DE GRUYTER OPEN Open Math. 2017; 15: 1623–1634

Open Mathematics

Research Article

Ronghui Liu and Jiang Zhou*

Weighted multilinear p-adic Hardy operators and commutators

https://doi.org/10.1515/math-2017-0139
Received June 26, 2017; accepted November 24, 2017.

Abstract: In this paper, the weighted multilinear p-adic Hardy operators are introduced, and their sharp bounds are obtained on the product of p-adic Lebesgue spaces, and the product of p-adic central Morrey spaces, the product of p-adic Morrey spaces, respectively. Moreover, we establish the boundedness of commutators of the weighted multilinear p-adic Hardy operators on the product of p-adic central Morrey spaces. However, it's worth mentioning that these results are different from that on Euclidean spaces due to the special structure of the p-adic fields.

Keywords: p-adic fields, Hardy operators, Weighted multilinear operators, Sharp bounds, Central Morrey spaces, Commutators

MSC: 42B25, 42B35, 46B25

1 Introduction

In recent years, p-adic analysis has gathered a lot of attention by its applications in many aspects of mathematical physics, such as quantum mechanics, the probability theory and the dynamical systems [1,2]. On the other hand, it plays a crucial role in pseudo-differential equations, wavelet theory and harmonic analysis, etc. (see [3-7,10]).

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}_p$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if any non-zero rational number x is represented as $x = p^{\gamma}\frac{m}{n}$, where γ is an integer and the integers m, n are indivisible by p, then $|x|_p = p^{-\gamma}$. It's not hard to see that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Moreover, if $|x|_p \neq |y|_p$, then $|x + y|_p = \max\{|x|_p, |y|_p\}$. It is well known that $\mathbb{Q}_p$ is a typical model of non-Archimedean local fields. From the standard p-adic analysis, we know that any non-zero element x of $\mathbb{Q}_p$ can be uniquely represented as a canonical form $x = p^{\gamma}(x_0 + x_1 p + x_2 p + \cdots)$, where $x_i \in \{0, 1, \dots, p-1\}$ and $x_0 \neq 0$, we then have $|x|_p = p^{-\gamma}$. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ be the class of all p-adic integrals in $\mathbb{Q}_p$ and denote $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash \{0\}$.

The space $\mathbb{Q}_p^n$ consists of elements $x = (x_1, x_2, \dots, x_n)$, where $x_i \in \mathbb{Q}_p$, $i = 1, 2, \dots, n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1\leq i\leq n}\{|x_i|_p\}, \quad x \in \mathbb{Q}_p^n.$$

Ronghui Liu: College of Mathematics and System Sciences, Xinjiang University, 830046, Urumqi, China, E-mail: liuronghuimath@126.com
***Corresponding Author: Jiang Zhou:** College of Mathematics and System Sciences, Xinjiang University, 830046, Urumqi, China, E-mail: zhoujiang@xju.edu.cn

Open Access. © 2017 Liu and Zhou, published by De Gruyter Open. This work is licensed under the Creative Commons Attribution-NonCommercial-NoDerivs 4.0 License.

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\}$, the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, and by $S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$, and we set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group with respect to addition, it follows from the standard analysis that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to a positive constant factor and is translation invariant, i.e., $d(x+a) = dx$. We normalize the measure dx such that

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|B|_H$ denotes the Haar measure of a measure subset B of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)|_H = p^{\gamma n}, \quad |S_\gamma(a)|_H = p^{\gamma n}(1-p^{-n}).$$

The classical Hardy operator $\mathcal{H}$ is defined by

$$\mathcal{H}f(x) := \frac{1}{x}\int_0^x f(t)dt, \quad x > 0,$$

where the function f is a nonnegative integrable function on $\mathbb{R}^+$. A celebrated integral inequality, due to Hardy [8], states that

$$\|\mathcal{H}f\|_{L^q(\mathbb{R}^+)} \le \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)},$$

holds for $1 < q < \infty$, and the constant factor $\frac{q}{q-1}$ is the best value and it is the norm of the operator $\mathcal{H}$, that is,

$$\|\mathcal{H}\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} = \frac{q}{q-1}.$$

N-dimensional Hardy operator was introduced by Christ and Grafakos in [9] as follows:

$$\mathcal{H}f(x) := \frac{1}{\Omega_n|x|^n}\int_{|t|\le|x|} f(t)dt, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. The norm of $\mathcal{H}$ on $L^q(\mathbb{R}^n)$ was evaluated and found to be equal to that of the classical Hardy operator.

In 2012, Fu et al. [10] defined the n-dimensional p-adic Hardy operator as follows:

$$\mathcal{H}^p f(x) := \frac{1}{|B(0,|x|_p)|_H}\int_{|t|_p\le|x|_p} f(t)dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\},$$

where f is a nonnegative measurable function on $\mathbb{Q}_p^n$, $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$, and they proved the sharp estimate of the p-adic Hardy operator on Lebesgue spaces with power weights.

In 1984, Carton-Lebrun and Fosset [11] defined the weighted Hardy average operator $\mathcal{H}_\varphi$ by

$$\mathcal{H}_\varphi(f)(x) := \int_0^1 f(tx)\varphi(t)dt, \quad x \in \mathbb{R}^n,$$

where $\varphi : [0,1] \to [0,\infty)$ is a function, and showed the boundedness of $\mathcal{H}_\varphi$ on Lebesgue and $BMO(\mathbb{R}^n)$ spaces. Evidently the operator $\mathcal{H}_\varphi$ deeply depends on the nonnegative function φ. For example, when $n = 1$ and $\varphi(x) = 1$ for $x \in [0,1]$, the operator $\mathcal{H}_\varphi$ is just reduced to the classical Hardy operator.

In 2006, Rim and Lee [13] defined the weighted p-adic Hardy operator $\mathcal{H}_\varphi^p$ by

$$\mathcal{H}_\varphi^p(f)(x) := \int_{\mathbb{Z}_p^*} f(tx)\varphi(t)dt, \quad x \in \mathbb{Q}_p^n,$$

where φ is a nonnegative function defined on $\mathbb{Z}_p^*$, and gave the characterization of function φ for which $\mathcal{H}_\varphi^p$ is bounded on $L^q(\mathbb{Q}_p^n)$, $1 \le q \le \infty$, they also obtained the corresponding operator norm.

Morrey [12] introduced the $L^{q,\lambda}(\mathbb{R}^n)$ spaces to study the local behavior of solutions to second order elliptic partial differential equations. The p-adic Morrey space is defined as follows.

Definition 1.1 ([13]). *Let $1 \le q < \infty$ and $\lambda \ge -1/q$. The p-adic Morrey space $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by*

$$\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n) = \{f \in L^q_{loc}(\mathbb{Q}_p^n) : \|f\|_{\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{a\in\mathbb{Q}_p^n, \gamma\in\mathbb{Z}} \Big(\frac{1}{|B_\gamma(a)|^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q\Big)^{1/q} < \infty.$$

Remark 1.2. *It is clear that $\mathcal{L}^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, $\mathcal{L}^{q,0}(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.*

In 2017, Wu and Fu [14] proved sufficient and necessary conditions of weighted functions, for which the weighted p-adic Hardy operators are bounded on p-adic central Morrey spaces.

The p-adic central Morrey space is defined as follows.

Definition 1.3. *Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The p-adic central Morrey space $B^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by*

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \Big(\frac{1}{|B_\gamma|^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q\Big)^{1/q} < \infty,$$

where $B_\gamma = B_\gamma(0)$. It is clear that $B^{q,-\frac{1}{q}}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, when $\lambda < -1/q$, the space $B^{q,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$, therefore, we can only consider the case $\lambda \ge -1/q$. If $1 \le q_1 \le q_2 < \infty$, by Hölder's inequality

$$B^{q_2,\lambda}(\mathbb{Q}_p^n) \subset B^{q_1,\lambda_1}(\mathbb{Q}_p^n)$$

for $\lambda \in \mathbb{R}$.

Definition 1.4 ([10]). *Let $1 \le q < \infty$. A function $f \in L^q_{loc}(\mathbb{Q}_p^n)$ is said to be $CMO^q(\mathbb{Q}_p^n)$, if*

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \Big(\frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |f(x) - f_{B_\gamma(0)}|^q dx\Big)^{1/q},$$

where

$$f_{B_\gamma(0)} = \frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} f(x)dx.$$

The study of multilinear averaging operators is traced to the multilinear singular integral operator theory [15], and motivated not only the generalization of the theory of linear ones but also their natural appearance in analysis. For a more complete account on multilinear opeartors, we refer to [16-19] and the references therein.

In this paper, we consider the multilinear version of weighted p-adic Hardy operators in the p-adic fields. Firstly, we introduce the weighted multilinear p-adic Hardy operators as follows.

Definition 1.5. *Let $m \in \mathbb{N}$, $x \in \mathbb{Q}_p^n$, and φ be a nonnegative integrable function on $\mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^*$. The weighted multilinear p-adic Hardy operator $\mathcal{H}^p_{\varphi,m}$ is defined as*

$$\mathcal{H}^p_{\varphi,m}(\vec{f})(x) = \int_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^m f_i(t_i x)\varphi(\vec{t})d\vec{t},$$

where $\vec{f} := (f_1, \ldots, f_m)$, $\vec{t} := (t_1, \ldots, t_m)$, $d\vec{t} := dt_1 \cdots dt_m$, and f_i $(i = 1, \ldots, m)$ are measurable functions on $\mathbb{Q}_p^n$. When $m = 1$, $\mathcal{H}^p_{\varphi,m}$ is reduced to the weighted p-adic Hardy operators $\mathcal{H}^p_\varphi$.

The outline of the paper is as follows. In Section 2, we furnish sharp estimate of weighted multilinear p-adic Hardy operator on the product of p-adic Lebesgue spaces, and then the result is extended to the product of p-adic central Morrey spaces, the product of p-adic Morrey spaces, respectively. In Section 3, we present the boundedness of commutators of the weighted multilinear p-adic Hardy operators.

2 Sharp estimates of weighted multilinear p-adic Hardy operator

We begin with the following sharp boundedness of $\mathcal{H}^p_{\varphi,m}$ on the product of p-adic Lebesgue spaces.

Theorem 2.1. *Let $1 < q, q_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then $\mathcal{H}^p_{\varphi,m}$ is bounded from $L^{q_1}(\mathbb{Q}_p^n) \times L^{q_2}(\mathbb{Q}_p^n) \times \cdots \times L^{q_m}(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$ if and only if*

$$\mathcal{A}_m := \int\limits_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^m |t_i|_p^{-n/q_i} \varphi(\vec{t}) d\vec{t} < \infty. \tag{1}$$

Moreover,

$$\|\mathcal{H}^p_{\varphi,m}\|_{L^{q_1}(\mathbb{Q}_p^n)\times L^{q_2}(\mathbb{Q}_p^n)\times\cdots\times L^{q_m}(\mathbb{Q}_p^n)\to L^q(\mathbb{Q}_p^n)} = \mathcal{A}_m.$$

Proof. Without loss of generality, we consider only the situation when $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$.

Suppose that (1) holds. Using Minkowski's inequality yields

$$\begin{aligned}
\|\mathcal{H}^p_{\varphi,2}(f_1, f_2)\|_{L^q(\mathbb{Q}_p^n)} &= \Big(\int\limits_{\mathbb{Q}_p^n} \Big| \int\limits_{(\mathbb{Z}_p^*)^2} f_1(t_1 x) f_2(t_2 x) \varphi(t_1, t_2) dt_1 dt_2 \Big|^q dx \Big)^{1/q} \\
&\le \int\limits_{(\mathbb{Z}_p^*)^2} \Big(\int\limits_{\mathbb{Q}_p^n} |f_1(t_1 x) f_2(t_2 x)|^q dx \Big)^{1/q} \varphi(t_1, t_2) dt_1 dt_2.
\end{aligned}$$

By Hölder's inequality with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we see that

$$\begin{aligned}
\|\mathcal{H}^p_{\varphi,2}(f_1, f_2)\|_{L^q(\mathbb{Q}_p^n)} &\le \int\limits_{(\mathbb{Z}_p^*)^2} \prod_{i=1}^2 \Big(\int\limits_{\mathbb{Q}_p^n} |f_i(t_i x)|^{q_i} dx \Big)^{1/q_i} \varphi(t_1, t_2) dt_1 dt_2 \\
&\le \Big(\prod_{i=1}^2 \|f_i\|_{L^{q_i}(\mathbb{Q}_p^n)} \Big) \int\limits_{(\mathbb{Z}_p^*)^2} \Big(\prod_{i=1}^2 |t_i|_p^{-n/q_i} \Big) \varphi(t_1, t_2) dt_1 dt_2.
\end{aligned}$$

Thus, $\mathcal{H}^p_{\varphi,2}$ maps the product of p-adic Lebesgue spaces $L^{q_1}(\mathbb{Q}_p^n) \times L^{q_2}(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$ and

$$\|\mathcal{H}^p_{\varphi,2}\|_{L^{q_1}(\mathbb{Q}_p^n)\times L^{q_2}(\mathbb{Q}_p^n)\times\to L^q(\mathbb{Q}_p^n)} \le \mathcal{A}_2. \tag{2}$$

To see the necessity, for any $0 < \varepsilon < 1$ and $|\varepsilon|_p > 1$, we take

$$f_i^\varepsilon(x) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-\frac{n}{q_i} - \frac{q_2 \varepsilon}{q_i}}, & |x_i|_p \ge 1. \end{cases} \tag{3}$$

An elementary calculation gives that

$$\|f_1^\varepsilon\|^{q_1}_{L^{q_1}(\mathbb{Q}_p^n)} = \|f_2^\varepsilon\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-\varepsilon q_2}}.$$

Consequently, we have

$$\begin{aligned}
&\|\mathcal{H}^p_{\varphi,2}(f_1^\varepsilon, f_2^\varepsilon)\|_{L^q(\mathbb{Q}_p^n)} \\
&= \Big\{ \int\limits_{\mathbb{Q}_p^n} |x|_p^{-n-q_2\varepsilon} \Big(\int\limits_{\frac{1}{|x|_p} \le |t_1|_p < 1} \int\limits_{\frac{1}{|x|_p} \le |t_2|_p < 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2 \varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \Big)^q dx \Big\}^{1/q} \\
&\ge \Big\{ \int\limits_{|x|_p \ge 1} |x|_p^{-n-q_2\varepsilon} \Big(\int\limits_{\frac{1}{|x|_p} \le |t_1|_p < 1} \int\limits_{\frac{1}{|x|_p} \le |t_2|_p < 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2 \varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \Big)^q dx \Big\}^{1/q}
\end{aligned}$$

$$\geq \Big\{ \int\limits_{|x|_p \geq |\varepsilon|_p} |x|_p^{-n-q_2\varepsilon} \Big(\int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_1|_p < 1} \int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_2|_p < 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2\varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \Big)^q dx \Big\}^{1/q}$$

$$= \Big(\int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_1|_p < 1} \int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_2|_p < 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2\varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \Big) \Big(\int\limits_{|x|_p \geq |\varepsilon|_p} |x|_p^{-n-q_2\varepsilon} dx \Big)^{1/q}$$

$$= \Big(\int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_1|_p < 1} \int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_2|_p < 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2\varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \Big) |\varepsilon|_p^{-\varepsilon q_2} \prod_{i=1}^{2} \|f_i^\varepsilon\|_{L^{q_i}(\mathbb{Q}_p^n)}.$$

Therefore,

$$\int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_1|_p \leq 1} \int\limits_{\frac{1}{|\varepsilon|_p} \leq |t_2|_p \leq 1} |t_1|_p^{-\frac{n}{q_1} - \frac{q_2\varepsilon}{q_1}} |t_2|_p^{-\frac{n}{q_1} - \varepsilon} \varphi(t_1, t_2) dt_1 dt_2 \leq \frac{C}{|\varepsilon|_p^{\varepsilon q_2}}.$$

Now take $\varepsilon = p^{-k}$, $k = 1, 2, \cdots$. Then $|\varepsilon|_p = p^k > 1$. Letting k approach to ∞, then ε approaches to 0 and $|\varepsilon|_p^{\varepsilon q_2} = p^{\frac{kq_2}{p^k}}$ approaches to 1. Then by Fatou's Lemma, we obtain

$$\int\limits_{\mathbb{Z}_p^*} \int\limits_{\mathbb{Z}_p^*} |t_1|_p^{-n_1/q_1} |t_2|_p^{-n_2/q_2} \varphi(t_1, t_2) dt_1 dt_2 < \infty.$$

and

$$\|\mathcal{H}_{\varphi,2}^p\|_{L^{q_1}(\mathbb{Q}_p^n) \times L^{q_2}(\mathbb{Q}_p^n) \times \to L^q(\mathbb{Q}_p^n)} \geq \mathcal{A}_2. \tag{4}$$

Combining (2) and (4) then finishes the proof. □

Next, we extend the result in Theorem 2.1 to the product of p-adic central Morrey spaces.

Theorem 2.2. *Let $1 < q < q_i < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, $\lambda = \lambda_1 + \cdots + \lambda_m$ and $-1/q_i \leq \lambda_i < 0$, $i = 1, \ldots, m$.*
(i) If

$$\widetilde{\mathcal{A}}_m := \int\limits_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^{m} |t_i|_p^{n\lambda_i} \varphi(\vec{t}) d\vec{t} < \infty. \tag{5}$$

Then, $\mathcal{H}_{\varphi,m}^p$ is bounded from $B^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times B^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times B^{q_m,\lambda_m}(\mathbb{Q}_p^n)$ to $B^{q,\lambda}(\mathbb{Q}_p^n)$ with its operator norm not more that $\widetilde{\mathcal{A}}_m$.

(ii) Assume that $\lambda_1 q_1 = \cdots = \lambda_m q_m$. In the case the condition (5) is also necessary for the boundedness of $\mathcal{H}_{\varphi,m}^p$: $B^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times B^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times B^{q_m,\lambda_m}(\mathbb{Q}_p^n) \to B^{q,\lambda}(\mathbb{Q}_p^n)$. Moreover,

$$\|\mathcal{H}_{\varphi,m}^p\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times B^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times B^{q_m,\lambda_m}(\mathbb{Q}_p^n) \to B^{q,\lambda}(\mathbb{Q}_p^n)} = \widetilde{\mathcal{A}}_m.$$

Proof. By similarity, we only give the proof in the case $m = 2$.

When $-1/q_i = \lambda_i$, $i = 1, 2$, then Theorem 2.2 is just Theorem 2.1.

Next we consider the case that $-1/q_i < \lambda_i < 0$, $i = 1, 2$. Let $\gamma \in \mathbb{Z}$, $t_i B_\gamma = B(0, |t_i|_p p^\gamma)$ and $\widetilde{\mathcal{A}}_2 < \infty$. Since $1/q = 1/q_1 + 1/q_2$, by Minkowski's inequality and Hölder's inequality, we see that, for all balls $B = B(0, p^\gamma)$,

$$\Big(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int\limits_{B_\gamma} |\mathcal{H}_{\varphi,2}^p(\vec{f})(x)|^q dx \Big)^{1/q}$$

$$\leq \int\limits_{(\mathbb{Z}_p^*)^2} \Big(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int\limits_{B_\gamma} |\prod_{i=1}^{2} f_i(t_i x)|^q dx \Big)^{1/q} \varphi(\vec{t}) d\vec{t}$$

$$\leq \int\limits_{(\mathbb{Z}_p^*)^2} \prod_{i=1}^{2} \Big(\frac{1}{|B_\gamma|_H^{1+\lambda_i q_i}} \int\limits_{B_\gamma} |f_i(t_i x)|^{q_i} dx \Big)^{1/q_i} \varphi(\vec{t}) d\vec{t}$$

$$= \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1} |t_2|_p^{n\lambda_2} \prod_{i=1}^{2} \Big(\frac{1}{|t_i B_\gamma|_H^{1+\lambda_i q_i}} \int\limits_{t_i B_\gamma} |f_i(x)|^{q_i} dx \Big)^{1/q_i} \varphi(\vec{t}) d\vec{t}$$

$$\leq \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t}.$$

This means that

$$\|\mathcal{H}^p_{\varphi,2}\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)\times B^{q_2,\lambda_2}(\mathbb{Q}_p^n)\to B^{q,\lambda}(\mathbb{Q}_p^n)} \leq \widetilde{\mathcal{A}}_2. \tag{6}$$

For the necessity when $\lambda_1 q_1 = \lambda_2 q_2$, let $f_1(x) = |x|_p^{n\lambda_1}$ and $f_2(x) = |x|_p^{n\lambda_2}$ for all $x \in \mathbb{Q}_p^n\backslash\{0\}$, and $f_1(0) = f_2(0) := 0$. Then for any $B = B(0, p^\gamma)$, we have

$$\begin{aligned}
&\Big(\frac{1}{|B_\gamma|_H{}^{1+\lambda_i q_i}} \int\limits_{B_\gamma} |f_i(x)|^{q_i}dx\Big)^{1/q_i}\\
&= \Big(p^{-n\gamma(1+\lambda_i q_i)} \sum_{k=-\infty}^{\gamma} \int\limits_{S_k} p^{nk\lambda_i q_i}dx\Big)^{1/q_i}\\
&= \Big((1-p^{-n})p^{-n\gamma(1+\lambda_i q_i)} \sum_{k=-\infty}^{\gamma} p^{nk(1+\lambda_i q_i)}\Big)^{1/q_i}\\
&= \Big(\frac{1-p^{-n}}{1-p^{-n(1+\lambda_i q_i)}}\Big)^{1/q_i},
\end{aligned}$$

where the series converge due to $\lambda_i > -1/q_i$. Then $f_i \in B^{q_i,\lambda_i}(\mathbb{Q}_p^n)$. Since $\lambda = \lambda_1 + \lambda_2$ and $-1/q_i \leq \lambda_i < 0$, $1 < q < q_i < \infty$, $i = 1, 2$, we have

$$\begin{aligned}
&\Big(\frac{1}{|B_\gamma|_H{}^{1+\lambda q}} \int\limits_{B_\gamma} |\mathcal{H}^p_{\varphi,2}(\vec{f})(x)|^q dx\Big)^{1/q}\\
&= \Big(\frac{1}{|B_\gamma|_H{}^{1+\lambda q}} \int\limits_{B_\gamma} |x|_p^{n\lambda q}dx\Big)^{1/q} \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t}\\
&= \Big(\frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}}\Big)^{1/q} \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t}\\
&= \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \frac{(1-p^{-n(1+\lambda_1 q_1)})^{1/q_1}(1-p^{-n(1+\lambda_2 q_2)})^{1/q_2}}{(1-p^{-n(1+\lambda q)})^{1/q}}\\
&\times \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t}\\
&= \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \int\limits_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t},
\end{aligned}$$

since $\lambda_1 q_1 = \lambda_2 q_2$. Then,

$$\widetilde{\mathcal{A}}_2 \leq \|\mathcal{H}^p_{\varphi,2}\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)\times B^{q_2,\lambda_2}(\mathbb{Q}_p^n)\to B^{q,\lambda}(\mathbb{Q}_p^n)} < \infty. \tag{7}$$

Combining (6) and (7) then concludes the proof. This finishes the proof of the Theorem 2.2.

We remark that Theorem 2.2 when $m = 1$ goes back to [14] Theorem 2.3. □

Next, we give sharp estimate of weighted multilinear p-adic Hardy operator on the product of p-adic Morrey spaces.

Theorem 2.3. *Let* $1 < q < q_i < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, $\lambda = \lambda_1 + \cdots + \lambda_m$ *and* $-1/q_i < \lambda_i < 0$, $i = 1, \ldots, m$.
(i) If

$$\mathcal{B}_m := \int\limits_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^m |t_i|_p^{n\lambda_i}\varphi(\vec{t})d\vec{t} < \infty. \tag{8}$$

Then, $\mathcal{H}^p_{\varphi,m}$ *is bounded from* $\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times \mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times \mathcal{L}^{q_m,\lambda_m}(\mathbb{Q}_p^n)$ *to* $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$ *with its operator norm not more that* $\mathcal{B}_m$.

(ii) Assume that $\lambda_1 q_1 = \cdots = \lambda_m q_m$. In the case the condition (8) *is also necessary for the boundedness of $\mathcal{H}^p_{\varphi,m}$: $\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times \mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times \mathcal{L}^{q_m,\lambda_m}(\mathbb{Q}_p^n) \to \mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$. Moreover,*

$$\|\mathcal{H}^p_{\varphi,m}\|_{\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n)\times\mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times\mathcal{L}^{q_m,\lambda_m}(\mathbb{Q}_p^n)\to\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)} = \mathcal{B}_m.$$

Proof. By similarity, we only give the proof in the case $m = 2$. Suppose $\mathcal{B}_2 < \infty$. Since $1/q = 1/q_1 + 1/q_2$, by Minkowski's inequality and Hölder's inequality, we see that

$$\begin{aligned}
&\Big(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}}\int_{B_\gamma(a)} |\mathcal{H}^p_{\varphi,2}(\vec{f})(x)|^q dx\Big)^{1/q}\\
&\le \int_{(\mathbb{Z}_p^*)^2}\Big(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}}\int_{B_\gamma(a)} |\prod_{i=1}^{2} f_i(t_i x)|^q dx\Big)^{1/q}\varphi(\vec{t})d\vec{t}\\
&\le \int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma(a)|_H^{1+\lambda_i q_i}}\int_{B_\gamma(a)} |f_i(t_i x)|^{q_i} dx\Big)^{1/q_i}\varphi(\vec{t})d\vec{t}\\
&= \int_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma(a)|_H^{1+\lambda_i q_i}}\int_{t_iB_\gamma(a)} |f_i(x)|^{q_i} dx\Big)^{1/q_i}\varphi(\vec{t})d\vec{t}\\
&\le \|f_1\|_{\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{\mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2} |t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\varphi(\vec{t})d\vec{t}.
\end{aligned}$$

This means that

$$\|\mathcal{H}^p_{\varphi,2}\|_{\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n)\times\mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n)\to B^{q,\lambda}(\mathbb{Q}_p^n)} \le \mathcal{B}_2. \tag{9}$$

For the necessity when $\lambda_1 q_1 = \lambda_2 q_2$, let $f_1(x) = |x|_p^{n\lambda_1}$ and $f_2(x) = |x|_p^{n\lambda_2}$ for all $x \in \mathbb{Q}_p^n\backslash\{0\}$, and $f_1(0) = f_2(0) := 0$. Then for any $B = B(a, p^\gamma)$, we need to show that $f_i \in \mathcal{L}^{q_i,\lambda_i}(\mathbb{Q}_p^n)$. Considering the following two cases.

(I) If $|a|_p > p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} > p^\gamma$. Since$-1/q_i \le \lambda_i < 0$, we have

$$\begin{aligned}
&\frac{1}{|B_\gamma(a)|_H^{1+\lambda_i q_i}}\int_{B_\gamma(a)} |x|_p^{n\lambda_i q_i} dx\\
&< \frac{1}{|B_\gamma(a)|_H^{1+\lambda_i q_i}}\int_{B_\gamma(a)} p^{\gamma n\lambda_i q_i} dx = 1.
\end{aligned}$$

(II) If $|a|_p \le p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} \le p^\gamma$. Therefore, $x \in B_\gamma(a)$. Recall that two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other [20]. So we have $B_\gamma(a) = B_\gamma$, thus

$$\begin{aligned}
&\frac{1}{|B_\gamma(a)|_H^{1+\lambda_i q_i}}\int_{B_\gamma(a)} |x|_p^{n\lambda_i q_i} dx\\
&= \frac{1}{|B_\gamma|_H^{1+\lambda_i q_i}}\int_{B_\gamma} |x|_p^{n\lambda_i q_i} dx\\
&= \frac{1-p^{-n}}{1-p^{-n(1+\lambda_i q_i)}}.
\end{aligned}$$

From the previous discussion, we can see that $f_i \in \mathcal{L}^{q_i,\lambda_i}(\mathbb{Q}_p^n)$. By the similar estimates to the method of Theorem 2.2, we have

$$\mathcal{B}_2 \le \|\mathcal{H}^p_{\varphi,2}\|_{\mathcal{L}^{q_1,\lambda_1}(\mathbb{Q}_p^n)\times\mathcal{L}^{q_2,\lambda_2}(\mathbb{Q}_p^n)\to\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty. \tag{10}$$

Combining (9) and (10) then yields the desired result. □

We remark that Theorem 2.3 when $m = 1$ goes back to [14] Theorem 2.1.

3 Boundedness of commutators of the weighted multilinear p-adic Hardy operators

Now we introduce the definition for the multilinear version of the commutator of the weighted p-adic Hardy operators. Let $m \geq 2$, and φ be a nonnegative integrable function on $\mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^*$, and b_i $(i = 1, \ldots, m)$ be locally integral functions on $\mathbb{Q}_p^n$. We define

$$\mathcal{H}_{\varphi,m}^{p,\vec{b}} := \int\limits_{(\mathbb{Z}_p^*)^m} \Big(\prod_{i=1}^m f_i(t_i x)\Big)\Big(\prod_{i=1}^m (b_i(x) - b_i(t_i x))\Big)\varphi(\vec{t})d\vec{t}, \quad x \in \mathbb{Q}_p^n.$$

Then we have the following multilinear result.

Theorem 3.1. *Let $1 < q < q_i < \infty$, $1 < \rho < \infty$, $-1/q_i < \lambda_i < 0$, $i = 1, \ldots, m$, such that $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m} + \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_m}$, $\lambda = \lambda_1 + \cdots + \lambda_m$. If*

$$\widetilde{\mathcal{B}}_m := \int\limits_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^m |t_i|_p^{n\lambda_i} \varphi(\vec{t})\Big(\prod_{i=1}^m \log_p \frac{1}{|t_i|_p}\Big)d\vec{t} < \infty.$$

Then $\mathcal{H}_{\varphi,m}^{p,\vec{b}}$ is bounded from $B^{q_1,\lambda_1}(\mathbb{Q}_p^n) \times B^{q_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times B^{q_m,\lambda_m}(\mathbb{Q}_p^n)$ to $B^{q,\lambda}(\mathbb{Q}_p^n)$ for all $\vec{b} = (b_1, b_2, \ldots, b_m) \in CMO^{\rho_1}(\mathbb{Q}_p^n) \times CMO^{\rho_2}(\mathbb{Q}_p^n) \times \cdots \times CMO^{\rho_m}(\mathbb{Q}_p^n)$.

Proof. By similarity, we only consider the case that $m = 2$, that is, we assume $\widetilde{\mathcal{B}}_2 < \infty$ and just need to show that

$$\|\mathcal{H}_{\varphi,2}^{p,\vec{b}}(\vec{f})\|_{B^{q,\lambda}(\mathbb{Q}_p^n)} \leq C\widetilde{\mathcal{B}}_2 \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)},$$

where $\vec{b} = (b_1, b_2) \in CMO^{\rho_1}(\mathbb{Q}_p^n) \times CMO^{\rho_2}(\mathbb{Q}_p^n)$. By Minkowski's inequality we have

$$\begin{aligned}
&\Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} |\mathcal{H}_{\varphi,2}^{p,\vec{b}}(\vec{f})(x)|^q dx\Big)^{1/q} \\
&\leq \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_i(t_i x)| \varphi(t_1, t_2)dt_1 dt_2\Big)^q dx\Big)^{1/q} \\
&\leq \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_i(t_i x)|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2 \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}$$

where

$$I_1 = \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(x) - b_{i,B_\gamma}|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2,$$

$$I_2 = \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_i(t_i x) - b_{i,t_i B_\gamma}|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2,$$

$$I_3 = \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \prod_{i=1}^2 |b_{i,B_\gamma} - b_{i,t_i B_\gamma}|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2,$$

$$I_4 = \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \sum_{D(i,j)} |b_i(x) - b_{i,B_\gamma}||b_{j,B_\gamma} - b_{j,t_j B_\gamma}|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2,$$

$$I_5 = \int\limits_{\mathbb{Z}_p^*}\int\limits_{\mathbb{Z}_p^*} \Big(\frac{1}{|B_\gamma|_H} \int\limits_{B_\gamma} \Big(\prod_{i=1}^2 |f_i(t_i x)| \sum_{D(i,j)} |b_i(x) - b_{i,B_\gamma}||b_j(t_j x) - b_{j,t_j B_\gamma}|\Big)^q dx\Big)^{1/q} \varphi(t_1, t_2)dt_1 dt_2,$$

$$I_6=\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\Big(\prod_{i=1}^{2}|f_i(t_ix)|\sum_{D(i,j)}|b_{i,B_\gamma}-b_{i,t_iB_\gamma}||b_j(t_jx)-b_{j,t_jB_\gamma}|\Big)^q dx\Big)^{1/q}\varphi(t_1,t_2)dt_1dt_2,$$

and

$$D(i,j):=\{(i,j):(1,2),(2,1)\},\qquad b_{i,B_\gamma}=\frac{1}{|B_\gamma|_H}\int_{B_\gamma}b_i,\quad i=1,2.$$

Choose $q<s_1<\infty$, $q<s_2<\infty$ such that $1/s_1=1/q_1+1/\rho_1$, $1/s_2=1/q_2+1/\rho_2$. Then by Hölder's inequality, we know that

$$\begin{aligned}I_1&\le\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|f_i(t_ix)|^{q_i}dx\Big)^{1/q_i}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_i(x)-b_{i,B_\gamma}|^{\rho_i}dx\Big)^{1/\rho_i}\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma|_H^{1+\lambda_iq_i}}\int_{t_iB_\gamma}|f_i(x)|^{q_i}dx\Big)^{1/q_i}\\&\quad\times\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_i(x)-b_{i,B_\gamma}|^{\rho_i}dx\Big)^{1/\rho_i}\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\\&\quad\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)dt_1dt_2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}I_2&\le\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|f_i(t_ix)|^{q_i}dx\Big)^{1/q_i}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_i(t_ix)-b_{i,t_iB_\gamma}|^{\rho_i}dx\Big)^{1/\rho_i}\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma|_H^{1+\lambda_iq_i}}\int_{t_iB_\gamma}|f_i(x)|^{q_i}dx\Big)^{1/q_i}\\&\quad\times\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma|_H}\int_{t_iB_\gamma}|b_i(x)-b_{i,t_iB_\gamma}|^{\rho_i}dx\Big)^{1/\rho_i}\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\\&\quad\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)dt_1dt_2.\end{aligned}$$

It follows from $1/q=1/s_1+1/s_2$ that $1=q/s_1+q/s_2$. From $1/s_1=1/q_1+1/\rho_1$, $1/s_2=1/q_2+1/\rho_2$ and Hölder's inequality, we deduce that

$$\begin{aligned}I_3&=\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\Big(\prod_{i=1}^{2}|f_i(t_ix)|\prod_{i=1}^{2}|b_{i,B_\gamma}-b_{i,t_iB_\gamma}|\Big)^q dx\Big)^{1/q}\varphi(t_1,t_2)dt_1dt_2\\&\le\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|f_i(t_ix)|^{s_i}dx\Big)^{1/s_i}\Big(\prod_{i=1}^{2}|b_{i,B_\gamma}-b_{i,t_iB_\gamma}|\Big)\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma|_H^{1+\lambda_iq_i}}\int_{t_iB_\gamma}|f_i(x)|^{q_i}dx\Big)^{1/q_i}\\&\quad\times\Big(\prod_{i=1}^{2}|b_{i,B_\gamma}-b_{i,t_iB_\gamma}|\Big)\varphi(t_1,t_2)dt_1dt_2\\&\le|B_\gamma|_H^{\lambda}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\int_{\{p^{-l-1}<|t_1|_p\le p^{-l}\}}\int_{\{p^{-k-1}<|t_2|_p\le p^{-k}\}}|t_1|_p^{n\lambda_1}|t_2|_p^{n\lambda_2}\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\sum_{j=0}^{l}|b_{1,p^{-j}B_\gamma}-b_{1,p^{-j-1}B_\gamma}|+|b_{1,p^{-l-1}B_\gamma}-b_{1,t_iB_\gamma}|\Big)\\
&\times\Big(\sum_{j=0}^{k}|b_{2,p^{-j}B_\gamma}-b_{2,p^{-j-1}B_\gamma}|+|b_{2,p^{-k-1}B_\gamma}-b_{2,t_iB_\gamma}|\Big)\varphi(t_1,t_2)dt_1dt_2\\
\le{}& C|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\\
&\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\log_p\frac{p}{|t_1|_p}\log_p\frac{p}{|t_2|_p}\varphi(t_1,t_2)dt_1dt_2,
\end{aligned}
$$

where we use the fact that

$$
\begin{aligned}
|b_{1,B_\gamma}-b_{1,t_1B_\gamma}|&=\sum_{j=0}^{l}|b_{1,p^{-j}B_\gamma}-b_{1,p^{-j-1}B_\gamma}|+|b_{1,p^{-l-1}B_\gamma}-b_{1,t_1B_\gamma}|\\
&\le C(l+1)\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\\
&\le C\log_p\frac{p}{|t_1|_p}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)},
\end{aligned}
$$

and

$$
|b_{2,B_\gamma}-b_{2,t_2B_\gamma}|\le C\log_p\frac{p}{|t_2|_p}\|b_2\|_{CMO^{\rho_2}(\mathbb{Q}_p^n)}.
$$

We now estimate I_4. Similarly, we choose $1<s<\infty$ such that $1/q=1/q_1+1/q_2+1/s$ and $1/s=1/q_1+1/q_2$. Using Minkowski's inequality and Hölder's inequality yields

$$
\begin{aligned}
I_4={}&\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\Big(\prod_{i=1}^{2}|f_i(t_ix)|\sum_{D(i,j)}|b_i(x)-b_{i,B_\gamma}||b_{j,B_\gamma}-b_{j,t_jB_\gamma}|\Big)^q dx\Big)^{1/q}\varphi(t_1,t_2)dt_1dt_2\\
\le{}&\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\Big[\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\Big(\Big(\prod_{i=1}^{2}|f_i(t_ix)|\Big)\Big(|b_1(x)-b_{1,B_\gamma}||b_{2,B_\gamma}-b_{2,t_2B_\gamma}|\Big)\Big)^q dx\Big)^{1/q}\\
&+\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\Big(\Big(\prod_{i=1}^{2}|f_i(t_ix)|\Big)\Big(|b_2(x)-b_{2,B_\gamma}||b_{1,B_\gamma}-b_{1,t_1B_\gamma}|\Big)\Big)^q dx\Big)^{1/q}\Big]\varphi(t_1,t_2)dt_1dt_2\\
\le{}&\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|f_i(t_ix)|^{q_i}dx\Big)^{1/q_i}\Big\{\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_1(x)-b_{1,B_\gamma}|^s dx\Big)^{1/s}\\
&\times|b_{2,B_\gamma}-b_{2,t_2B_\gamma}|+\Big(\int_{B_\gamma}|b_2(x)-b_{2,B_\gamma}|^s dx\Big)^{1/s}|b_{1,B_\gamma}-b_{1,t_1B_\gamma}|\Big\}\varphi(t_1,t_2)dt_1dt_2\\
\le{}&|B_\gamma|_H^{\lambda}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\prod_{i=1}^{2}\Big(\frac{1}{|t_iB_\gamma|_H^{1+\lambda_iq_i}}\int_{t_iB_\gamma}|f_i(x)|^{q_i}dx\Big)^{1/q_i}\\
&\times\Big\{\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_1(x)-b_{1,B_\gamma}|^s dx\Big)^{1/s}|b_{2,B_\gamma}-b_{2,t_2B_\gamma}|\\
&+\Big(\int_{B_\gamma}|b_2(x)-b_{2,B_\gamma}|^s dx\Big)^{1/s}|b_{1,B_\gamma}-b_{1,t_1B_\gamma}|\Big\}\varphi(t_1,t_2)dt_1dt_2\\
\le{}&|B_\gamma|_H^{\lambda}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\\
&\times\Big\{\Big(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|b_1(x)-b_{1,B_\gamma}|^s dx\Big)^{1/s}|b_{2,B_\gamma}-b_{2,t_2B_\gamma}|\\
&+\Big(\int_{B_\gamma}|b_2(x)-b_{2,B_\gamma}|^s dx\Big)^{1/s}|b_{1,B_\gamma}-b_{1,t_1B_\gamma}|\Big\}\varphi(t_1,t_2)dt_1dt_2
\end{aligned}
$$

$$\leq C|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}$$
$$\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)\Big(\prod_{i=1}^{2}\log_p\frac{p}{|t_i|_p}\Big)dt_1dt_2.$$

It is apparent from the estimates of I_1, I_2, I_3 and I_4 that

$$I_5\leq|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}$$
$$\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)dt_1dt_2,$$

and

$$I_6\leq|B_\gamma|_H^{\lambda}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}$$
$$\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)\Big(\prod_{i=1}^{2}\log_p\frac{p}{|t_i|_p}\Big)dt_1dt_2.$$

Combining the estimates of I_1, I_2, I_3, I_4, I_5 and I_6 gives

$$\Big(\frac{1}{|B_\gamma|_H{}^{1+\lambda q}}\int_{B_\gamma}|\mathcal{H}_{\varphi,2}^{p,\vec{b}}(\vec{f})(x)|^q dx\Big)^{1/q}$$
$$\leq C\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|b_1\|_{CMO^{\rho_1}(\mathbb{Q}_p^n)}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}$$
$$\times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p^*}\prod_{i=1}^{2}|t_i|_p^{n\lambda_i}\varphi(t_1,t_2)\Big(\prod_{i=1}^{2}\log_p\frac{1}{|t_i|_p}\Big)dt_1dt_2.$$

This finishes the proof of Theorem 3.1. □

Acknowledgement: The authors would like to thank the anonymous reviewer for his/her comments which help to develop the manuscript. The research was supported by the National Natural Science Foundation of China (Grant No. 11661075).

References

[1] Vladimirov V.S., Volovich I.V., Zelenov E.I., p-adic Analysis and Mathematical Physics, World Scientific, Singapore, 1994
[2] Vladimirov V.S., Volovich I.V., p-adic quantum mechanics, Comm. Math. Phys., 1989, 123, 659-676
[3] Chuong N.M., Ciarlet P.G., Lax P., Mumford D. and Phong D.H., Advances in deterministic and stochastic snalysis, World Scientific, Singapore, 2007
[4] Chuong N.M., Egorov Yu.V., Khrennikov A., Meyer Y. and Mumford D., Harmonic, wavelet and p-adic analysis, World Scientific, Singapor, 2007
[5] Taibleson, M., Fourier analysis on local fields, Princeton Univ. Press, Princeton, 1975
[6] Rim Y.C., Carleson measures and the BMO space on the p-adic vector space, Math. Z., 2009, 282, 1470-1477
[7] Wu Q.Y., Fu Z.W., Sharp estimates of m-linear p-adic Hardy and Hardy-Littlewood-Pólya operators, J. Appl. Math., 2011, 2011,137-150
[8] Hardy G.H., Note on a theorem of Hilbert, Math. Z., 1920, 6, 314-317
[9] Christ M., Grafakos, L., Best constants for two nonconvolution inequalities, Proc. Am. Math. Soc., 1995, 123, 1687-1693
[10] Fu Z.W., Wu Q.Y., Lu S.Z., et al., Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya operators, Acta Math. Sin. (Engl. Ser.), 2012 29, 137-150
[11] Carton-Lebrun C., Fosset M., Moyennes et quotients de Taylor dans BMO, Bull. Soc. Roy. Sci. Liege., 1984, 53(2), 85-87
[12] Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 1938, 43, 126-166
[13] Rim K.S., Lee J., Estimates of weighted Hardy-Littlewood averages on the p-adic vector space, J. Math. Anal. Appl., 2006, 324, 1470-1477

[14] Wu Q.Y., Fu Z.W., Weighted p-adic Hardy operators and their commutators on p-adic central Morrey spaces, Bull. Malays. Math. Sci. Soc., 2017, 40, 1-20

[15] Coifman R.R., Meyer Y., On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc., 1975, 212, 315-331

[16] Bényi Á., Oh T., Best constants for certain multilinear integral operators, J. Inequal. Appl. 2006, Article ID 28582

[17] Wang S.M., Lu S.Z., Yan D.Y., Explicit constants for Hardy's inequality with power weight on n-dimensional product spaces, Sci. China Math., 2012, 55, 2469-2480

[18] Muckenhoupt B., Weighted norm inequalities for classical operators, Proc. Symp. Pure Math., 1979, 35, 69-83

[19] Pachpatte B.G., On multivariate Hardy type inequalities, An. stiint. Univ. Al.I. Cuza Iasi, Mat., 1992, 38, 355-361

[20] Albeverio S., Khrennikov A.Yu. and Shelkovich M., Theory of p-adic distributions: linear and nonlinear models, Lodon Mathematical Lecture Note series, Vol. 370, Canbridge University Press, Cambridge, 2010