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A modified quasi-boundary value method for an abstract ill-posed biparabolic problem

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Abstract: In this paper, we are concerned with the problem of approximating a solution of an ill-posed biparabolic problem in the abstract setting. In order to overcome the instability of the original problem, we propose a modified quasi-boundary value method to construct approximate stable solutions for the original ill-posed boundary value problem. Finally, some other convergence results including some explicit convergence rates are also established under a priori bound assumptions on the exact solution. Moreover, numerical tests are presented to illustrate the accuracy and efficiency of this method.

Keywords: Ill-posed problems, Biparabolic problem, Regularization

MSC: 47A52, 65J22

1 Formulation of the problem

Throughout this paper H denotes a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, $\mathcal{L}(H)$ stands for the Banach algebra of bounded linear operators on H .

Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a positive, self-adjoint operator with compact resolvent, so that A has an orthonormal basis of eigenvectors $(\phi_n) \subset H$ with real eigenvalues $(\lambda_n) \subset \mathbb{R}_+$, i.e.,

$$A\phi_n = \lambda_n\phi_n, n \in \mathbb{N}^*, \quad \langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

$$0 < \nu \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty,$$

$$\forall h \in H, \quad h = \sum_{n=1}^{\infty} h_n \phi_n, \quad h_n = \langle h, \phi_n \rangle.$$

In this paper, we consider the following inverse source problem of determining the unknown source term $u(0) = f$ and the temperature distribution $u(t)$ for $0 \leq t < T$, of the following biparabolic problem

$$\begin{cases} \mathcal{B}^2 u = \left(\frac{d}{dt} + A \right)^2 u(t) = u''(t) + 2Au'(t) + A^2 u(t) = 0, & 0 < t < T, \\ u(T) = g, \quad u'(0) = 0, \end{cases} \quad (1)$$

where $0 < T < \infty$ and g is a given H -valued function.

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To our knowledge, the literature devoted to this class of problems is quite scarce, except the papers [1–6]. The study of this case is caused not only by theoretical interest, but also by practical necessity. In particular, the bipolarabolic model is used in mathematical modeling to describe special features of the dynamics of deformable water-saturated porous media during their filtration consolidation subject to applied loads [7–9].

It is well-known that the classical heat equation does not accurately describe the conduction of heat [10, 11]. Numerous models have been proposed for better describing this phenomenon. Among them, we can cite the bipolarabolic model proposed in [12], the fractional bipolarabolic model [13], for a more adequate mathematical description of heat and diffusion processes than the classical heat equation. For physical motivation and other models we refer the reader to [14–21].

This work is a continuity of the work developed recently by Lakhdari and Boussetila [2], where the strategy of regularization which will be used is completely different that used in [2]. Our new strategy is motivated by the simplicity of the method, as well as the numerical results obtained, which are better compared to those obtained using a variant of an iterative regularization [2]. More precisely, we propose an improved modified quasi-boundary-value method with two parameters $\alpha > 0$ and $r \geq 0$, where the parameter α is introduced to filter the high frequencies, and the second parameter r to include the regularity of the solution of the original problem. The advantage of the multi-parameter regularization is such that it gives more freedom in attaining order optimal accuracy [22–29].

The quasi-boundary value method, also called non-local auxiliary boundary condition, introduced and developed by Showalter [30, 31], is a regularization technique by replacing the final condition or boundary condition by a nonlocal condition such that the perturbed problem is well-posed.

The main advantage of the quasi-boundary-value method is that it gives a well-posed problem, where the differential equation has not been changed, only the boundary values have been modified. Therefore, we can exploit various numerical methods to approach the problem in question, for arbitrary geometry $[0, T] \times \Omega$, where Ω is a sub-set of $\mathbb{R}^n$, $n \geq 1$.

This method has been used to solve some ill-posed problems for parabolic, hyperbolic and elliptic equations; for more details, see [22, 32–44] and the references therein.

2 Ill-posedness of the problem and a conditional stability result

We point out here some results established in [2].

Let us consider the following well-posed problem.

$$\begin{cases} \mathcal{B}^2 w = \left(\frac{d}{dt} + A \right)^2 w(t) = w''(t) + 2Aw'(t) + A^2 w(t) = 0, & 0 < t < T, \\ w(0) = \xi, \quad w'(0) = 0, \end{cases} \quad (2)$$

where $\xi \in \mathcal{D}(A)$.

Theorem 2.1 ([2]). *For any $\xi \in \mathcal{D}(A)$, problem (2) admits a unique solution*

$w \in C^2([0, +\infty[; H) \cap C^1([0, +\infty[; H) \cap C([0, +\infty[; \mathcal{D}(A)) \cap C^1([0, +\infty[; \mathcal{D}(A)) \cap C^2([0, +\infty[; \mathcal{D}(A^2))$
given by

$$w(t) = \mathcal{R}(t; A)\xi = (I + tA)e^{-tA}\xi = \sum_{n=1}^{\infty} (1 + t\lambda_n)e^{-t\lambda_n} \langle \xi, \phi_n \rangle \phi_n. \quad (3)$$

Remark 2.2. *It is easy to check that*

$$\|\mathcal{R}(t; A)\| = \sup_{\lambda \geq \lambda_1} (1 + t\lambda)e^{-t\lambda} \leq (1 + t\lambda_1)e^{-t\lambda_1}, \quad (4)$$

$$\sup_{0 \leq t \leq T} \|\mathcal{R}(t; A)\| = \sup_{0 \leq t \leq T} (1 + t\lambda_1)e^{-t\lambda_1} = 1. \quad (5)$$

2.1 Ill-posedness of the problem (1)

Theorem 2.3 ([2]). *Let $g \in H$, then the unique formal solution of the problem (1) is given by*

$$u(t) = \sum_{n=1}^{\infty} \left(\frac{1+t\lambda_n}{1+T\lambda_n} \right) e^{(T-t)\lambda_n} \langle g, \phi_n \rangle \phi_n. \quad (6)$$

In this case,

$$f = u(0) = \sum_{n=1}^{\infty} \frac{1}{1+T\lambda_n} e^{T\lambda_n} \langle g, \phi_n \rangle \phi_n. \quad (7)$$

From this representation we see that $u(t)$ is unstable in $[0, T[$. This follows from the high-frequency

$$\sigma(t, \lambda_n) = \left(\frac{1+t\lambda_n}{1+T\lambda_n} \right) e^{(T-t)\lambda_n} \longrightarrow +\infty, \quad n \longrightarrow +\infty.$$

Remark 2.4.

- In the classical backward parabolic problem

$$v_t + Av = 0, \quad 0 < t < T, \quad v(T) = g, \quad (8)$$

the unique formal solution is given by

$$v(t) = \sum_{n=1}^{\infty} \theta_n(t, \lambda_n) \langle g, \phi_n \rangle \phi_n, \quad (9)$$

where

$$\theta_n(t, \lambda_n) = e^{(T-t)\lambda_n} \longrightarrow +\infty, \quad n \longrightarrow +\infty.$$

In this case, the high-frequency $\theta_n(t, \lambda_n)$ is equal to $e^{(T-t)\lambda_n}$ and the problem is severely ill-posed.

- In the case of biparabolic model, we have $\sigma_n = r_n \theta_n$, where

$$r_n = \left(\frac{1+t\lambda_n}{1+T\lambda_n} \right),$$

is the **relaxation coefficient** resulting from the **hyperbolic character** of the biparabolic model.

Observe that

$$\frac{t}{T} \leq r_n \leq \frac{1+t\lambda_1}{1+T\lambda_1} \leq 1, \quad (10)$$

and

$$u(t) = R(t)v(t), \quad (11)$$

where

$$\|R(t)\| = \sup_{n \geq 1} \{r_n\} = r_1 = \frac{1+t\lambda_1}{1+T\lambda_1}. \quad (12)$$

From this remark, we observe that the degree of ill-posedness in the biparabolic model is relaxed compared to the classical parabolic case.

2.2 Conditional stability estimate

We would like to have estimates of the form

$$\|u(t)\| \leq \Psi(\|g\|),$$

for some function $\Psi(\cdot)$ such that $\Psi(s) \longrightarrow 0$ as $s \longrightarrow 0$.

Since the problem of determining $u(t)$ from the knowledge of $\{u(T) = g, u'(0) = 0\}$ is ill-posed, an estimate such as the above will not be possible unless we restrict the solution $u(t)$ to certain source set $\mathcal{M} \subset H$.

In our model, we will see that we can employ the method of logarithmic convexity to identify this source set:

$$M_\rho = \{w(t) \in H : w \text{ obeys (1) and } \|Aw(0)\| \leq \rho < \infty\}. \quad (13)$$

On the basis $\{\phi_n\}$ we introduce the Hilbert scale $(H^s)_{s \in \mathbb{R}}$ (resp. $(\mathfrak{E}^s)_{s \in \mathbb{R}}$) induced by A as follows

$$H^s = \mathcal{D}(A^s) = \{h \in H : \|h\|_{H^s}^2 = \sum_{n=1}^{\infty} \lambda_n^{2s} |\langle h, \phi_n \rangle|^2 < +\infty\},$$

$$\mathfrak{E}^s = \mathcal{D}(e^{sTA}) = \{h \in H : \|h\|_{\mathfrak{E}^s}^2 = \sum_{n=1}^{\infty} e^{2Ts\lambda_n} |\langle h, \phi_n \rangle|^2 < +\infty\},$$

We give here a result of conditional stability. The demonstration is given in the paper [2].

Theorem 2.5. *The problem 1 is conditionally well-posed on the set*

$$M = \{w(t) \in H : \|Aw(0)\| < \infty\}$$

if and only if

$$g \in \mathfrak{E}^1 = \{h \in H : \sum_{n=1}^{\infty} e^{2T\lambda_n} |\langle h, \phi_n \rangle|^2 < \infty\}.$$

Moreover, if $u(t) \in M_\rho$, then we have the following Hölder continuity

$$\|u(t)\| \leq \Psi(\|g\|) = \gamma \left(\rho^{\frac{T-t}{T}} \right) \|g\|^{\frac{t}{T}}, \quad (14)$$

where $\gamma = \left(\frac{1+T\lambda_1}{\lambda_1} \right)^{\frac{T-t}{T}}$.

3 Regularization and error estimates

In this work, we propose a modified quasi-boundary value method (MQBVM) to solve the inverse problem 1, i.e., replacing the final condition $u(T) = g$ with the functional time nonlocal condition,

$$\alpha A^r u(0) + u(T) = g, \quad (15)$$

to form an approximate regularized problem

$$\begin{cases} \left(\frac{d}{dt} + A \right)^2 u(t) = u''(t) + 2Au'(t) + A^2 u(t) = 0, & 0 < t < T, \\ \alpha A^r u(0) + u(T) = g, \quad u'(0) = 0, \end{cases} \quad (16)$$

where $r > 0$ is a real parameter and $\alpha > 0$ is the regularization parameter.

Remark 3.1. *The case $r = 0$, corresponds to the classical quasi-boundary value method.*

Denoting by $u_\alpha(t)$ the solution of 16. By the separation of variables and the formula (3), we show the well-posedness of (16), and its solution can be expressed by

$$u_\alpha(t) = (I + tA) \left[\alpha A^r + (1 + TA) e^{-TA} \right]^{-1} e^{-tA} g = \sum_{n=1}^{+\infty} \frac{(1 + t\lambda_n)}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} e^{-t\lambda_n} \langle g, \phi_n \rangle \phi_n \quad (17)$$

Theorem 3.2. *For all $g \in D(A)$ and $r > 0$, the approximate problem 16 admits an unique solution u_α given by*

$$u_\alpha(t) = (I + tA) \left[\alpha A^r + (1 + TA) e^{-TA} \right]^{-1} e^{-tA} g = \sum_{n=1}^{+\infty} \frac{(1 + t\lambda_n)}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} e^{-t\lambda_n} g_n \phi_n, \quad g_n = \langle g, \phi_n \rangle \quad (18)$$

Moreover, the following inequality holds

$$\sup_{0 \leq t \leq T} \|u_\alpha(t)\| \leq \|u_\alpha(0)\| \leq C_3 \kappa(\alpha) \|g\|, \quad (19)$$

where

$$\kappa(\alpha) = \begin{cases} \frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}, & 0 < r < 1, \\ \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}, & r \geq 1, \end{cases} \quad (20)$$

and $C_3 = \max(C_1, C_2)$, $C_1 = (rT)^r$, $C_2 = r$.

Proof. We compute

$$\|u_\alpha(t)\|^2 = \sum_{n=1}^{+\infty} \left[\frac{(1+t\lambda_n) e^{-t\lambda_n}}{\alpha \lambda_n^r + (1+T\lambda_n) e^{-T\lambda_n}} \right]^2 |g_n|^2$$

Putting

$$G(\lambda_n) = (1+t\lambda_n) e^{-t\lambda_n} \leq \sup_{n \geq 1} G(\lambda_n) = (1+t\lambda_1) e^{-t\lambda_1} \leq 1, \quad (21)$$

and

$$\tilde{H}(\lambda_n) = \frac{1}{\alpha \lambda_n^r + (1+T\lambda_n) e^{-T\lambda_n}}. \quad (22)$$

Our goal here is to prove that

$$\sup_{\lambda \geq \lambda_1} \tilde{H}(\lambda) = \kappa(\alpha) = \begin{cases} \frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}, & 0 < r < 1, \\ \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}, & r \geq 1, \end{cases} \quad (23)$$

Indeed, we have

$$H(\lambda_n) = \frac{(1+t\lambda_n) e^{-t\lambda_n}}{\alpha \lambda_n^r + (1+T\lambda_n) e^{-T\lambda_n}} \leq \frac{1}{\alpha \lambda_n^r + (1+T\lambda_n) e^{-T\lambda_n}} = \tilde{H}(\lambda_n).$$

Now to estimate $\tilde{H}(\lambda)$, we proceed as follows

$$\sup_{\lambda \geq \lambda_1} \tilde{H}(\lambda) \leq \max\{A, B\}, \text{ where}$$

$$\begin{cases} A = \sup_{\lambda \leq \lambda^*} \tilde{H}(\lambda), \\ B = \sup_{\lambda \geq \lambda^*} \tilde{H}(\lambda), \\ \lambda^* = \frac{1}{Tr} \ln\left(\frac{1}{\alpha}\right), \quad 0 < \alpha < 1. \end{cases}$$

For $0 < \nu \leq \lambda \leq \lambda^*$, we have

$$\tilde{H}(\lambda_n) \leq \frac{1}{(1+T\lambda_n) e^{-T\lambda_n}} = \frac{e^{T\lambda_n}}{1+T\lambda_n}$$

We denote $s = T\lambda_n$ and the function

$$f(s) = \frac{e^s}{1+s}.$$

The function attains its maximum at λ^* ,

$$\sup_{\lambda \leq \lambda^*} \tilde{H}(\lambda) = \frac{e^{T\lambda^*}}{1+T\lambda^*} \leq \frac{e^{T\lambda^*}}{T\lambda^*} = \frac{e^{T(\frac{1}{Tr} \ln(\frac{1}{\alpha}))}}{T(\frac{1}{Tr} \ln(\frac{1}{\alpha}))}.$$

Hence, we get

$$\sup_{\lambda \leq \lambda^*} \tilde{H}(\lambda) \leq \frac{r}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}. \quad (24)$$

If $\lambda \geq \lambda^*$, we can write

$$\tilde{H}(\lambda) = \frac{1}{\alpha \lambda^r + (1+T\lambda) e^{-T\lambda}} \leq \frac{1}{\alpha} \left(\frac{1}{\lambda}\right)^r \leq \frac{1}{\alpha} \left(\frac{1}{\lambda^*}\right)^r$$

which implies that

$$\sup_{\lambda \geq \lambda^*} \tilde{H}(\lambda) \leq \frac{1}{\alpha} \left(\frac{1}{\frac{1}{r} \ln(\frac{1}{\alpha})} \right)^r = (rT)^r \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}. \quad (25)$$

Putting $C_1 = (rT)^r$, $C_2 = r$, $C_3 = \max(C_1, C_2)$.

- If $0 < r < 1$ and $0 < \alpha < 1$, we observe that

$$\begin{aligned} \left[\frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})} \right] \left[\frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r} \right]^{-1} &= \frac{\alpha [\ln(\frac{1}{\alpha})]^r}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})} \\ &= \alpha^{\frac{r-1}{r}} \left[\ln\left(\frac{1}{\alpha}\right) \right]^{r-1} \\ &= \frac{1}{\alpha^{\frac{1-r}{r}} [\ln(\frac{1}{\alpha})]^{1-r}} = \gamma(\alpha) \longrightarrow +\infty, \text{ as } \alpha \longrightarrow 0. \end{aligned}$$

$\lim_{\alpha \rightarrow 0} \gamma(\alpha) = +\infty \implies$ for $\varepsilon = 1$, $\exists \alpha_0$ such as $\alpha \leq \alpha_0 \Rightarrow \gamma(\alpha) \geq \varepsilon = 1$. Then, for α sufficiently small, we have

$$\frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})} \geq \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}.$$

Therefore,

$$\max(A, B) = C_3 \frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}. \quad (26)$$

- If $r \geq 1$ and $0 < \alpha < 1$, we have $\lim_{\alpha \rightarrow 0} \gamma(\alpha) = 0 \implies$ for $\varepsilon = 1$, $\exists \alpha_0$ such that $\alpha \leq \alpha_0 \Rightarrow \gamma(\alpha) \leq \varepsilon = 1$. Then, for α sufficiently small, we have

$$\frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})} \leq \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r},$$

and thus

$$\max(A, B) = C_3 \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}. \quad (27)$$

From (26) and (27), we obtain the desired estimate:

$$\sup_{0 \leq t \leq T} \|u_\alpha(t)\| \leq \|u_\alpha(0)\| \leq C_3 \kappa(\alpha) \|g\|,$$

where

$$\kappa(\alpha) = \begin{cases} \frac{1}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}, & 0 < r < 1, \\ \frac{1}{\alpha [\ln(\frac{1}{\alpha})]^r}, & r \geq 1. \end{cases} \quad \square$$

Theorem 3.3. If $u(0) \in H$ and $u(0) \in \mathcal{D}(A)$, i.e., $\|u(0)\| + \|Au(0)\| < \infty$, then we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \{ \|u(t) - u_\alpha(t)\| + \|A(u(t) - u_\alpha(t))\| \} &\leq \\ \{ \|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\| \} &\rightarrow 0, \text{ as } \alpha \rightarrow 0 \end{aligned} \quad (28)$$

Remark 3.4. We recall here that

$$\|u(0)\| + \|Au(0)\| < \infty \iff g \in \mathfrak{C}^1.$$

Proof. We have

$$\begin{aligned} u(0) &= \sum_{n=1}^{\infty} \frac{1}{1 + T\lambda_n} e^{T\lambda_n} \langle g, \phi_n \rangle \phi_n, \\ u_\alpha(0) &= \sum_{n=1}^{+\infty} \frac{1}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \langle g, \phi_n \rangle \phi_n, \end{aligned}$$

and

$$\|u(0) - u_\alpha(0)\|^2 = \sum_{n=1}^{+\infty} \left[\frac{[\alpha \lambda_n^r e^{T\lambda_n}]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} \right]^2 |g_n|^2.$$

From this equality we can write

$$\begin{aligned} \|u(t) - u_\alpha(t)\|^2 &= \sum_{n=1}^{+\infty} \left[\frac{(1 + t\lambda_n)}{(1 + T\lambda_n)} e^{(T-t)\lambda_n} - \frac{(1 + t\lambda_n)}{\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n}} e^{-t\lambda_n} \right]^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left[\frac{(1 + t\lambda_n) \left[(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n}) e^{T\lambda_n} - (1 + T\lambda_n) \right]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} e^{-t\lambda_n} \right]^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left[\frac{(1 + t\lambda_n) [\alpha \lambda_n^r e^{T\lambda_n}]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} e^{-t\lambda_n} \right]^2 |g_n|^2 \\ &\leq \sum_{n=1}^{+\infty} \left[\frac{[\alpha \lambda_n^r e^{T\lambda_n}]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} \right]^2 |g_n|^2 \\ &= \|u(0) - u_\alpha(0)\|^2, \end{aligned}$$

and

$$\begin{aligned} \|A(u(t) - u_\alpha(t))\|^2 &= \sum_{n=1}^{+\infty} \left[\lambda_n \left(\frac{(1 + t\lambda_n)}{(1 + T\lambda_n)} e^{(T-t)\lambda_n} - \frac{(1 + t\lambda_n)}{\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n}} e^{-t\lambda_n} \right) \right]^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left[\frac{\lambda_n (1 + t\lambda_n) [\alpha \lambda_n^r e^{T\lambda_n}]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} e^{-t\lambda_n} \right]^2 |g_n|^2 \\ &\leq \sum_{n=1}^{+\infty} \left[\frac{\lambda_n [\alpha \lambda_n^r e^{T\lambda_n}]}{(1 + T\lambda_n)(\alpha \lambda_n^r + (1 + T\lambda_n)e^{-T\lambda_n})} \right]^2 |g_n|^2 \\ &= \|A(u(0) - u_\alpha(0))\|^2, \end{aligned}$$

thus we get

$$\sup_{0 \leq t \leq T} \{\|u(t) - u_\alpha(t)\| + \|A(u(t) - u_\alpha(t))\|\} \leq \{\|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\|\}.$$

Now, we show that

$$\{\|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\|\} \rightarrow 0, \text{ as } \alpha \rightarrow 0.$$

We have

$$u(0) = (I + TA)^{-1} e^{TA} g = \sum_{n=1}^{+\infty} \frac{e^{T\lambda_n}}{(1 + T\lambda_n)} \langle g, \phi_n \rangle \phi_n,$$

and

$$u_\alpha(0) = [\alpha A^r + (I + TA) e^{-TA}]^{-1} g = \sum_{n=1}^{+\infty} \frac{1}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \langle g, \phi_n \rangle \phi_n,$$

then we get

$$\begin{aligned} \|u(0) - u_\alpha(0)\|^2 &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r e^{T\lambda_n}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}](1 + T\lambda_n)} \right)^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \right)^2 \left[\frac{e^{T\lambda_n}}{(1 + T\lambda_n)} \right]^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \right)^2 |u_n(0)|^2 \end{aligned}$$

$$= \sum_{n=1}^{+\infty} (\tilde{F}(\lambda_n))^2 |u_n(0)|^2,$$

where

$$\tilde{F}(\lambda_n) = \frac{\alpha \lambda_n^r}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}}.$$

We assume that $u(0) \in H$.

$$u(0) \in H \iff \sum_{n=1}^{+\infty} \left(\frac{e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2 = \sum_{n=1}^{+\infty} |u_n(0)|^2 = \|u(0)\|^2 < +\infty$$

For $\varepsilon > 0$, we choose $N > 0$ such that $\sum_{n=N}^{+\infty} \left(\frac{e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2 \leq \frac{\varepsilon^2}{2}$. Thus

$$\|u(0) - u_\alpha(0)\|^2 = \sum_{n=1}^N (\tilde{F}(\lambda_n))^2 |u_n(0)|^2 + \sum_{n=N}^{+\infty} (\tilde{F}(\lambda_n))^2 |u_n(0)|^2. \quad (29)$$

We observe that $\tilde{F}(\lambda_n) = \frac{\alpha \lambda_n^r}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \leq 1$, then we can write

$$\sum_{n=N}^{+\infty} [\tilde{F}(\lambda_n)]^2 |u_n(0)|^2 \leq \sum_{n=N}^{+\infty} |u_n(0)|^2 \leq \frac{\varepsilon^2}{2}.$$

The other quantity can be estimated as follows

$$\begin{aligned} \sum_{n=1}^N \left(\frac{\alpha \lambda_n^r e^{T\lambda_n}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}] (1 + T\lambda_n)} \right)^2 |g_n|^2 &= \sum_{n=1}^N \left[\frac{\alpha \lambda_n^r}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}]} \right]^2 |u_n(0)|^2 \\ &= \sum_{n=1}^N [\tilde{F}(\lambda_n)]^2 |u_n(0)|^2 \\ &\leq \left[\sup_{1 \leq n \leq N} \tilde{F}(\lambda_n) \right]^2 \sum_{n=1}^N |u_n(0)|^2. \end{aligned}$$

It is clear that

$$\tilde{F}(\lambda_n) = \frac{\alpha \lambda_n^r}{\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}} \leq \frac{\alpha \lambda_n^r}{(1 + T\lambda_n) e^{-T\lambda_n}} \leq \frac{\alpha \lambda_n^r e^{T\lambda_n}}{1 + T\lambda_n},$$

and $\lambda_n \leq \lambda_N$ implies that

$$\tilde{F}(\lambda_n) \leq \frac{\alpha \lambda_n^r}{(1 + T\lambda_n)} e^{T\lambda_n} \leq \alpha \lambda_N^r e^{T\lambda_N}.$$

It follows that

$$\sup_{1 \leq n \leq N} \tilde{F}(\lambda_n) \leq \alpha \lambda_N^r e^{T\lambda_N},$$

and consequently

$$\sum_{n=1}^N (\tilde{F}(\lambda_n))^2 |u_n(0)|^2 \leq [\alpha \lambda_N^r e^{T\lambda_N}]^2 \|u(0)\|^2.$$

If we choose the parameter α such that $\alpha \lambda_N^r e^{T\lambda_N} \|u(0)\| \leq \frac{\varepsilon}{\sqrt{2}}$, we obtain

$$\begin{aligned} \|u(0) - u_\alpha(0)\|^2 &\leq [\alpha \lambda_N^r e^{T\lambda_N}]^2 \|u(0)\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \end{aligned}$$

Which shows that

$$u_\alpha(0) \rightarrow u(0), \text{ as } \alpha \rightarrow 0.$$

To complete the proof, it remains to show that

$$\|A(u(0) - u_\alpha(0))\| \rightarrow 0, \text{ as } \alpha \rightarrow 0.$$

We compute

$$\begin{aligned}\|A(u(0) - u_\alpha(0))\|^2 &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^{r+1} e^{T\lambda_n}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}] (1 + T\lambda_n)} \right)^2 |g_n|^2 \\ &= \sum_{n=N}^{+\infty} [\tilde{F}(\lambda_n)]^2 |\lambda_n u_n(0)|^2.\end{aligned}$$

We have

$$u(0) \in D(A) \iff \|Au(0)\|^2 = \sum_{n=1}^{+\infty} \left(\frac{\lambda_n e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2 < +\infty.$$

For $\varepsilon > 0$, we choose $N > 0$ such that

$$\sum_{n=N}^{+\infty} |\lambda_n u_n(0)|^2 = \sum_{n=N}^{+\infty} \left(\frac{\lambda_n e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2 < \frac{\varepsilon^2}{2}.$$

Then, we can write

$$\sum_{n=N}^{+\infty} (\tilde{F}(\lambda_n))^2 |\lambda_n u_n(0)|^2 \leq \sum_{n=N}^{+\infty} |\lambda_n u_n(0)|^2 \leq \frac{\varepsilon^2}{2}.$$

and

$$\begin{aligned}\sum_{n=1}^N (\tilde{F}(\lambda_n))^2 |\lambda_n u_n(0)|^2 &\leq \sup_{1 \leq n \leq N} [\tilde{F}(\lambda_n)] \sum_{n=1}^N |\lambda_n u_n(0)|^2 \\ &\leq [\alpha \lambda_N^r e^{T\lambda_N}]^2 \|Au(0)\|^2.\end{aligned}$$

If we choose the parameter α such that $[\alpha \lambda_N^r e^{T\lambda_N}] \|Au(0)\| \leq \frac{\varepsilon}{\sqrt{2}}$, we get

$$\begin{aligned}\|A(u(0) - u_\alpha(0))\|^2 &\leq [\alpha \lambda_N^r e^{T\lambda_N}]^2 \|Au(0)\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.\end{aligned}$$

Which shows that

$$A(u(0) - u_\alpha(0)) \rightarrow 0, \text{ as } \alpha \rightarrow 0.$$

In conclusion,

$$\begin{aligned}\sup_{0 \leq t \leq T} \{\|u(t) - u_\alpha(t)\| + \|A(u(t) - u_\alpha(t))\|\} &\leq \\ \{\|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\|\} &\rightarrow 0, \alpha \rightarrow 0\end{aligned} \quad (30)$$

Theorem 3.5. If $u(0) \in \mathcal{D}(A^{(\theta+1)})$ such that $\|A^{(\theta+1)}u(0)\| \leq E_\theta$, and $1 \leq r \leq \theta$, then we have the following estimate

$$\begin{aligned}\sup_{0 \leq t \leq T} \{\|u(t) - u_\alpha(t)\| + \|A(u(t) - u_\alpha(t))\|\} &\leq \\ \{\|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\|\} &\leq \left[\frac{C_5}{\ln\left(\frac{C_4}{\alpha}\right)} \right]^\theta E_\theta\end{aligned} \quad (31)$$

Proof. We have

$$\begin{aligned}\|u(0) - u_\alpha(0)\|^2 &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r e^{T\lambda_n} \lambda_n^{-(\theta+1)} \lambda_n^{(\theta+1)}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}] (1 + T\lambda_n)} \right)^2 |g_n|^2 \\ &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r \lambda_n^{-(\theta+1)}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}]} \right)^2 \left(\frac{\lambda_n^{(\theta+1)} e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^{-(\theta+1)}}{[\alpha + (1 + T\lambda_n) \lambda_n^{-r} e^{-T\lambda_n}]} \right)^2 \left(\frac{\lambda_n^{(\theta+1)} e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 |g_n|^2 \\
&= \sum_{n=1}^{+\infty} (\alpha \widehat{G}_\alpha(\lambda_n))^2 \left| \lambda_n^{(\theta+1)} u_n(0) \right|^2,
\end{aligned}$$

where

$$\widehat{G}_\alpha(\lambda_n) = \frac{\lambda_n^{-(\theta+1)}}{[\alpha + (1 + T\lambda_n) \lambda_n^{-r} e^{-T\lambda_n}]} = \frac{1}{\lambda_n [\alpha \lambda_n^\theta + (1 + T\lambda_n) \lambda_n^{\theta-r} e^{-T\lambda_n}]}.$$

If $r \leq \theta$, then

$$\begin{aligned}
\widehat{G}_\alpha(\lambda_n) &\leq \frac{1}{\lambda_1 [\alpha \lambda_n^\theta + (1 + T\lambda_n) \lambda_1^{\theta-r} e^{-T\lambda_n}]} \\
&= \frac{1}{\lambda_1^{\theta+1-r} \left[\frac{\alpha}{\lambda_1^{\theta-r}} \lambda_n^\theta + (1 + T\lambda_n) e^{-T\lambda_n} \right]} = \frac{1}{\lambda_1^{\theta+1-r}} \widetilde{H}_\beta(\lambda_n),
\end{aligned}$$

where $\beta = \frac{\alpha}{\lambda_1^{\theta-r}}$. Now by (23), we conclude that

$$\begin{aligned}
\sup_{n \geq 1} \frac{1}{\lambda_1^{\theta+1-r} \left[\frac{\alpha}{\lambda_1^{\theta-r}} \lambda_n^\theta + (1 + T\lambda_n) e^{-T\lambda_n} \right]} &= \frac{1}{\lambda_1^{\theta+1-r}} \sup_{\lambda \geq \lambda_1} \widetilde{H}_\beta(\lambda) = \\
&\begin{cases} \frac{\lambda_1^{-\theta} \left[\lambda_1^{1-\frac{1}{\theta}} \right]^r}{\alpha^{\frac{1}{\theta}} \ln \left(\frac{\lambda_1^{\theta-r}}{\alpha} \right)}, & 0 < \theta < 1, \\ \frac{\lambda_1^{-1}}{\alpha \left[\ln \left(\frac{\lambda_1^{\theta-r}}{\alpha} \right) \right]^\theta}, & \theta \geq 1. \end{cases} \quad (32)
\end{aligned}$$

If $\theta \geq 1$, we can write

$$\|u(0) - u_\alpha(0)\|^2 \leq \sum_{n=1}^{+\infty} (\alpha \widehat{G}_\alpha(\lambda_n))^2 \left| \lambda_n^{(\theta+1)} u_n(0) \right|^2 \quad (33)$$

$$\leq \left\{ \frac{\lambda_1^{-1}}{\left[\ln \left(\frac{\lambda_1^{\theta-r}}{\alpha} \right) \right]^\theta} \right\}^2 \sum_{n=1}^{+\infty} \left| \lambda_n^{\theta+1} u_n(0) \right|^2 \quad (34)$$

$$\leq \left\{ \frac{\lambda_1^{-1}}{\left[\ln \left(\frac{\lambda_1^{\theta-r}}{\alpha} \right) \right]^\theta} \right\}^2 \|A^{(\theta+1)} u(0)\|^2, \quad (35)$$

and

$$\begin{aligned}
\|A(u(0) - u_\alpha(0))\|^2 &= \sum_{n=1}^{+\infty} \left(\frac{\alpha \lambda_n^r \lambda_n^{-\theta}}{[\alpha \lambda_n^r + (1 + T\lambda_n) e^{-T\lambda_n}]} \right)^2 \left(\frac{\lambda_n^{\theta+1} e^{T\lambda_n}}{(1 + T\lambda_n)} \right)^2 |g_n|^2 \\
&\leq \sum_{n=1}^{+\infty} \left(\frac{\alpha}{[\alpha \lambda_n^\theta + (1 + T\lambda_n) \lambda_n^{\theta-r} e^{-T\lambda_n}]} \right)^2 \left| \lambda_n^{\theta+1} u_n(0) \right|^2 \\
&\leq \sum_{n=1}^{+\infty} \left(\frac{\alpha}{\lambda_1^{\theta-r} \left[\frac{\alpha}{\lambda_1^{\theta-r}} \lambda_n^\theta + (1 + T\lambda_n) e^{-T\lambda_n} \right]} \right)^2 \left| \lambda_n^{\theta+1} u_n(0) \right|^2 \\
&\leq \alpha \left(\sup_{n \geq 1} \frac{1}{\lambda_1^{\theta-r} \left[\frac{\alpha}{\lambda_1^{\theta-r}} \lambda_n^\theta + (1 + T\lambda_n) e^{-T\lambda_n} \right]} \right)^2 \sum_{n=1}^{+\infty} \left| \lambda_n^{\theta+1} u_n(0) \right|^2.
\end{aligned}$$

By virtue of (23), we obtain

$$\|A(u(0) - u_\alpha(0))\|^2 \leq \frac{1}{\left[\ln\left(\frac{\lambda_1^{\theta-r}}{\alpha}\right)\right]^{2\theta}} \|A^{\theta+1}u(0)\|^2. \quad (36)$$

Combining (33) and (36), we obtain

$$\sup_{0 \leq t \leq T} \{\|u(t) - u_\alpha(t)\| + \|A(u(t) - u_\alpha(t))\|\} \leq \quad (37)$$

$$\begin{aligned} \{\|u(0) - u_\alpha(0)\| + \|A(u(0) - u_\alpha(0))\|\} &\leq \frac{1}{\left[\ln\left(\frac{\lambda_1^{\theta-r}}{\alpha}\right)\right]^\theta} (\lambda_1^{-1} + 1) \|A^{\theta+1}u(0)\| \\ &\leq \frac{C_5}{\left[\ln\left(\frac{C_4}{\alpha}\right)\right]^\theta} E_\theta, \end{aligned} \quad (38)$$

where $C_4 = \lambda_1^{\theta-r}$ and $C_5 = \lambda_1^{-1} + 1$. $\square$

We conclude this paper by constructing a family of regularizing operators for the problem 1.

Definition 3.6. A family $\{R_\alpha(t), \alpha > 0, t \in [0, T]\} \subset \mathcal{L}(H)$ is called a family of regularizing operators for the problem (1) if for each solution $u(t)$, $0 \leq t \leq T$ of (1) with final element g , and for any $\eta > 0$, there exists $\alpha(\eta) > 0$, such that

$$\alpha(\eta) \rightarrow 0, \quad \eta \rightarrow 0, \quad (39)$$

$$\|R_{\alpha(\eta)}(t)g_\eta - u(t)\| \rightarrow 0, \quad \eta \rightarrow 0, \quad (40)$$

for each $t \in [0, T]$ provided that g_η satisfies $\|g_\eta - g\| \leq \eta$.

Define $R_\alpha(t) = (I + tA) [\alpha A^r + (1 + tA) e^{-tA}]^{-1} e^{-tA}$. It is clear that $R_\alpha(t) \in \mathcal{L}(H)$ (see (19)). In the following we will show that $R_\alpha(t)$ is a family of regularizing operators for the problem 1.

Theorem 3.7. Under the assumption $g \in \mathfrak{E}^1$, the condition (40) holds.

Proof. We have

$$\Delta_\alpha(t) = \|R_\alpha(t)g_\eta - u(t)\| \leq \|R_\alpha(t)(g_\eta - g)\| + \|R_\alpha(t)g - u(t)\| = \Delta_1(t) + \Delta_2(t),$$

where

$$\begin{aligned} \Delta_1(t) &= \|R_\alpha(t)(g_\eta - g)\| \leq \kappa(\alpha)\eta, \\ \Delta_2(t) &= \|R_\alpha(t)g - u(t)\|. \end{aligned}$$

We observe that

$$\Delta_1(t) \leq \begin{cases} \frac{\eta}{\alpha^{\frac{1}{r}} \ln(\frac{1}{\alpha})}, & 0 < r < 1, \\ \frac{\eta}{\alpha [\ln(\frac{1}{\alpha})]^r}, & r \geq 1. \end{cases}$$

Choose $\alpha = \eta^{\frac{r}{1-r}}$ if $0 < r < 1$, and $\alpha = \sqrt{\eta}$ if $r \geq 1$, it follows

$$\Delta_1(t) \leq \begin{cases} \frac{\sqrt{\eta}}{\ln(\frac{1}{\eta^{1/r}})}, & 0 < r < 1, \\ \frac{\sqrt{\eta}}{[\ln(\frac{1}{\sqrt{\eta}})]^r}, & r \geq 1, \end{cases} \rightarrow 0, \quad \text{as } \eta \rightarrow 0. \quad (41)$$

Now, by Theorem 3.3 we have

$$\Delta_2(t) = \|u_{\alpha(\eta)}(t) - u(t)\| \rightarrow 0, \quad \text{as } \eta \rightarrow 0, \quad (42)$$

uniformly in t . Combining (41) and (42) we obtain

$$\sup_{0 \leq t \leq T} \|R_\alpha(t)g_\eta - u(t)\| \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

This shows that $R_\alpha(t)$ is a family of regularizing operators for the problem 1. $\square$

4 Numerical results

In this section we give a two-dimensional numerical test to show the feasibility and efficiency of the proposed method. Numerical experiments were carried out using MATLAB.

We consider the following inverse problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right)^2 u(x, t) = 0, & x \in (0, \pi), t \in (0, 1), \\ u(0, t) = u(\pi, t) = 0, & t \in (0, 1), \\ u(x, 1) = g(x), u_t(x, 0) = 0, & x \in [0, \pi], \end{cases} \quad (43)$$

where $f(x) = u(x, 0)$ is the unknown initial condition and $u(x, 1) = g(x)$ is the final condition.

It is well known that the operator

$$A = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A) = H_0^1(0, \pi) \cap H^2(0, \pi) \subset H = L^2(0, \pi),$$

is positive, self-adjoint with compact resolvent (A is diagonalizable).

The eigenpairs (λ_n, ϕ_n) of A are

$$\lambda_n = n^2, \quad \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \quad n \in \mathbb{N}^*.$$

In this case, the formula (7) takes the form

$$f(x) = u(x, 0) = \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{1}{1+n^2} e^{n^2} \left(\int_0^\pi g(x) \sin(nx) dx \right) \sin(nx). \quad (44)$$

In the following, we consider an example which has an exact expression of solutions $(u(x, t), f(x))$.

Example. If $u(x, 0) = \phi_1(x) = \sqrt{\frac{2}{\pi}} \sin(x)$, then the function

$$u(x, t) = \sum_{n=1}^{\infty} (1 + t\lambda_n) e^{-t\lambda_n} \langle \phi_1, \phi_n \rangle \phi_n(x) = (1 + t\lambda_1) e^{-t\lambda_1} \phi_1(x) = \sqrt{\frac{2}{\pi}} (1 + t\lambda_1) e^{-t\lambda_1} \sin(x)$$

is the exact solution of the problem (43). Consequently, the data function is $g(x) = u(x, 1) = \sqrt{\frac{2}{\pi}} \frac{2}{e} \sin(x)$.

By using the central difference with step length $h = \frac{\pi}{N+1}$ to approximate the first derivative u_x and the second derivative u_{xx} , we can get the following semi-discret problem (ordinary differential equation):

$$\begin{cases} \left(\frac{d}{dt} - \mathbb{A}_h \right)^2 u(x_i, t) = 0, & x_i = ih, i = 1, \dots, N, t \in (0, 1), \\ u(x_0, t) = u(x_{N+1}, t) = 0, & t \in (0, 1), \\ u(x_i, 0) = g(x_i), u_t(x_i, 0) = 0, & x_i = ih, i = 1, \dots, N, \end{cases} \quad (45)$$

where $\mathbb{A}_h$ is the discretisation matrix stemming from the operator $A = -\frac{d^2}{dx^2}$:

$$\mathbb{A}_h = \frac{1}{h^2} \text{Tridiag}(-1, 2, -1) \in \mathcal{M}_N(\mathbb{R})$$

is a symmetric, positive definite matrix. We assume that it is fine enough so that the discretization errors are small compared to the uncertainty δ of the data; this means that $\mathbb{A}_h$ is a good approximation of the differential operator $A = -\frac{d^2}{dx^2}$, whose unboundedness is reflected in a large norm of $\mathbb{A}_h$. The eigenpairs (μ_k, e_k) of $\mathbb{A}_h$ are given by

$$\mu_k = 4 \left(\frac{N+1}{\pi} \right)^2 \sin^2 \left(\frac{k\pi}{2(N+1)} \right), \quad e_k = \left(\sin \left(\frac{jk\pi}{N+1} \right) \right)_{j=1}^N, \quad k = 1 \dots N.$$

Adding a random distributed perturbation (obtained by the Matlab command `randn`) to each data function, we obtain the vector g^δ :

$$g^\delta = g + \varepsilon \text{randn}(\text{size}(g)),$$

where ε indicates the noise level of the measurement data and the function "randn(.)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "randn(size(g))" returns an array of random entries that is the same size as g . The bound on the measurement error δ can be measured in the sense of Root Mean Square Error (**RMSE**) according to

$$\delta = \|g^\delta - g\|_* = \left(\frac{1}{N} \sum_{i=1}^N (g(x_i) - g^\delta(x_i))^2 \right)^{1/2}.$$

The discret approximation of (18) takes the form

$$u_\alpha^\delta(x_j, 0) = f^{\alpha, \delta}(x_j) = (\alpha \mathbb{A}_h^r + (\mathbb{I}_N + \mathbb{A}_h) e^{-\mathbb{A}_h})^{-1} g^\delta(x_j), \quad j = 1 \dots N, \quad (46)$$

where $\mathbb{I}_N$ is the identity matrix.

In our numerical computations we always take $N = 40$ and consider only the cases when $\varepsilon = 0.001, 0.01$. The regularization parameter (α, r) is chosen in the following way: for any fixed $r \in \{0, 1, 2, 3\}$, we try to find a satisfactory error by varying the second parameter $\alpha = \varepsilon^s$ with step length $s = 0.1$. We note α_0 one of the best choice which gives this result. Now, for $\alpha = \alpha_0$ fixed, we try to find an acceptable error by varying the first parameter $r = 0, 1, 2, 3$ in order to obtain the best possible convergence rate. It is important to note that this choice is of heuristic nature and the multiparameter discrepancy principle is quite scarce in the literature.

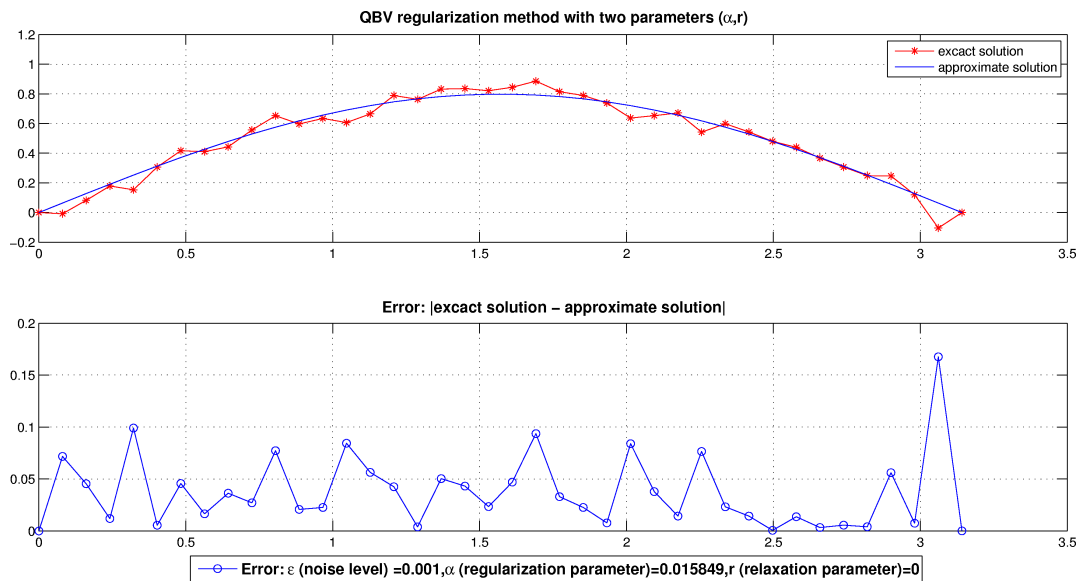
The relative error $RE(f)$ is given by

$$RE(f) = \frac{\|f^{\alpha, \delta} - f\|_*}{\|f\|_*}.$$

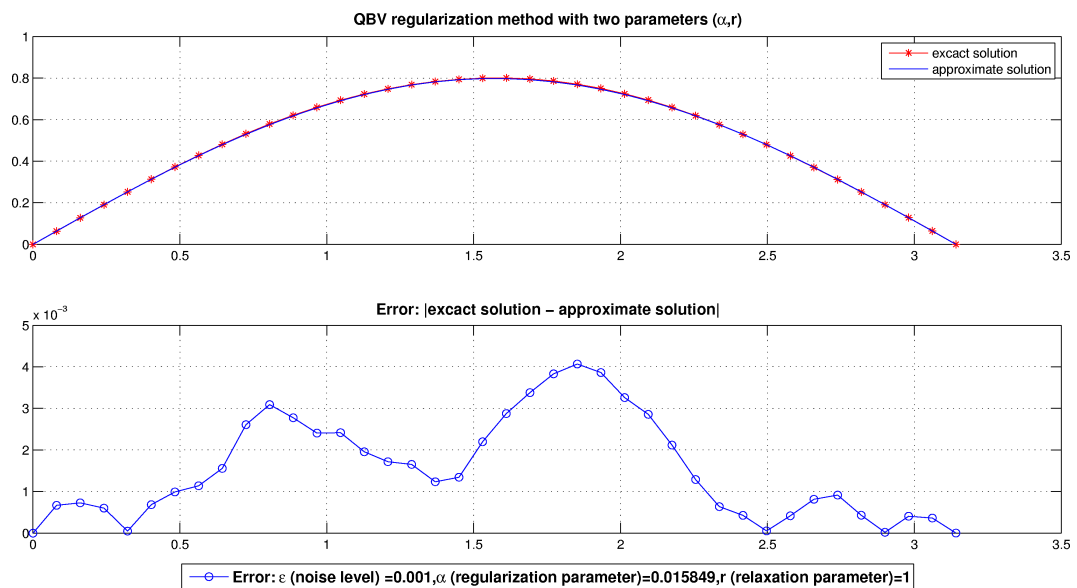
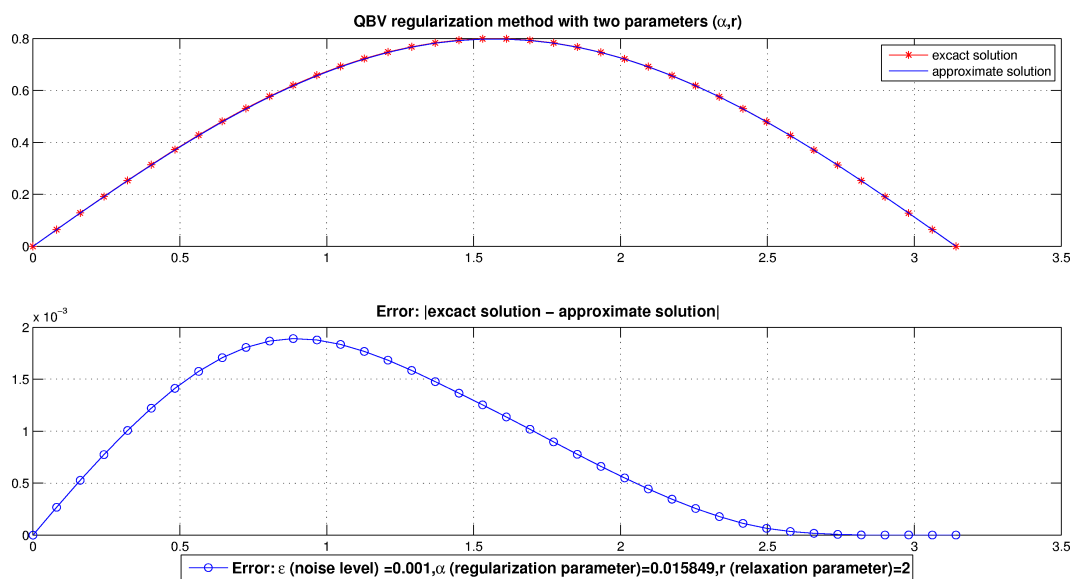
Conclusion and discussion

Numerical results are shown in Figures 1-8 and Tables 1-2.

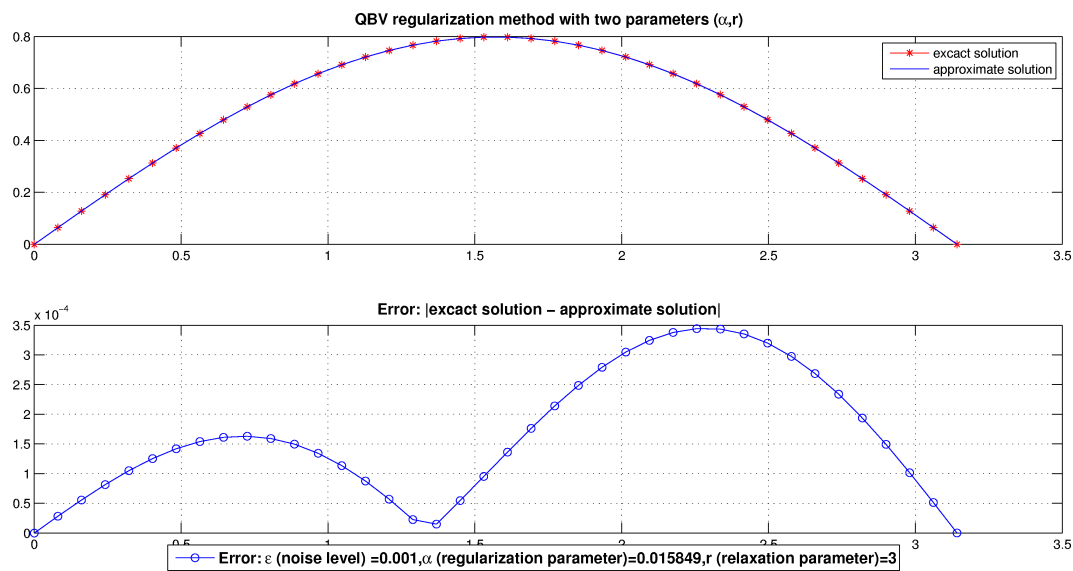
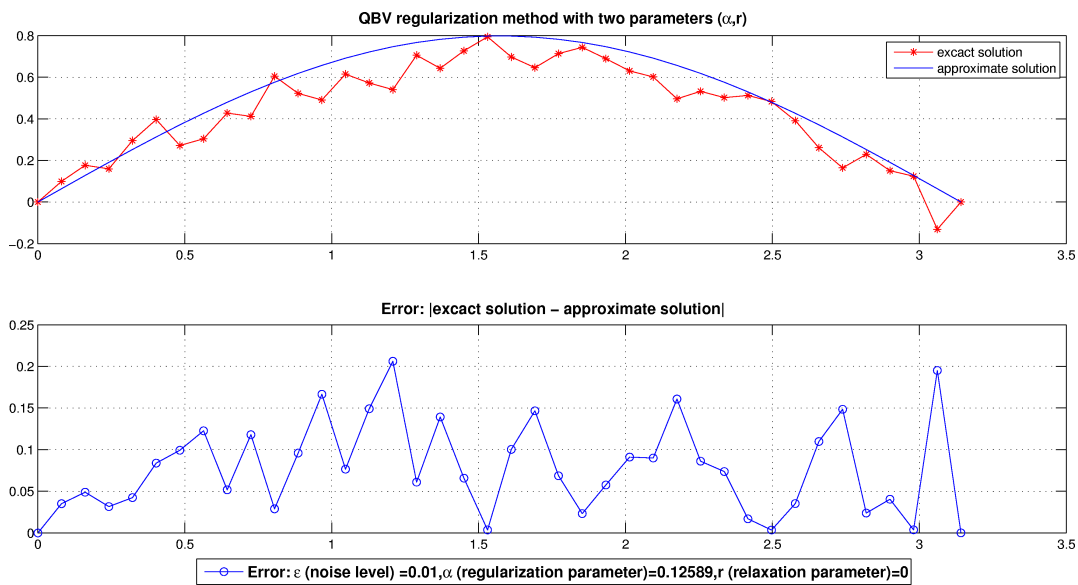
Fig. 1. ε (noise level) = 0.001, α (regularization parameter) = 0.015849, r (relaxation parameter) = 0



In this paper, we have proposed an improved two-parameter regularization method (MQBVM) to solve an ill-posed biparabolic problem. The convergence and stability estimates have been obtained under a priori bound assumptions for the exact solution. Finally, some numerical tests show that our proposed regularization method is effective and stable.

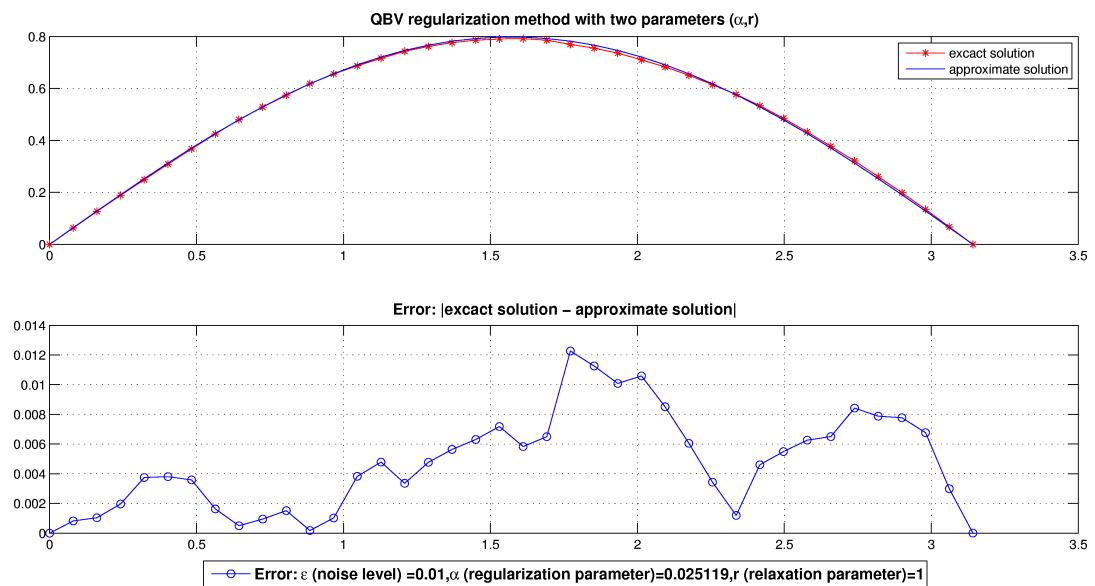
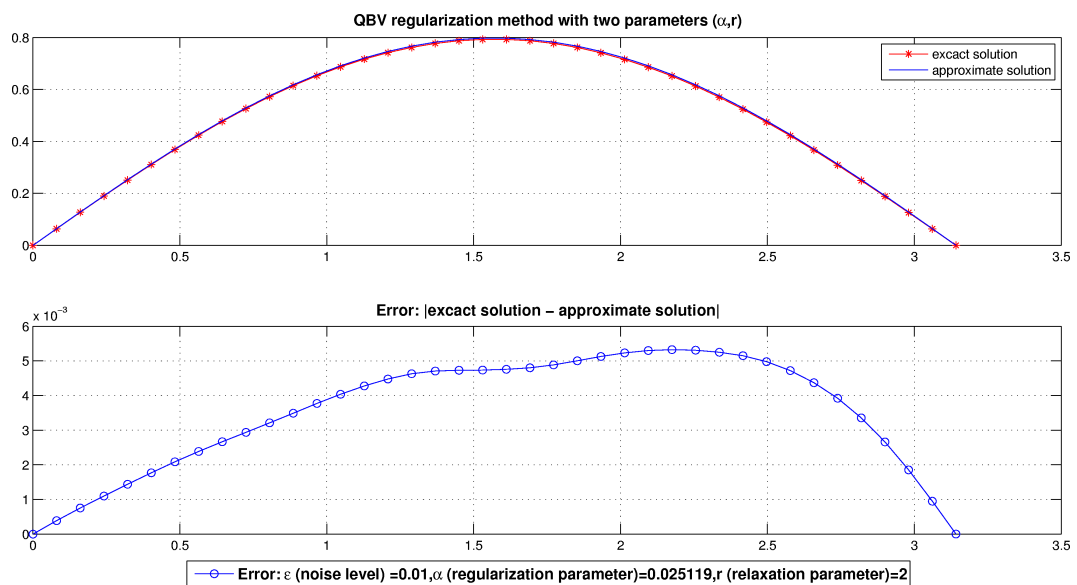
Fig. 2. ε (noise level) = 0.001, α (regularization parameter) = 0.015849, r (relaxation parameter) = 1**Fig. 3.** ε (noise level) = 0.001, α (regularization parameter) = 0.015849, r (relaxation parameter) = 2**Table 1.** The absolute errors E_{ra} for fixed α and for various value of r

N	ε	α	r	RE
40	0.001	0.015849	0	0.0920
40	0.001	0.015849	1	0.0035
40	0.001	0.015849	2	0.0019
40	0.001	0.015849	3	0.0003501

Fig. 4. ε (noise level) = 0.001, α (regularization parameter) = 0.015849, r (relaxation parameter) = 3Fig. 5. ε (noise level) = 0.01, α (regularization parameter) = 0.025119, r (relaxation parameter) = 0Table 2. The absolute errors E_{ra} for fixed α and for various value of r

N	ε	α	r	RE
40	0.01	0.025119	0	0.1707
40	0.01	0.025119	1	0.0103
40	0.01	0.025119	2	0.0070
40	0.01	0.025119	3	0.0061

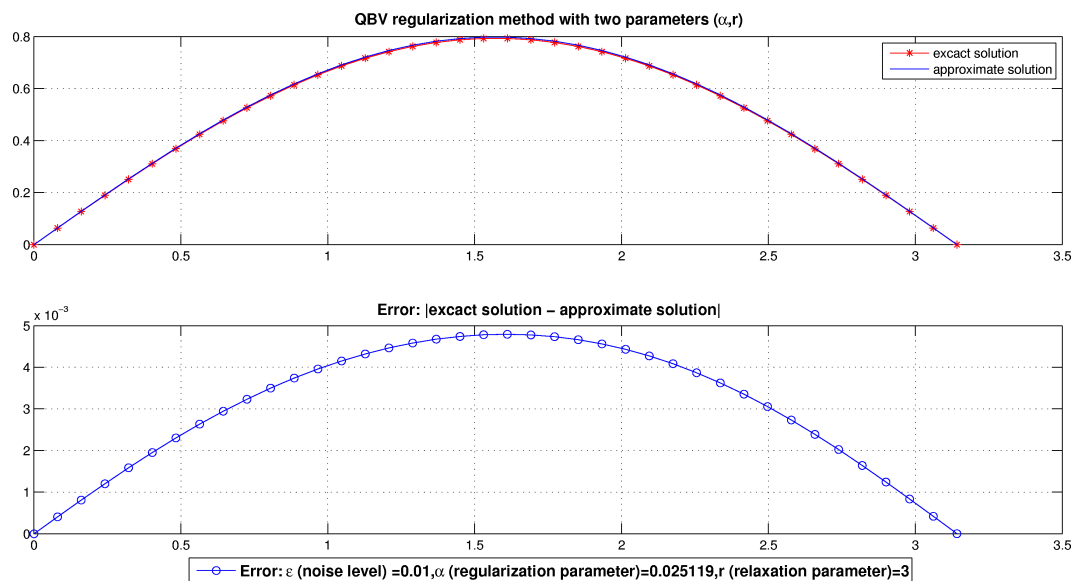
According to the numerical tests, we observe the following regularizing effect:

Fig. 6. ε (noise level) = 0.01, α (regularization parameter) = 0.025119, r (relaxation parameter) = 1**Fig. 7.** ε (noise level) = 0.01, α (regularization parameter) = 0.025119, r (relaxation parameter) = 2

• In the case $r = 0$, $\varepsilon = 0.001$ and $\alpha = 0.015849$ (resp. $r = 0$, $\varepsilon = 0.01$ and $\alpha = 0.025119$), the approximate solution is far from the exact solution. But for the case $r = 1, 2, 3$, we observe that the solution becomes precise and very near to the exact solution (in particular for $r = 2, 3$). This shows that our approach has a nice regularizing effect and gives a better approximation with comparison to the classical QBV-method.

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Fig. 8. ε (noise level) = 0.01, α (regularization parameter) = 0.025119, r (relaxation parameter) = 3



References

- [1] Atakhadzaev M.A., Egamberdiev O.M., The Cauchy problem for the abstract bicaloric equation, *Sibirskii Matematicheskii Zhurnal*, 1990, 31(4), 187-191.
- [2] Lakhdari A., Boussetila N., An iterative regularization method for an abstract ill-posed biparabolic problem, *Boundary Value Problems*, 2015, 55, 1-17.
- [3] Egamberdiev O.M., An ill-posed problem for an abstract bicaloric equation, *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk.*, 1988, 3, 42-46 (in Russian).
- [4] Orynbasarov M.O., A nonlocal boundary value problem of Samarskii type for a biparabolic equation, *Izv. Nats. Akad. Nauk Resp. Kaz. Ser. Fiz.-Mat.*, 2008, 5, 8-13 (in Russian).
- [5] Egamberdiev O.M., A continuation problem for an abstract bicaloric equation, *Uzbek. Mat. Zh.*, 2011, 4, 211-217 (in Russian).
- [6] Zouyed F., Djemoui S., An Iterative Regularization Method for Identifying the Source Term in a Second Order Differential Equation, *Mathematical Problems in Engineering*, 2015, Article ID 713403, 1-9.
- [7] Bulavatsky V. M., Biparabolic mathematical model of the filtration consolidation problem, *Dopov. NAN Ukrainy*, 1997, 8, 13-17.
- [8] Bulavatsky V.M., Mathematical modeling of filtrational consolidation of soil under motion of saline solutions on the basis of biparabolic model, *J. Autom. Inform. Sci.*, 2003, 35(8), 13-22.
- [9] Bulavatsky V.M., Skopetsky V.V., Generalized mathematical model of the dynamics of consolidation processes with relaxation, *Cybern. Syst. Analysis*, 2008, 44(5), 646-654.
- [10] Fichera G., Is The Fourier Theory of Heat Propagation Paradoxical?, *Rendicont 1 Del Circolo Matematico Di Palermo*, 1992, Serie II, Tomo XLI, 5-28.
- [11] Joseph L., Preziosi D.D., Heat Waves, *Rev.Mod. Physics*, 1989, 41-73.
- [12] Fushchich V.L., Galitsyn A.S., Polubinskii A.S., A new mathematical model of heat conduction processes, *Ukrainian Math. J.*, 1990, 42, 210-216.
- [13] Bulavatsky V.M., Fractional differential analog of biparabolic evolution equation and some its applications, *Cybernetics and Systems Analysis*, 2016, 52(5), 337-347.
- [14] Radu P., Todorova G., Yordanov B., Diffusion phenomenon in Hilbert spaces and applications, *J. Differential Equations*, 2001, 250, 4200-4218.
- [15] Ekoue F., D'Halloy A.F., Gigon D., Plantamp G., Zajdman E., Maxwell-Cattaneo Regularization of Heat Equation, *World Academy of Science, Engineering and Technology*, 2013, 7, 05-23.
- [16] Auriault J.-L., The paradox of fourier heat equation: A theoretical refutation, *International Journal of Engineering Science*, 2017, 118, 82-88.

- [17] Eidelman S.D., Ivasyshen S.D., Kochubei A.N., *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, Springer Basel AG 2004.
- [18] Carasso A.S., Bochner Subordination, Logarithmic Diffusion Equations, and Blind Deconvolution of Hubble Space Telescope Imagery and Other Scientific Data, *SIAM J. Imaging Sciences*, 2010, 3(4), 954-980.
- [19] Payne L.E., On a proposed model for heat conduction, *IMA Journal of Applied Mathematics*, 2006, 71, 590-599.
- [20] Saporaa A., Codegonea M., Barbero G., Diffusion phenomenon in the hyperbolic and parabolic regimes, *Physics Letters A*, 2013, 377, 2416-2421.
- [21] Wang L., Zhou X., Wei X., *Heat Conduction: Mathematical Models and Analytical Solutions*, Springer-Verlag 2008.
- [22] Boussetila N., Rebbani F., Optimal regularization method for ill-posed Cauchy problems, *Electron. J. Differential Equations*, 2006, 147, 1-15.
- [23] Boussetila N., Hamida S., Rebbani F., Spectral regularization methods for an abstract ill-posed elliptic problem, *Abstract and Applied Analysis*, 2013, Article ID 947379, 1-11.
- [24] Boussetila N., Rebbani F., A modified quasi-reversibility method for a class of ill-posed Cauchy problems, *Georgian Mathematical Journal*, 2007, 14(4), 627-642.
- [25] Benrabah A., Boussetila N., Rebbani F., Regularization method for an ill-posed Cauchy problem for elliptic equations, *Journal of Inverse and Ill-Posed Problems*, 2017, 25(3), 01-19.
- [26] Hoa N.V., Khanh T.Q., Two-parameter regularization method for an axisymmetric inverse heat problem, *Boundary Value Problems*, 2017, 25, 1-12.
- [27] Chen Z., Lu Y., Xu Y., Yang H., Multi-parameter Tikhonov regularization for linear ill-posed operator equations, *J. Comput. Math.*, 2008, 26, 37-55.
- [28] Lu S., Pereverzev S.V., Shao Y., Tautenhahn U., Discrepancy curves for multi-parameter regularization, *J. Inverse Ill-Posed Prob.*, 2010, 18, 655-676.
- [29] Lu S., Pereverzev S.V., Multi-parameter regularization and its numerical realization, *Numer. Math.*, 2011, 118, 1-31.
- [30] Showalter R.E., The final value problem for evolution equations, *J. Math. Anal. Appl.*, 1974, 47, 563-572.
- [31] Showalter R.E., Quasi-reversibility of first and second order parabolic evolution equations, In: *Improperly Posed Boundary Value Problems*. Pitman, London, 1975, 76-84.
- [32] Abdulkrimov L.S., Regularization of an ill-posed Cauchy problem for evolution equations in a Banach space, *Azerbaij. Gos. Univ. Ucen. Zap. Fiz. Mat.*, 1977, 1, 32-36 (in Russian).
- [33] Ames K.A., Payne L.E., Asymptotic for two regularizations of the Cauchy problem for the backward heat equation, *Math. Models Methods Appl. Sci.*, 1998, 8, 187-202.
- [34] Feng X.L., Eldén L., Fu C.L., A quasi-boundary-value method for the Cauchy problem for elliptic equations with nonhomogeneous Neumann data, *J. Inv. Ill-Posed Problems*, 2010, 18, 617-645.
- [35] Hào D.N., Duc N.V., Lesnic D., A non-local boundary value problem method for the Cauchy problem for elliptic equations, *Inverse Problems*, 2009, 25, 1-27.
- [36] Hào D.N., Duc N.V., Lesnic D., Regularization of parabolic equations backward in time by a non-local boundary value problem method, *IMA J. Appl. Math.*, 2010, 75, 291-315.
- [37] Hào D.N., Nguyen V.D., Sahli H., A non-local boundary value problem method for parabolic equations backward in time, *J. Math. Anal. Appl.*, 2008, 345, 805-815.
- [38] Melnikova I.V., Regularization of ill-posed differential problems, *Siberian Math. J.*, 1992, 33, 289-298.
- [39] Melnikova I.V., Freiberg A.YU., Regularization of a boundary value problem for an oscillation equation, *U.S.S.R. Comput. Maths. Matb. Phys.*, 1985, 25(3), 90-94.
- [40] Quan P.H., Trong D.D., Triet Le.M., Tuan N.H., A modified quasi-boundary value method for regularizing of a backward problem with time-dependent coefficient, *Inverse Problems in Science and Engineering*, 2011, 19(3), 409-423.
- [41] Samarskii A.A., Vabishchevich P.N., *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, Inverse and Ill-Posed Problems Series, Walter de Gruyter, Berlin. New York, 2007.
- [42] Zhang H.W., Modified quasi-boundary value method for cauchy problems of elliptic equations with variable coefficients, *Electronic Journal of Differential Equations*, 2011, 106, 1-10.
- [43] Zhang H.W., Wei T., An improved non-local boundary value problem method for a cauchy problem of the Laplace equation, *Numer. Algor.*, 2012, 59, 249-269.
- [44] Zhang H. W., Modified quasi-boundary value method for a Cauchy problem of semi-linear elliptic equation, *International Journal of Computer Mathematics*, 2012, 89(12), 1689-1703.