{\infty} F_k$ where F_k is the vector space generated by the products of at most k elements of $\mathcal{H}$. The generalized enveloping algebra is the algebra of the elements of the form $\sum_{k\leq n} u_k$ where u_k is an element of weight k of $\mathcal{UH}$. The product of two such elements is defined by:

$$\sum_{k=-\leq n} u_k . \sum_{k\leq m} v_k = \sum_{k\in\mathbb{Z}} w_k,$$

where $w_k = \sum_{i\in\mathbb{Z}} u_i . v_{k_i}$ which is a finite sum. Let $\omega_1, \dots, \omega_n$ be two-cocycles on the Lie algebra $\mathcal{H}$, let $\widetilde{\mathcal{H}}$ be the central extension associated with and let $e_1, \dots, e_n$ be the central elements associated with these cocycles.

The modified generalized enveloping algebra $\mathcal{U}^{\mathcal{H}}_{\omega_1,\dots,\omega_n}$ is defined to be the quotient of the generalized enveloping algebra of $\widetilde{\mathcal{H}}$ by the ideal generated by the elements $\{e_1 - 1, \dots, e_n - 1\}$. We denote again by 1 the neutral element of $\mathcal{U}^{\mathcal{G}}_{\lambda}$. The algebra $\mathcal{U}^{\mathcal{H}}_{\omega_1,\dots,\omega_n}$ is by construction a graded algebra and a filtered algebra. We denote by F_n, $n \in \mathbb{N}$ its filtration. Let us recall shortly the main properties of the modified generalized enveloping algebra. Let V is be a module over $\widetilde{\mathcal{H}}$ such that for any $v \in V$, there exists $n_0 \in \mathbb{Z}$ such that for any $n > n_0$ and any $h \in \widetilde{\mathcal{H}}_n$ we have $h.v = 0$. Such modules are called representations of positive energy, and e_i acts on V by $\lambda_i Id$. Then V is a module over $\mathcal{U}^{\mathcal{H}}_{\omega_1,\dots,\omega_n}$. Such modules are named modules of positive energy. The anticommutator provides a structure of Lie algebra on $\mathcal{U}^{\mathcal{H}}_{\omega_1,\dots,\omega_n}$. For this bracket F_1 is a Lie sub-algebra isomorphic to the central extension of $\mathcal{H}$ by the cocycle $\omega = \sum_{i=1}^{n} \omega_i$. We denote by i be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}^{\mathcal{G}}_{\omega}$ given by this identification.

5.1 Decomposition of the enveloping algebra of a semi-direct product

In some very particular cases, the modified generalized enveloping algebra of a semi-direct product $\mathcal{K} \ltimes \mathcal{H}$ of two Lie algebras is isomorphic to the tensor product of some modified generalized enveloping algebras of $\mathcal{K}$ and of $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ be the central extension of $\mathcal{H}$ with the two-cocycle $\omega_{\mathcal{H}}$. Denote by $\cdot$ the action of the Lie algebra $\mathcal{K}$ on the Lie algebra $\widetilde{\mathcal{H}}$. Let us introduce the semi-direct product $\mathcal{K} \ltimes \widetilde{\mathcal{H}}$ which is a central extension of $\mathcal{K} \ltimes \mathcal{H}$ by a two-cocycle $\omega'_{\mathcal{H}}$ with

$$\omega'_{\mathcal{H}}\big((0, h_1), (0, h_2)\big) = \omega_{\mathcal{H}}(h_1, h_2).$$

A two-cocycle $\omega_{\mathcal{K}}$ on $\mathcal{K}$ defines also a two-cocycle $\omega'_{\mathcal{K}}$ by

$$\omega'_{\mathcal{K}}\big((g_1, h_1), (g_2, h_2)\big) = \omega_{\mathcal{K}}(g_1, g_2),$$

of $\mathcal{K} \ltimes \mathcal{H}$. Let I be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$ and J be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H})$.

We call the action of $\mathcal{K}$ on $\mathcal{H}$ *realizable* in $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$ when there exists a map $F : \mathcal{K} \to \mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$ and a two-cocycle α on $\mathcal{K}$ such that for any pair (g_1, g_2) in $\mathcal{K}^2$

$$F([g_1, g_2]) = [F(g_1), F(g_2)] + \alpha(g_1, g_2)1,$$

and the map F satisfies the *compatibility condition*, i.e., for any $g \in \mathcal{K}$ and $h \in \widetilde{\mathcal{H}}$ with the anti–commutator $[F(g), I(h)] = I(g \cdot h)$, of the algebra $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$.

Theorem 5.1. *If the action of $\mathcal{K}$ is realizable in $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$ then*

$$\mathcal{U}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H}) \simeq \mathcal{U}_{\omega_{\mathcal{K}}-\alpha}(\mathcal{K}) \otimes \mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H}).$$

Proof. Let $\mathcal{U}_g = \{\widehat{g} | g \in \mathcal{K}\}$ with be the unitary subalgebra of $\mathcal{U}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H})$ generated by the elements $\widehat{g} = g - F(g)$, and $\mathcal{U}_j = \{j(h), h \in \widetilde{\mathcal{H}}\}$ be the unitary subalgebra of $\mathcal{U}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H})$. For any (g, h) this implies that the generators of $\mathcal{U}_g$ and $\mathcal{U}_j$ commute, i.e., $[\widehat{g}, j(h)] = 0$. The subalgebras $\mathcal{U}_g$ and $\mathcal{U}_j$ therefore commute. The subalgebra $\mathcal{U}_g$ is isomorphic to $\mathcal{U}_{\omega_{\mathcal{K}}-\alpha}(\mathcal{K})$. Let us check that the generators $\{\widehat{g} | g \in \mathcal{K}\}$ of this algebra satisfy the relations of the generators of $\mathcal{U}_{\omega_{\mathcal{K}}-\alpha}(\mathcal{K})$:

$$[\widehat{g}_1, \widehat{g}_2] = [g_1, g_2] + \omega_{\mathcal{K}}(g_1, g_2)1 + [F(g_1), F(g_2)] - [F(g_1), g_2] - [g_1, F(g_2)].$$

Since $F(g_1)$ is an element of $\mathcal{U}_j$ and since the algebras $\mathcal{U}_g$ and $\mathcal{U}_j$ commute $[F(g_1), g_2] = [F(g_1), F(g_2)]$ and $[g_1, F(g_2)] = [F(g_1), F(g_2)]$. Therefore:

$$[\widehat{g}_1, \widehat{g}_2] = [g_1, g_2] + \omega_{\mathcal{K}}(g_1, g_2)1 - [F(g_1), F(g_2)],$$

and finally

$$[\widehat{g}_1, \widehat{g}_2] = [g_1, g_2] - F([g_1, g_2]) + \big(\omega_{\mathcal{K}}(g_1, g_2) - \alpha(g_1, g_2)\big)1.$$

The subalgebra $\mathcal{U}_j$ is obviously isomorphic to $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$. The generalized modified enveloping algebra $\mathcal{U}_{\omega'_{\mathcal{K}}+\omega'_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H})$ is therefore isomorphic to the tensor product over $\mathbb{C}$ of $\mathcal{U}_{\omega_{\mathcal{K}}-\alpha}(\mathcal{K})$ with $\mathcal{U}_{\omega_{\mathcal{H}}}(\mathcal{H})$. □

5.2 The case of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$

Let $\mathcal{G}$ be a simple complex Lie algebra and C_φ its dual Coxeter number. Introduce the $\{K_1, \ldots, K_n\}$ a basis of $\mathcal{G}$, and the dual basis $\{K_1^*, \ldots, K_n^*\}$ with respect to the Killing form $\langle ., . \rangle$. We apply Theorem 5.1 for $\mathcal{K} = Vect(S^1)$, $\mathcal{H} = \mathcal{LG}$, $\omega_{\mathcal{K}} = \xi\omega_{Vir}$, and $\omega_{\mathcal{H}} = \beta\omega_{K-M}$. In this case, $\omega'_{\mathcal{H}} = \beta\omega_{K-M}$. For $\eta = \beta + C_\varphi \neq 0$, the Sugawara construction, delivers a map $F : Vect(S^1)_{\mathbb{C}} \to \mathcal{U}_{\omega_{\mathcal{G}}}(\mathcal{LG}_{\mathbb{C}})$ defined by

$$(\beta + \eta)F(L_n) = K \cdot K^*,$$

where

$$K \cdot K^* = \sum_{i \in \mathbb{Z},\, j=1,\ldots,n} :(K_j)_i (K_j^*)_{n-i}:,$$

(here dots denote the normal ordering), i.e., the action of $Vect(S^1)$ is realizable in $\mathcal{U}_{\beta\omega_{K-M}}(\mathcal{LG})$, with $\alpha = \beta\omega_{Vir}/12\eta$. Thus we obtain

Proposition 5.2. *If* $\eta \neq 0$, *then* $\mathcal{U}_{\xi\omega_{Vir},\beta\omega_{K-M}}(\mathcal{SU}_{\mathbb{C}}\mathcal{G}) \simeq \mathcal{U}_{\beta\omega_{K-M}}(Vect(S^1)_{\mathbb{C}(\xi-\alpha)}) \otimes \mathcal{U}(\mathcal{LG})$.

The Lie algebra $Vect_{\mathbb{C}}(S^1)$ acts on the Heisenberg algebra by

$$L_n.a_m = m a_{n+m} + \delta_{n,-m} m^2 c_{K-M}.$$

In this case, on has $\omega'_{\mathcal{H}} = \beta\omega_{\mathcal{H}} + \gamma\omega_{sp}$. The map $F : Vect(S^1)_{\mathbb{C}} \to \widetilde{\mathcal{SU}_{\mathbb{C}}(\mathbb{C})}$ defined by

$$\beta F(L_n) = \frac{1}{2} \sum_{i \in \mathbb{Z}} : a_i a_{n-i} : + \gamma a_n,$$

for a cocycle $\widehat{\alpha} = (\alpha + \gamma^2\beta^{-1})\omega_{Vir}$. For $\widetilde{\mathcal{SU}_{\mathbb{C}}(\mathcal{C})}$ we obtain

Proposition 5.3. *For* $\beta \neq 0$, *we have*

$$\mathcal{U}_{\xi\omega_{Vir},\beta\omega_{K-M},\gamma\omega_{sp}}(\widetilde{\mathcal{SU}_{\mathbb{C}}(\mathbb{C})}) \simeq \mathcal{U}_{\theta\omega_{Vir}}(Vect(S^1)_{\mathbb{C}}) \otimes \mathcal{U}_{\omega_{K-M}}(\mathcal{LG}),$$

with $\theta = \xi - \gamma^2/\beta - 1/12$.

5.3 Representations of $\mathcal{SU}(\mathcal{G})$

Proposition 5.4. *A positive energy representation V of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ with non-vanishing βId-action of the cocyle c_{K-M} brings about a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras.*

This proposition determines whether a $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ Verma module is a sub-module of another Verma module of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$. Let $\mathfrak{h}$ be a Cartan algebra of $\mathcal{G}$ with a basis $\{h_1, \dots, h_k\}$. The Lie subalgebra $\mathfrak{k}$ of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ is generated by the elements $\{c_{Vir}, c_{K-M}, u_0, (h_1)_0, \dots, (h_k)_0\}$. A Verma module $V_\lambda(\mathcal{SU}_{\mathbb{C}}(\mathcal{G}))$ of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ is associated to any linear form $\lambda \in \mathfrak{h}^*$. Verma modules V_ν^{Vir}, V_μ^{K-M}, are associated to linear forms ν, μ over the spaces generated by c_{Vir} and u_0, c_{K-M} and $\{(h_1)_0, \dots, (h_k)_0\}$ correspondingly. For any $\lambda \in \mathfrak{k}^*$, the Verma module $V_\lambda(\mathcal{SU}_{\mathbb{C}}(\mathcal{G}))$ is a positive energy representation. Thus, $V_\lambda(\mathcal{SU}_{\mathbb{C}}(\mathcal{G}))$ is Virasoro and affine Kac–Moody algebra module. The generator e of $V_\lambda(\mathcal{SU}_{\mathbb{C}}(\mathcal{G}))$ brings about a Verma module V_ν^{Vir} for Virasoro algebra. It generates also a Verma module V_ν^{Vir} for the affine Kac–Moody algebra. The linear form ν satisfies $\nu(u_0)e = \lambda(u_0 - F(u_0))e$, i.e.,

$$(u_0 - (\beta + \eta)^{-1} K \cdot K^* e = \nu(u_0)e.$$

Suppose the action of a Casimir element of $\mathcal{G}$ is given by acts by $D(\lambda)Id$ for $D(\lambda) \in \mathbb{C}$. We then have

$$(u_0 - (\beta + \eta)^{-1} K \cdot K^* .e = (u_0 - (\beta + \eta)^{-1} \sum_{j=1\dots n} :(K_j)_0 (K_j^*)_0 :).e,$$

$(\lambda(u_0) - \frac{D(\lambda)}{2\eta})e$. This implies $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$. The other values of μ and ν can be computed by the same method.

Proposition 5.5. *Let λ be a linear form over $\mathfrak{h}$ with non-vanishing $\lambda(c_{K-M})$. Then*

$$V_\lambda(\mathcal{SU}_{\mathbb{C}}(\mathcal{G})) \simeq V_\nu^{Vir} \otimes V_{\mathbb{C}}^{K-M}\mu,$$

where $\mu(e_i) = \lambda(e_i)$, $i = 1, \dots, n$, defines μ, $\mu(c_{K-M}) = \lambda(c_{K-M})$, and $\nu(c_{Vir}) = \lambda(c_{Vir}) - \frac{\beta}{12\eta}$ defines ν, $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$.

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