DE GRUYTER OPEN Open Math. 2018; 16: 9–15

Open Mathematics

Research Article

Jürgen Socolowsky*

On a viscous two-fluid channel flow including evaporation

A short version of this article has been presented at the 7ECM in Berlin in July, 2016.

https://doi.org/10.1515/math-2018-0001
Received November 2, 2016; accepted January 4, 2018.

Abstract: In this contribution a particular plane steady-state channel flow including evaporation effects is investigated from analytical point of view. The channel is assumed to be horizontal. The motion of two heavy viscous immiscible fluids is governed by a free boundary value problem for a coupled system of Navier-Stokes and Stephan equations. The flow domain is unbounded in two directions and the free interface separating partially both liquids is semi-infinite, i.e. infinite in one direction. The free interface begins in some point Q where the half-line Σ_1 separating the two parts of the channel in front of Q ends. Existence and uniqueness of a suitable solution in weighted HÖLDER spaces can be proved for small data (i.e. small fluxes) of the problem.

Keywords: Navier-Stokes equations, Stephan equations, Free boundary value problem, Semi-infinite inner channel wall

MSC: 35R35, 35Q30, 76D03, 76D05

1 Introduction

In this paper we are concerned with the investigation of a particular free **boundary value problem** (= BVP) for a two-fluid non-isothermal channel flow. The infinite channel is assumed to be horizontal and it contains a partial inner wall (cf. the thin red line Σ_1 in Figure 1) which is semi-infinite. The flow problem is assumed to be stationary and 2D. In Figure 1 the blue line denotes the lower channel wall which moves with constant speed R in x_1-direction. The red line Σ_2 denotes the upper channel wall that is at rest. Finally, by the cyan curve Γ we understand the *a priori unknown* free interface between the two fluid layers. It has the representation $x_2 = \varphi(x_1)$ where the function φ has to be found as well as the flow fields for velocity $\boldsymbol{v}(\boldsymbol{x})$, for the pressure $p(\boldsymbol{x})$ and for the temperature $\theta(\boldsymbol{x})$.

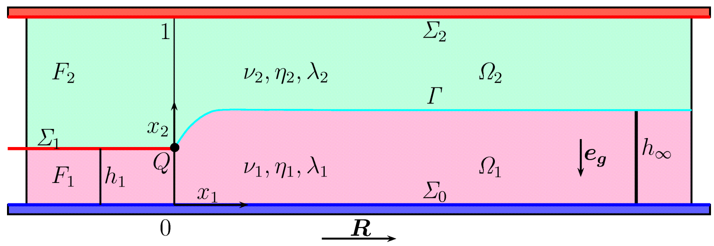
Models of the described kind are quite important in many technological and scientific applications. Corresponding examples may be found in the field of materials science, particularly in coating and solidification processes with evaporation or in crystal-growth processes (cf. [1-12]). The investigations of such problems are performed from technical point of view as well as from analytical and/or numerical point of view. It was our main objective to obtain statements about the existence and/or uniqueness of free BVP for evaporation problems.

The flow describes a coupled heat-and mass transfer (Stephan equations). The (positive) fluxes F_m are prescribed in each fluid layer $\Omega_m (m = 1, 2)$ (cf. Fig. 1). The lower liquid layer is characterized by red color whereas the upper one is marked by green color. Both liquids are heavy, viscous, heat-conducting,

***Corresponding Author: Jürgen Socolowsky:** TH Brandenburg, Department of Engineering, PO Box 2132, 14737 Brandenburg, Germany, E-mail: socolowsky@th-brandenburg.de

incompressible and immiscible. Therefore, the mathematical model can incorporate evaporation effects. The surface tension $\sigma(\theta)$ is temperature-depending in a known manner.

Fig. 1. Flow domain of a two-fluid channel flow



By ν_m, η_m and λ_m $(m = 1, 2)$ we understand the domain-wise (i.e. regional) constant values of the kinematic viscosity, of the density and of the thermal conductivity, respectively, of the m-th fluid. By h_∞ we denote the (asymptotic) position of the free interface Γ when x_1 goes to $+\infty$. By $\boldsymbol{n}$, $\boldsymbol{\tau}$, respectively, the unit normal and the unit tangential vectors with respect to Γ are denoted. Their orientation (direction) is the same as for x_1, x_2. By g and $\boldsymbol{e}_g$ we understand the acceleration and the direction of gravity, respectively. Concerning the interface tension σ we suppose the following linear function of temperature θ

$$\sigma(\theta) = \widehat{a} - \widehat{b}\,\theta, \qquad (\widehat{a}, \widehat{b} > 0)$$

which is frequently used in the literature. This leads to an effect which is called Benard-Marangoni-effect or thermo-capillary convection. Finally, the following symbols and abbreviations have been used throughout this paper: $\delta_j(t) := \{x_1 = t\} \cap \Omega_j,\ j = 1, 2\}$ is some cross section of Ω_j. The frictional stress tensor has the subsequent elements: $S_{jk} = \nu\eta(\partial_k v_j + \partial_j v_k) \quad (j, k = 1, 2)$. The symbol $\big[w(\boldsymbol{x}_0)\big]\,|_\Gamma$ represents the jump of the field w crossing the interface Γ from below to above:

$$\big[w(\boldsymbol{x}_0)\big]\,|_\Gamma := \lim_{\boldsymbol{y} \to \boldsymbol{x}_0} w(\boldsymbol{y}) - \lim_{\boldsymbol{x} \to \boldsymbol{x}_0} w(\boldsymbol{x}), \qquad (\boldsymbol{x}_0 \in \Gamma,\ \boldsymbol{y} \in \Omega_1,\ \boldsymbol{x} \in \Omega_2),$$

2 Mathematical model

The governing equations (Navier - Stokes & Stephan) of the problem which yield in $\Omega := \Omega_1 \cup \Omega_2$ read as follows

$$\begin{cases} (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} - \nu\,\boldsymbol{\nabla}^2 \boldsymbol{v} + \frac{1}{\eta}\,\boldsymbol{\nabla} p = g\,\boldsymbol{e}_g, \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \\ (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\theta - \lambda\,\boldsymbol{\nabla}^2 \theta = 0. \end{cases} \tag{1}$$

They are supplied by the boundary conditions at the lower moving wall Σ_0:

$$\boldsymbol{v}|_{\Sigma_0} = \boldsymbol{R}, \qquad \theta|_{\Sigma_0} = \theta_0 = 0. \tag{2}$$

Let us emphasize that the value 0 in Eq. (2) does not represent the absolute temperature but some dimensionless value which is in fact the difference to some reference temperature related to a characteristic temperature difference.

The boundary conditions at the walls at rest $\Sigma_k (k = 1, 2)$ look like:

$$\boldsymbol{v}|_{\Sigma_k} = \boldsymbol{0}, \qquad \theta|_{\Sigma_k} = \theta_k. \tag{3}$$

Let us explain that the boundary conditions (3) for $k = 1$ mean **both sides** $\Sigma_1^\pm$ of the partial inner wall Σ_1. Finally, the conditions at the free interface Γ are:

$$\begin{cases} [\theta]|_\Gamma = 0, & [\boldsymbol{v}]|_\Gamma = \boldsymbol{0}, \\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma^-} = \left[\lambda \frac{\partial\theta}{\partial n}\right]\Big|_\Gamma, & [\boldsymbol{\tau}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}]|_\Gamma = 0, \\ \frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\varphi'(x_1)}{\sqrt{1+\varphi'(x_1)^2}} = \frac{1}{\sigma(\theta)}\,[-p+\boldsymbol{n}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}]|_\Gamma, \\ \int_{\delta_k(t)} \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}x_2 = F_k. \quad (k=1,2) \end{cases} \tag{4}$$

As a consequence one gets the relation: $\lim_{x_1\to+\infty}\varphi(x_1) = \text{const.} = h_\infty$.

In order to prove the unique solvability of the BVP in appropriate functional spaces the following two-cycle iteration scheme was applied.

$$\begin{aligned} &[\Gamma^{(0)} \to \Omega^{(0)}] \to (\boldsymbol{v}^{(0)}, p^{(0)}, \theta^{(0)}) \to [\Gamma^{(1)} \to \Omega^{(1)}] \\ &\to (\boldsymbol{v}^{(1)}, p^{(1)}, \theta^{(1)}) \to \ldots \to [\Gamma^{(k)} \to \Omega^{(k)}] \to (\boldsymbol{v}^{(k)}, p^{(k)}, \theta^{(k)}) \to \ldots \end{aligned} \tag{5}$$

This scheme was introduced by V.V. Pukhnachev and V.A. Solonnikov about 45 years ago (cf. e.g. [13, 14] or [15]). The two-cycle iteration scheme was also applied in the papers [14, 15] and by the author in [9, 16]. In the references [11, 17, 18] other methods are used to handle different free BVP.

The scheme (5) is very senseful in cases where the free boundary is semi-infinite. In a first cycle the three flow fields $\boldsymbol{v}, p, \theta$ are computed in a flow domain with fixed boundaries neglecting one of the boundary conditions - mostly the normal stress condition $(4)_5$, i.e. the 5th equation in (4). This first cycle is then divided into several steps: The linear problem with fixed boundary containing the corresponding estimates for the solution, a model problem at the separation point Q for the determination of the weight functions, the regularity of the solutions at infinity and then the nonlinear problem with fixed boundary.

In a second stage the neglected boundary condition is used in order to compute a new shape of the free boundary (and simultaneously a new shape of the entire flow domain). This equation is usually

$$K(x_1) := \frac{1}{\sigma(\theta)}\,[-p(\boldsymbol{x}) + \boldsymbol{n}\cdot\boldsymbol{S}(\boldsymbol{v})\boldsymbol{n}]|_\Gamma,$$

where $K(x_1)$ denotes the curvature of Γ in x_1 and it is equal to the left-hand side of Eq. $(4)_5$. In both cycles a related linear problem is solved and the continuous dependence of the solutions on the boundary data is also proved. Then BANACH‘s fixed point argument related to some contraction operator $\mathfrak{B}$ shows the remaining parts for small data.

3 Function spaces

First of all we define some weighted HÖLDER spaces. Let B be an arbitrary domain in $\mathbb{R}^2$ and $N \subset \bar{B}$ a manifold of dimension $\bar{n} < 2$. Define further $\varrho_N(x) := \operatorname{dist}(x, N)$. By $\beta = (\beta_1, \beta_2)$ we understand a multiindex, and $\lfloor r \rfloor$ is the integer part of r. Then by $C^r(B)(r > 0$, non-integer) we mean the well-known HÖLDER space with a finite norm $|u|_B^{(r)}$. Now we obtain the subsequent weighted HÖLDER space $\overset{\circ}{C}{}^r_s(B, N)$ of functions with the finite norm

$$|u|_{\overset{\circ}{C}{}^r_s(B,N)} = \sum_{|\beta|<r} \sup_{x\in B\setminus N} \varrho_N^{|\beta|-s}(x)|D^\beta u(x)| + \sum_{|\beta|=\lfloor r\rfloor} \sup_{x\in B\setminus N} \varrho_N^{r-s}(x) \sup_{|x-y|<\frac{1}{2}\varrho_N(x)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^{r-\lfloor r\rfloor}}. \tag{6}$$

Let us remark that the weight functions in (6) represent some kind of power functions with respect to the distance from the singularity points. For $(r > s > 0; r, s$ non-integer) we get the space $C^r_s(B, N)$ having the norm

$$|u|_{C^r_s(B,N)} := |u|_B^{(s)} + \sum_{s<|\beta|<r} \sup_{x\in B\setminus N} \varrho_N^{|\beta|-s}(x)|D^\beta u(x)| + \sum_{|\beta|=\lfloor r\rfloor} \sup_{x\in B\setminus N} \varrho_N^{r-s}(x) \sup_{|x-y|<\frac{1}{2}\varrho_N(x)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^{r-\lfloor r\rfloor}}. \tag{7}$$

The spaces $C^r_s(B_1 \cup B_2, N)$ represent the natural generalization of the last ones to the case of two separate subdomains B_k as we have.

Furthermore, $C^\infty_0(\Omega, \Gamma)$ is the set of functions v vanishing for $|x| \gg 1$ and satisfying the boundary conditions

$$v|_{\Sigma_j} = 0 \qquad (j = 0, 1, 2), \qquad v \cdot n|_{\Gamma^-} = 0, \qquad [v]|_\Gamma = 0.$$

Also, $J^\infty_0(\Omega, \Gamma) := \{v \in C^\infty_0(\Omega, \Gamma),\ \operatorname{div} v = 0\}$ and we need the solenoidal spaces with DIRICHLET - norm like $D(\Omega) := \overline{C^\infty_0(\Omega, \Gamma)}$ and $H(\Omega) := \overline{J^\infty_0(\Omega, \Gamma)}$. The DIRICHLET norm is then defined by

$$\| u_x \|^2_\Omega := \| u_x \|^2_{L_2(\Omega)} = \int\limits_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial u_i}{\partial x_j} \right)^2 \mathrm{d}x.$$

By $C^\infty_{\theta,0}(\Omega, \Gamma)$ we understand the set of scalar fields $\theta(\boldsymbol{x})$ vanishing for $|\boldsymbol{x}| \gg 1$ and $\theta|_{\Sigma_j} = 0$ ($j = 0, 1, 2$), $[\theta] \|_\Gamma = 0$. The symbol $D_\theta(\Omega)$ is an abbreviation for the set $\overline{C^\infty_{\theta,0}(\Omega, \Gamma)}$. The spaces to which the solution belongs are of the subsequent type. First, define some subdomains of the infinite channel (with $m = 1, 2$), i.e. $I^0 = \{x_1 \in \mathbb{R}, 0 < x_1 < 2\}$, $I^+ = \{x_1 \in \mathbb{R}, x_1 > 1\}$, $\Omega^0_m = \{\boldsymbol{x} \in \Omega_m, |x_1| < 2\}$, $\Omega^+_m = \{\boldsymbol{x} \in \Omega_m, |x_1 > 1\}$, $\Omega^-_m = \{\boldsymbol{x}, |x_1 < -1\}$.

Finally, the weighted HÖLDER spaces containing the generalized solutions, are

$$C^r_{s,z}(\Omega_m) = \{u(x), u|_{\Omega^0_m} \in C^r_s(\Omega^0_m, Q),\ \exp(zx_1)u(x)|_{\Omega^+_m} \in C^r(\Omega^+_m),\ \exp(-zx_1)u(x)|_{\Omega^-_m} \in C^r(\Omega^-_m)\}.$$

They are essentially used throughout this paper and their norms are given by

$$\| u \|^{r,z}_{\Omega_m,s} := |u|_{C^r_s(\Omega^0_m, Q)} + |\exp(zx_1)u|^{(r)}_{\Omega^+_m} + |\exp(-zx_1)u|^{(r)}_{\Omega^-_m}. \tag{8}$$

The weight functions here in formula (8) are exponential functions and they decay at the infinities. As above, we obtain for our double channel $C^r_{s,z}(\Omega) := C^r_{s,z}(\Omega_1 \cup \Omega_2, Q)$.

At the end, for functions of one real variable we deal with the space $C^r_{s,z}(\mathbb{R}^1_+)$ supplied with the norm $\| f \|^{r,z}_{\mathbb{R}^1_+,s} = |f|_{C^r_s(I^0,0)} + |f(x_1)\exp(zx_1)|^{(r)}_{I^+}$.

4 On the Basic Flow for Large x_1

In this section we are interested in getting an approriate starting (or initial) solution for the iteration scheme (5). For this purpose, and under the assumptions

$$v_2 \equiv 0, \qquad \frac{\partial v_1}{\partial x_1} \equiv 0, \qquad \frac{\partial \theta}{\partial x_1} \equiv 0, \tag{9}$$

we calculate for given values $F_1, F_2, R, \theta_0 = 0, \theta_1, \theta_2$ and associated rheological parameters the flow fields and values $\boldsymbol{v}(\boldsymbol{x}), p(\boldsymbol{x}), \theta(\boldsymbol{x}), p_0, h_\infty, \theta_\infty$. The value θ_∞ which has not been defined before describes the (asymptotic) value of the temperature θ at the free interface when x_1 goes to $+\infty$. Let us emphasize that the assumptions guarantee solution fields that are uniform and unidirectional (not depending on main-stream direction x_1).

Under the assumptions (9) the governing Eqs. (1) take the subsequent reduced form:

$$-\nu\, \nabla^2 v_1 + \frac{1}{\eta} \frac{\partial p}{\partial x_1} = 0, \qquad \frac{\partial p}{\partial x_2} = -\eta\, g, \qquad -\lambda\, \nabla^2 \theta = 0,$$

where the second equation replaces the continuity Eq. $(1)_2$ Now it is possible to divide the original problem into three independent problems for the flow fields. Let us start with the problem for velocities $\boldsymbol{v}$:

$$\begin{cases} \nu_1\eta_1 \frac{\mathrm{d}^2 v_1^{(1)}}{\mathrm{d}x_2^2} = \frac{\partial p^{(1)}}{\partial x_1}, & \nu_2\eta_2 \frac{\mathrm{d}^2 v_1^{(2)}}{\mathrm{d}x_2^2} = \frac{\partial p^{(2)}}{\partial x_1}, \\ v_1^{(1)}(0) = R, & v_1^{(2)}(1) = 0, \\ v_1^{(1)}|_{x_2=h_\infty} = v_1^{(2)}|_{x_2=h_\infty}, \ \nu_1\eta_1 \left.\frac{\mathrm{d}v_1^{(1)}}{\mathrm{d}x_2}\right|_{x_2=h_\infty} & = \nu_2\eta_2 \left.\frac{\mathrm{d}v_1^{(2)}}{\mathrm{d}x_2}\right|_{x_2=h_\infty}, \\ \int_0^{h_\infty} v_1(x_2)\,\mathrm{d}x_2 = F_1, & \int_{h_\infty}^1 v_1(x_2)\,\mathrm{d}x_2 = F_2. \end{cases} \tag{10}$$

For the pressure p one obtains the following equations

$$\begin{cases} \frac{\partial p^{(1)}}{\partial x_2} = -\eta_1\, g, & \frac{\partial p^{(2)}}{\partial x_2} = -\eta_2\, g, \\ p^{(1)}|_{x_2=h_\infty} = p^{(2)}|_{x_2=h_\infty}. \end{cases} \tag{11}$$

Finally, the problem for temperature θ reads

$$\begin{cases} \frac{\mathrm{d}^2\theta^{(1)}}{\mathrm{d}x_2^2} = 0, & \frac{\mathrm{d}^2\theta^{(2)}}{\mathrm{d}x_2^2} = 0, \\ \theta^{(1)}|_{x_2=0} = \theta_0 = 0, & \theta^{(2)}|_{x_2=1} = \theta_2, \\ \theta^{(1)}|_{x_2=h_\infty} = \theta^{(2)}|_{x_2=h_\infty}, \ \lambda_1 \left.\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x_2}\right|_{x_2=h_\infty} = \lambda_2 \left.\frac{\mathrm{d}\theta^{(2)}}{\mathrm{d}x_2}\right|_{x_2=h_\infty}, \end{cases} \tag{12}$$

In Eqs. (10), (11), (12) the superscripts $(k), (k = 1, 2)$ or $(+)$ denote the corresponding fluid layer and the subregion $x_1 \geqslant 1$. The solutions of these three (independent) problems are of NUSSELT type (cf. also [19]) and allow the representation

$$\begin{cases} v_1^{(+)}(x_2) = \begin{cases} 0.5a_1x_2^2 + b_1x_2 + R, & 0 \leqslant x_2 \leqslant h_\infty \\ 0.5a_2(x_2^2-1) + b_2(x_2-1), & h_\infty \leqslant x_2 \leqslant 1 \end{cases}, \\ v_2^{(+)}(x_2) \equiv 0, \qquad p_0 = a_1 v_1 \eta_1 = a_2 v_2 \eta_2, \qquad r = (v_1\eta_1)/(v_2\eta_2), \end{cases} \tag{13}$$

The coefficients in (13) are given by

$$a_1 = \left[-3\frac{F_1 - Rh_\infty}{h_\infty^2} - 3\frac{F_2}{r(1-h_\infty^2)}\right], \quad a_2 = r\,a_1,$$

$$b_1 = \left[(2+h_\infty)\frac{F_1 - Rh_\infty}{h_\infty^2} + h_\infty\frac{F_2}{r(1-h_\infty^2)}\right], \quad b_2 = r\,b_1.$$

Note, that the values h_∞ and p_0 are already known for these expressions (see Eq. (16) below). That is why it follows $\theta_\infty = (\lambda_2\theta_2h_\infty)/[\lambda_1(1-h_\infty)+\lambda_2h_\infty]$ and for the complete temperature and pressure fields one obtains

$$\theta^{(+)}(x_2) = \begin{cases} \frac{\theta_\infty}{h_\infty}\, x_2, & 0 \leqslant x_2 \leqslant h_\infty \\ \theta_\infty + \frac{\theta_2-\theta_\infty}{1-h_\infty}\, x_2 - \frac{\theta_2-\theta_\infty}{1-h_\infty}\, h_\infty, & h_\infty \leqslant x_2 \leqslant 1, \end{cases} \tag{14}$$

$$p^{(+)}(\boldsymbol{x}) = \begin{cases} p_0x_1 - \eta_1gx_2 + k, \\ p_0x_1 - \eta_2g(x_2-h_\infty) - \eta_1gh_\infty + k, \end{cases} \tag{15}$$

Since the associated linear problem is completely decomposed, we got the same polynomial equation for the determination of the value h_∞ as in the former paper [17].

$$\begin{aligned} 0 &= r(r-1)Rh_\infty^5 + \left[-4r(r-1)R - r(r-1)F_1 - (r-1)F_2\right] h_\infty^4 \\ &+ \left[r(6r-5)R + 2r(2r-3)F_1 - 2rF_2\right] h_\infty^3 + [2r(-2r+1)R \\ &+3r(-2r+3)F_1 + 3rF_2]h_\infty^2 + \left[r^2R + 4r(r-1)F_1\right] h_\infty - r^2F_1. \end{aligned} \tag{16}$$

In [17] the subsequent two lemmas were proved.

Lemma 4.1. *If* $F_1F_2 > 0$, *then Eq.* (16) *has at least one root* h_∞ *within the open interval* $]0, 1[$.

Lemma 4.2. *If* $F_1F_2 \geqslant 0$, *then Eq.* (16) *has at most three different roots* $h_\infty \in]0, 1[$.

Note that in the subregion $\Omega^{(-)} := \Omega_1^{(-)} \cup \Omega_2^{(-)}$, i.e. for $x_1 \leqslant -1$, the corresponding problems are even simpler due to the fact that there is no free boundary. In order not to repeat simple things we restrict the presentation to the basic solution $\boldsymbol{v}^{(-)}, p^{(-)}, \theta^{(-)}$ in the double - channel which can also be determined very simple by straightforward calculations in the left part Ω^- of the (double) - channel. The corresponding velocities and temperatures do not depend on x_1. In Ω_1^- one obtains

$$\begin{cases} v_1^{(-)}(x_2) = \left(\frac{3R}{h_1^2} - \frac{6F_1}{h_1^3}\right) x_2^2 + \left(-\frac{4R}{h_1} + \frac{6F_1}{h_1^2}\right) x_2 + R, \\ v_2^{(-)}(\boldsymbol{x}) \equiv 0, \qquad \theta^{(-)}(x_2) = \frac{x_2\theta_1}{h_1}, \\ p^{(-)}(\boldsymbol{x}) = 2v_1\eta_1 \left(\frac{3R}{h_1^2} - \frac{6F_1}{h_1^3}\right) x_1 - \eta_1gx_2 + k_1. \end{cases} \tag{17}$$

In Ω_2^- one gets, respectively,

$$\begin{cases} v_1^{(-)}(x_2) = -\frac{6F_2}{(1-h_1)^3}x_2^2 + \frac{6(1+h_1)F_2}{(1-h_1)^3}x_2 - \frac{6h_1F_2}{(1-h_1)^3}, \\ v_2^{(-)}(\boldsymbol{x}) \equiv 0, \qquad \theta^{(-)}(x_2) = \theta_1 + \frac{(x_2-h_1)}{(1-h_1)}(\theta_2-\theta_1), \\ p^{(-)}(\boldsymbol{x}) = -\frac{12\nu_2\eta_2F_2}{(1-h_1)^3}x_1 - \eta_2 g x_2 + k_2. \end{cases} \tag{18}$$

It is well-known that the pressure p can be determined only up to an additive constant in channel flows (cf. k_1, k_2 in formulae (17), (18)).

5 The free interface equation

Eq. (16) coincides with equation (A.13) from [17] for horizontal channels. Recall that the final thickness h_∞ is a function of F_1, F_2, R and of the rheological parameters of the fluids. It can have up to three different values in the open interval $]0,1[$ for the same parameter set (cf. [17]). Furthermore, by $\varphi^{(+)}(x_1)$ we denote the infinitely differentiable solution of the following free BVP.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\varphi'(x_1)}{\sqrt{1+\varphi'(x_1)^2}} - \frac{\eta_1-\eta_2}{\widehat{a}}g\varphi(x_1) = -\frac{\eta_1-\eta_2}{\widehat{a}}gh_\infty, \\ \varphi(0) = h_1, \qquad \lim_{x_1\to+\infty}\varphi(x_1) = h_\infty. \end{cases} \tag{19}$$

which can be obtained from the 5-th condition $(4)_5$ of (4) by setting $\boldsymbol{v} = \boldsymbol{0}$, $p = \text{const.}$, $\theta = 0$ as the initial solution for $F_1 = F_2 = R = \theta_0 = \theta_2 = 0$. Let $\xi = \xi(x_1)$ be a smooth cut-off function vanishing for $|x_1| \leqslant 1$ and being equal to 1 for $|x_1| \geqslant 2$. Finally, assume that $\eta_1 > \eta_2$ is satisfied. This makes physically sense.

Now, the difference function $\omega(x_1) := (\varphi(x_1) - \varphi^{(+)}(x_1))$ is equivalent to $\exp(-\sqrt{g(\eta_1-\eta_2)/\widehat{a}}\ x_1)$ as $x_1 \to +\infty$. For the unknown function $\omega(x_1)$ we get a two-point BVP like BVP (8.8) from [20] subtracting Eq. (19) from Eq. $(4)_5$. A difference to BVP (8.8) consists in the following. We have to replace β_1 by $g(\eta_1-\eta_2)/\widehat{a}$ everywhere and, furthermore, we have to introduce the operator $\mathfrak{T}^{(3)}$ by

$$\mathfrak{T}^{(3)}\omega := \frac{\widehat{b}\,\theta}{\sigma(\theta)}\ \omega = \frac{\sigma(0)-\sigma(\theta)}{\sigma(\theta)}\ \omega. \tag{20}$$

The remaining part of the proof of the main theorem is a slightly modified repetition of the proof of Theorem 8.1 in [20]. First of all, one has to study the dependence of the solution to the nonlinear auxiliary problem with fixed boundary on small variations of the boundary. After getting the corresponding estimates one applies BANACH's fixed point principle to the subsequent operator equation. Instead of the operator Eq. (8.10) from [20] we have to study the following one:

$$\omega = \mathfrak{L}(\mathfrak{T}^{(1)}\omega + \mathfrak{T}^{(2)}\omega + \mathfrak{T}^{(3)}\omega) =: \mathfrak{B}\,\omega$$

with $\mathfrak{T}^{(3)}$ given in (20) and the other parts taken from [20]. Since $\mathfrak{T}^{(3)}$ is a contraction operator for small θ, we can conclude as in [20] that $\mathfrak{B}$ is a contraction operator in the ball $\|\ \omega\ \|^{3+s,z}_{\mathbb{R}^1_+,1+s} < \varepsilon$. Consequently, we have proved the main result of this paper.

6 Results

Let us formulate the main result of this contribution. A sketch of the proof has been given before. A very detailed application of this method can be found in the thesis [16] as well as in the article [20].

Theorem 6.1. *There exist positive real numbers $\bar{s}, \bar{z} \leqslant \min(z_0, \sqrt{1/\widehat{a}})\,l$ such that for arbitrary $s \in]0,\bar{s}[$, $z \in]0,\bar{z}[$ and for sufficiently small values $(|F_1|, |F_2|, |R|, |\theta_1|, |\theta_2|)$ and for values h_∞ fulfilling the condition*

$$|h_\infty - h_1| < \sqrt{\frac{2\widehat{a}}{g(\eta_1-\eta_2)}}$$

the complete mathematical model has a unique solution $\{\boldsymbol{v}, p, \theta, \varphi\}$ *which can be represented in the form*

$$\boldsymbol{v} = \xi(-x_1)\boldsymbol{v}^{(-)} + \xi(x_1)\boldsymbol{v}^{(+)} + \boldsymbol{w}, \quad \varphi(x_1) = \varphi^{(+)}(x_1) + \omega(x_1),$$

$$p = \xi(-x_1)p^{(-)} + \xi(x_1)p^{(+)} + q; \quad \theta = \xi(-x_1)\theta^{(-)} + \xi(x_1)\theta^{(+)} + \vartheta_0,$$

where ξ *is the cut-off function described above,* $(\boldsymbol{v}^{(-)}, p^{(-)}, \theta^{(-)})$ *is the basic exact solution given by* (17), (18) *in both channels on left-hand side. The function* $\varphi^{(+)}$ *is the solution to the free* BVP (19). *Moreover,* $\vartheta_0, \boldsymbol{w} \in C^{s+2}_{s,z}(\Omega)$, $q \in C^{s+1}_{s-1,z}(\Omega^0 \cup \Omega^+)$, $\nabla q \in C^{s}_{s-2,z}(\Omega)$ *and* $\omega \in C^{3+s}_{1+s,z}(\mathbb{R}^1_+)$ *hold.*

Remark 6.2. *If Eq.* (16) *has more than one real root* h_∞ *between 0 and 1 then the statements of Theorem 6.1 remain true in the neighbourhood of each value.*

References

[1] Bänsch, E., Basting, S., Krahl, R., Numerical simulation of two-phase flows with heat and mass transfer, Discrete Contin. Dyn. Syst. 35, (2015) 6, 2325-2347, doi: 10.3934/dcds.2015.35.2325.

[2] Burelbach J.P., Bankhoff, S.G., Davis, S.H., Nonlinear stability of evaporating condensing liquid films, J. Fluid Mech. 195, (1988), 463-494, http://dx.doi.org/10.1017/S0022112088002484.

[3] Denk, R., Geissert, M., Hieber, M., Saal, J., Sawada, O., The spin-coating process: analysis of the free boundary value problem, Commun. Partial Differ. Equations 36, (2011) 7-9, 1145-1192, https://doi.org/10.1080/03605302.2010.546469.

[4] Erunova, I.B., Rivkind, V.Ya., Investigation of the problem of liquid evaporation, Vestnik St. Petersburg University - Mathematics 24 (1991), 25-30.

[5] Fontelos, M.A., Hong, S.H., Hwang, H.J., A stable self-similar singularity of evaporating drops: ellipsoidal collapse to a point, Arch. Ration. Mech. Anal. 217, (2015) 2, 373-411, doi: 10.1007/s00205-014-0834-x.

[6] Köhne, M., Prüss, J., Wilke, M., Qualitative behaviour of solutions for the two-phase Navier-Stokes equations with surface tension, Math. Ann. 356, (2013) 2, 737-792, https://doi.org/10.1007/s00208-012-0860-7.

[7] Larina, I.N., Rykov, V.A., Shakov, E.M., Evaporation from a surface and vapor flow through a plane channel into a vacuum, Fluid Dyn. 31, (1996), 127-133, http://dx.doi.org/10.1007/BF02230757.

[8] Neittaanmäki, P., Rivkind, V.Ya., Mathematical modelling of liquid drops evaporation, Jyväskylä – St. Petersburg Seminar on Partial Differential Equations and Numerical Methods, Ber. Univ. Jyväskylä Math. Inst. 56, (1993), 89-100.

[9] Socolowsky, J., The solvability of a two-layer slot coating flow with evaporation, J. Coupled Syst. Multisc. Dyn. 1, (2013), No. 4, 434-441, doi: 10.1166/jcsmd.2013.1033.

[10] Taylor, Sh.D., Two-Layer Slot Coating: Study of Die Geometry and Interfacial Region (1997). Open Access Dissertations and Theses. Paper 5146. http://digitalcommons.mcmaster.ca/opendissertations/

[11] Xu, L., Zhang, Zh., On the free boundary problem to the two viscous immiscible fluids, J. Differ. Equations 248, (2010) 5, 1044-1111.

[12] van de Fliert, B.W., A free boundary problem for evaporating layers, Nonlinear Anal., Theory Methods Appl. 47, (2001) 3, 1785-1796, http://dx.doi.org/10.1016/S0362-546X(01)00310-8.

[13] Pukhnachov, V.V., Plane stationary free boundary problem for Navier-Stokes equation, Zh. Prikl. Mekh. i Tekhn. Fiz. 3, (1972), 91-102 (in Russian), English Transl. in J. Appl. Mech. Techn. Phys. 13, (1972).

[14] Solonnikov, V.A., Solvability of a problem on the plane motion of a heavy viscous incompressible capillary liquid partially filling a container, Izv. Akad. Nauk SSSR - Ser. Math. 43, (1979) 1, 203-236 (in Russian), Engl. Transl. in Math. USSR, Izv. 14, (1980), 193-221, http://dx.doi.org/10.1070/IM1980v014n01ABEH001069.

[15] Solonnikov, V.A., Solvability of a three-dimensional boundary value problem with a free surface for the stationary Navier-Stokes equations, Zapiski Nauchn. Sem. LOMI 84, (1979), 252-285 (in Russian).

[16] Socolowsky, J., Mathematische Untersuchungen freier Randwertaufgaben der Hydrodynamik viskoser Flüssigkeiten, Habilitation thesis , Techn. Hochschule Merseburg, Merseburg (GERMANY), 1989, pp.1-218. (in German)

[17] Pileckas, K., Socolowsky, J., Viscous two-fluid flows in perturbed unbounded domains, Math. Nachrichten 278, (2005) 5, 589-623, http://dx.doi.org/10.1002/mana.200310260.

[18] Prüss, J., Simonett, G., On the two-phase Navier-Stokes equations with surface tension, Interfaces Free Bound. 12, (2010) 3, 311-345, doi: 10.4171/IFB/237.

[19] Socolowsky, J., On the Nusselt solution of a non-isothermal two-fluid inclined film flow, International Journal of Mathematics and Mathematical Sciences (IJMMS) (2009), Article ID 981983, 8 pages. [**Electronic Journal**], doi:10.1155/2009/981983 (*Online and free access Journal*)

[20] Socolowsky, J., Solvability of a stationary problem on the plane motion of two viscous incompressible liquids with non-compact free boundaries, Z. Angew. Math. Mech.(ZAMM) 72, (1992) 251-268, doi: 10.1002/zamm.19920720706.