Open Mathematics

Research Article

Chonghui Huang and Lijing Zheng*

Simple modules over Auslander regular rings

https://doi.org/10.1515/math-2017-0138
Received January 2, 2017; accepted November 28, 2017.

Abstract: In this paper, we discuss the properties of simple modules over Auslander regular rings with global dimension at most 3. Using grade theory, we show the right projective dimension of $\mathrm{Ext}^1_\Lambda(S,\Lambda)$ is equal to 1 for any simple Λ-module S with $\mathrm{gr}\, S = 1$. As a result, we give some equivalent characterization of diagonal Auslander regular rings.

Keywords: Auslander regular ring, Diagonal ring, Simple module, Grade

MSC: 16E05, 16E10

1 Introduction

Recall from [1, Theorem 3.7] that an Artin algebra Λ is called to be *n-Gorenstein* if $\mathrm{l.pd}\, I_i(\Lambda) \le i$ holds for any integer $0 \le i \le n-1$. Λ is an *Auslander-regular* ring if Λ is n-Gorenstein for any integer $n \ge 0$ and $\mathrm{gl.dim}\,\Lambda < \infty$. It is known to all that n-Gorenstein rings admit left-right symmetric properties and this kind of rings play an important role in homological algebras and representation theory of algebras as well as many other fields (see [2-9] among others). In [9], O. Iyama introduced a kind of special n-Gorenstein rings–diagonal Auslander regular rings, that is, Auslander regular ring Γ is said to be *diagonal* if any non-zero direct summand I of $I_i(\Gamma)$ satisfies the left flat dimension of $I_i(\Gamma) = i$. These rings are closely related to the representation theory of partially ordered sets, vector space categories, and orders over complete discrete valuation rings (See [10-11]).

Simple modules are important research objects and many properties of rings are determined by them. So, it is nice to study properties of simple modules over Auslander regular rings. Moreover, motivated by the work of O. Iyama, we use grade theory to study the properties of diagonal Auslander regular rings. In section 2, we study the properties of simple modules over Auslander regular ring with $\mathrm{gl.dim}\,\Lambda \le 3$. As an application, we give some equivalent characterization of diagonal Auslander regular rings. Our results will be helpful to understand Auslander regular rings better.

Throughout this paper, Λ is an artin algebra and all the modules we considered are finitely generated. We use $\mathrm{mod}\,\Lambda$(resp. $\mathrm{mod}\,\Lambda^{op}$) to denote the category of finitely generated Λ-modules(resp. Λ^{op}-modules). We put $(\ )^* := \mathrm{Hom}_\Lambda(\ ,\Lambda)$ and for any integer $n \ge 1$

$$\mathcal{E}_n := \mathrm{Ext}^n_\Lambda(\ ,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op},$$
$$\mathcal{F}_n := \mathrm{soc}\,\mathrm{Ext}^n_\Lambda(\ ,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op}.$$

We use the following symbols:

Chonghui Huang: College of Mathematics and Physics, University of South China, Hengyang 421001, China, E-mail: huangch78@163.com
***Corresponding Author: Lijing Zheng:** College of Mathematics and Physics, University of South China, Hengyang 421001, China, E-mail: zhenglijing817@163.com

gl.dim Λ:	the global dimension of Λ
l.pd M:	the projective dimension of a left module M
r.pd M:	the projective dimension of a right module M
$I_i(M)$:	the $(i+1)$-th term in a minimal injective resolution of a module M
gr M:	the grade of M, defined by $\inf\{i \geq 0 \mid \operatorname{Ext}^i_\Lambda(M,\Lambda) \neq 0\}$
s.gr M:	the strong grade of M, defined by $\inf\{\operatorname{gr} N$ for any $N \subseteq M\}$

2 Properties of simple modules and diagonal Auslander regular ring

The following results are useful in our statements, for the sake of the readability, we list them as lemmas.

Lemma 2.1 ([12, Lemma1]). *Let R be any ring. M is a R-module and $0 \to M \to I_0(M) \to I_1(M) \to \cdots$ a minimal injective resolution for M. Then, for a simple R-module S and $n > 0$, $\operatorname{Ext}^n_R(S,M) \cong \operatorname{Hom}_R(S, I_n(M))$.*

Lemma 2.2 ([12, Proposition 7]). *Let R be a Gorenstein ring with self-injective dimension n. Then for a simple left R-module S, if S is a submodule of $I_n(R)$, S never appears in $\operatorname{soc}(I_i(R))$ for all $i \neq n$.*

Lemma 2.3 ([8, Theorem 1.2]). *Let Λ be an n-Gorenstein noetherian algebra and $0 \leq l \leq n-2$. Then $\mathcal{E}_l$ gives a duality between $C_l(\Lambda)$ and $C_l(\Lambda^{op})$, where $C_l(\Lambda)$ is the subcategory consisting of all module $X = \mathcal{E}_l(Y)$ for some $Y \in \operatorname{mod} \Lambda^{op}$ with $\operatorname{gr} Y \geq l$, and $\mathcal{E}_l \circ \mathcal{E}_l$ is isomorphic to the identity functor.*

Lemma 2.4 ([8, Theorem 1.3]). *Let Λ be an n-Gorenstein noetherian algebra and $0 \leq l \leq n$. Then $\mathcal{F}_l$ gives a duality between simple Λ-modules X with $\operatorname{gr} X = l$ and that of Λ^{op}, and $\mathcal{F}_l \circ \mathcal{F}_l$ is isomorphic to the identity functor. Moreover, $\operatorname{gr} \mathcal{E}_l(X)/\mathcal{F}_l(X) > l$ holds.*

For any $A \in \operatorname{mod} \Lambda$, there exists a projective presentation in $\operatorname{mod} \Lambda$:

$$P_1 \xrightarrow{f} P_0 \to A \to 0.$$

Then we get an exact sequence

$$0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Coker} f^* \to 0$$

in $\operatorname{mod} \Lambda^{op}$. Recall from [13] that $\operatorname{Coker} f^*$ is called a *transpose* of A, and denoted by $\operatorname{Tr} A$. We remark that the transpose of A depends on the choice of the projective presentation of A, but it is unique up to projective equivalence (see [13]). By [13, Proposition 2.6], we have the following two useful exact sequences:

$$0 \to \operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} A, \Lambda) \to A \xrightarrow{\sigma_A} A^{**} \to \operatorname{Ext}^2_{\Lambda^{op}}(\operatorname{Tr} A, \Lambda) \to 0 \tag{1}$$

$$0 \to \operatorname{Ext}^1_{\Lambda}(A, \Lambda) \to \operatorname{Tr} A \xrightarrow{\sigma_{\operatorname{Tr} A}} (\operatorname{Tr} A)^{**} \to \operatorname{Ext}^2_{\Lambda}(A, \Lambda) \to 0 \tag{2}$$

The following result plays an important role in this paper.

Proposition 2.5. *Let Λ be an Auslander regular ring with $\operatorname{gl.dim} \Lambda \leq 3$ and S be a simple Λ-module with $\operatorname{gr} S = 1$. Then $\operatorname{r.pd} \mathcal{E}_1(S) = \operatorname{gr} \mathcal{E}_1(S) = 1$.*

Proof. Let $\cdots \to P_1 \to P_0 \to \mathcal{E}_1(S) \to 0$ be a projective resolution of $\mathcal{E}_1(S)$ in $\operatorname{mod} \Lambda^{op}$ and $L = \operatorname{Im}(P_1 \to P_0)$. Then by [1, Theorem 3.7], $\operatorname{gr} \mathcal{E}_1(S) \geq 1$ and we get an exact sequence $0 \to P_0^* \to L^* \to \mathcal{E}_1\mathcal{E}_1(S) \to 0$ in $\operatorname{mod} \Lambda$. Taking $(\, , \Lambda)$, we have the following exact sequence in $\operatorname{mod} \Lambda^{op}$:

$$0 \to L^{**} \to P_0^{**} \to \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S) \to \mathcal{E}_1(L^*) \to 0\,.$$

Put $H = \mathrm{Coker}(L^{**} \to P_0^{**})$. We get the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\sigma_L"] & P_0 \arrow[r] \arrow[d, "\sigma_{P_0}"] & \mathcal{E}_1(S) \arrow[r] \arrow[d, "h"] & 0, \\ 0 \arrow[r] & L^{**} \arrow[r] & P_0^{**} \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

where h is induced by the natural morphisms σ_L and σ_{P_0}. It is easy to see the morphism h is an epimorphism. Moreover, there is an exact sequence $0 \to \mathcal{E}_2(\mathrm{Tr}\,L) \to \mathcal{E}_1(S) \to H \to 0$ by snake lemma and the sequence (1).

Denote $S_r = \mathcal{F}_1(S)$, we claim that $\mathcal{E}_2(\mathrm{Tr}\,L) = 0$. By Lemma 2.4, $\mathcal{F}_1$ is a duality between simple Λ-module S with $\mathrm{gr}\,S = 1$ and that of Λ^{op}. So we have $\mathrm{gr}\,S_r = 1$. If $\mathcal{E}_2(\mathrm{Tr}\,L) \neq 0$, S_r will be a submodule of $\mathcal{E}_2(\mathrm{Tr}\,L)$ and hence $\mathrm{gr}\,S_r \geq 2$ by [1, Theorem 3.7] and [9, Lemma 2.3(2)], a contradiction will arise. So we have $L \cong L^{**}$ and $H \cong \mathcal{E}_1(S)$.

By Lemma 2.3, we get $\mathcal{E}_1(S) \cong \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S)$. This derives that $\mathcal{E}_3(\mathrm{Tr}\,L) \cong \mathcal{E}_1(L^*) = 0$ by the exactness of the sequence $0 \to H \to \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S) \to \mathcal{E}_1(L^*) \to 0$. Then the facts $\mathcal{E}_i(\mathrm{Tr}\,L) = 0$ for any $i \geq 1$ and $\mathrm{gl.dim}\,\Lambda \leq 3$ yield that $\mathrm{Tr}\,L$ is projective. So we have L is projective and $\mathrm{r.pd}\,\mathcal{E}_1(S) \leq 1$. It follows that $\mathrm{r.pd}\,\mathcal{E}_1(S) = \mathrm{gr}\,\mathcal{E}_1(S) = 1$. □

We have the following useful result.

Corollary 2.6. *Let Λ be an Auslander regular ring with $\mathrm{gl.dim}\,\Lambda \leq 3$ and S be a simple Λ-module with $\mathrm{gr}\,S = 1$. Then $\mathrm{l.pd}\,\mathcal{E}_1\mathcal{E}_1(S) = \mathrm{gr}\,\mathcal{E}_1\mathcal{E}_1(S) = 1$.*

Proof. By Proposition 2.5, we get $\mathrm{r.pd}\,\mathcal{E}_1(S) = \mathrm{gr}\,\mathcal{E}_1(S) = 1$. Let $0 \to Q_1 \to Q_0 \to \mathcal{E}_1(S) \to 0$ be any projective resolution of $\mathcal{E}_1(S)$. Taking $(\ ,\Lambda)$, we have a projective resolution of $\mathcal{E}_1\mathcal{E}_1(S)$ in $\mathrm{mod}\,\Lambda$: $0 \to Q_0^* \to Q_1^* \to \mathcal{E}_1\mathcal{E}_1(S) \to 0$. Because $\mathrm{gr}\,\mathcal{E}_1(S) = 1$, we get that $\mathrm{l.pd}\,\mathcal{E}_1\mathcal{E}_1(S) = 1$. Moreover, we have $\mathrm{gr}\,\mathcal{E}_1\mathcal{E}_1(S) \geq 1$ by [1, Theorem 3.7]. So we are done. □

Let S be any simple module with $\mathrm{gr}\,S = 1$. Lemma 2.4 shows that the socle of $\mathcal{E}_1(S)$ is simple, denoted by S_r, and there is an exact sequence $0 \to S_r \to \mathcal{E}_1(S) \to N \to 0$ with $\mathrm{gr}\,N \geq 2$. On the other hand, we always have $0 \to S \to S^{**} \to \mathcal{E}_2(\mathrm{Tr}\,S) \to 0$ for any S with $\mathrm{gr}\,S = 0$. It is easy to see that $\mathrm{gr}\,\mathcal{E}_2(\mathrm{Tr}\,S) \geq \mathrm{gr}\,S + 2$. Then, it is natural to ask: does this hold true for any simple module with $\mathrm{gr}\,S = 1$? The following result shows the relationship between S and $\mathcal{E}_1\mathcal{E}_1(S)$. That is, we have

Theorem 2.7. *Let Λ be an Auslander regular ring with $\mathrm{gl.dim}\,\Lambda \leq 3$ and S be a simple Λ-module with $\mathrm{gr}\,S = 1$. Then there exists $K \in \mathrm{mod}\,\Lambda$ with $\mathrm{gr}\,K \geq 3$ such that $0 \to S \to \mathcal{E}_1\mathcal{E}_1(S) \to K \to 0$ is exact.*

Proof. Put $S_r = \mathcal{F}_1(S)$ and $N = \mathcal{E}_1(S_r)/\mathcal{F}_1(S_r)$. By Lemma 2.4, we have $\mathrm{gr}\,S_r = 1$ and $\mathrm{gr}\,N \geq 2$. Applying $(-,\Lambda)$ on the exact sequence $0 \to S \xrightarrow{i_S} \mathcal{E}_1(S_r) \to N \to 0$, since $\mathrm{gr}\,N \geq 2$, we get the following exact sequence in $\mathrm{mod}\,\Lambda^{op}$:

$$0 \to \mathcal{E}_1\mathcal{E}_1(S_r) \to \mathcal{E}_1(S) \to \mathcal{E}_2(N) \to 0\,.$$

Taking $(-,\Lambda)$ again, we get an exact sequence $0 \to \mathcal{E}_1\mathcal{E}_1(S) \to \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S_r) \to \mathcal{E}_2\mathcal{E}_2(N) \to 0$ in $\mathrm{mod}\,\Lambda$ by the fact $\mathrm{gr}\,\mathcal{E}_2(N) \geq 2$. By Lemma 2.3, $\mathcal{E}_1(S) \cong \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S)$. Suppose f and g are the isomorphisms $\mathcal{E}_1(S) \to \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S)$ and $\mathcal{E}_1\mathcal{E}_1(S) \to \mathrm{Ker}(\mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S_r) \to \mathcal{E}_2\mathcal{E}_2(N))$, respectively. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, "u"] & \mathcal{E}_1(S_r) \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & \mathcal{E}_1\mathcal{E}_1(S) \arrow[r] & \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S_r) \arrow[r] & \mathcal{E}_2\mathcal{E}_2(N) \arrow[r] & 0, \end{tikzcd}$$

where $u = g^{-1} \circ f \circ i_S$ and v is the induced morphism. Put $K = \mathrm{Ker}\,v$, then we have exact sequences in $\mathrm{mod}\,\Lambda$:

$$0 \to S \to \mathcal{E}_1\mathcal{E}_1(S) \to K \to 0 \qquad (*)$$

$$0 \to K \to N \to \mathcal{E}_2\mathcal{E}_2(N) \to 0 \qquad (**)$$

by snake lemma. By [9, Lemma 2.3(2)] and the exactness of sequence $(**)$, we have $\operatorname{gr} K \geq 2$. Applying $(-, \Lambda)$ on the exact sequence $(*)$, we get the following long exact sequence in mod Λ^{op}:

$$0 \to \mathcal{E}_1(K) \to \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S) \to \mathcal{E}_1(S) \to \mathcal{E}_2(K) \to \mathcal{E}_2\mathcal{E}_1\mathcal{E}_1(S) \to \cdots.$$

By Corollary 2.6, we have $\operatorname{l.pd} \mathcal{E}_1\mathcal{E}_1(S) = 1$ and, therefore, $\mathcal{E}_2\mathcal{E}_1\mathcal{E}_1(S) = 0$. So the facts $\mathcal{E}_1(K) = 0$ and $\mathcal{E}_1\mathcal{E}_1\mathcal{E}_1(S) \cong \mathcal{E}_1(S)$ yield that $\mathcal{E}_2(K) = 0$. That is, $\operatorname{gr} K \geq 3$. □

Now we fix our sight on diagonal rings. We have the following

Theorem 2.8. *Let Λ be an Auslander regular ring. Then the following statements are equivalent.*
(1) Λ is diagonal.
(2) Any simple Λ-module S with $\operatorname{gr} S = k$ satisfies $\operatorname{l.pd} S = k$ for any $k \geq 0$.
(3) Any simple Λ-module S with $\operatorname{l.pd} S = k$ satisfies $S \cong \mathcal{E}_k\mathcal{E}_k(S)$ for any $k \geq 0$.

Proof. $(1) \Leftrightarrow (2)$ is trivial by the definition of diagonal ring and Lemma 2.1. We need to show $(1) \Leftrightarrow (3)$.

$(1) \Rightarrow (3)$ Suppose Λ is diagonal. Let S be any simple Λ-module with $\operatorname{l.pd} S = k$. Then one gets that $\mathcal{E}_k(S) \neq 0$. By Lemma 2.1, S can be embedded into $I_k(\Lambda)$. So, S will never appear in $I_i(\Lambda)$ for any $i \neq k$. Applying Lemma 2.1 again, we get $\operatorname{gr} S = k$. That is, $S \cong \mathcal{E}_k\mathcal{E}_k(S)$.

$(3) \Rightarrow (1)$ Suppose any simple Λ-module S with $\operatorname{l.pd} S = k$ satisfies $S \cong \mathcal{E}_k\mathcal{E}_k(S)$ for any $k \geq 0$. Then by [1, Theorem 3.7], we get $\operatorname{gr} S \geq k$. Hence S can not be embedded into any term of the minimal injective resolution except for $I_k(\Lambda)$. That is to say, Λ is diagonal. □

Actually, we have a strong version of the above result.

Proposition 2.9. *Let Λ be an Auslander regular ring with global dimension n. Then, Λ is diagonal if and only if any simple Λ-module S with $\operatorname{gr} S = k$ satisfies $\operatorname{l.pd} S = k$ for any $k \leq n - 2$.*

Proof. We need only to show the necessity. It is known that any simple module S with $\operatorname{gr} S = n$ must have a projective dimension n. Let S be any simple module with $\operatorname{gr} S = n-1$. Then the projective dimension of S must not be n by Lemma 2.2. So we have $\operatorname{l.pd} S = n - 1$. That is, Λ is diagonal by Proposition 2.5 (2). □

Let Λ be an Auslander regular ring with global dimension at most 3. By Corollary 2.6, we get that any simple Λ-module S with $\operatorname{gr} S = 1$ satisfies $\operatorname{l.pd}(\mathcal{E}_1\mathcal{E}_1(S)) = 1$. So we have the following

Corollary 2.10. *Let Λ be an Auslander regular ring with global dimension at most 3. Then the following statements are equivalent.*
(1) Λ is diagonal.
(2) Any simple Λ-module S with $\operatorname{gr} S = 0$ is projective and any simple Λ-module S with $\operatorname{gr} S = 1$ satisfies $S \cong \mathcal{E}_1\mathcal{E}_1(S)$.

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgement: This research was partially supported by NSFC (Grant No. 11701488, 11201220, 11101217, 11126169) and Hunan Provincial Natural Science Foundation of China (Grant Nos. 14JJ3099, 2016JJ6124).

References

[1] Fossum R. M., Griffith P., Reiten I., *Trivial extensions of abelian categories. Homological algebra of trivial extensions of abelian categories with applications to ring theory*, Lecture Note in Mathematics, Vol. 456, Springer-Verlag, Berlin, New York, 1975. MR52:10810.

[2] Auslander M., Reiten I., *Syzygy modules for Noetherian rings*, J. Algebra, 1990, 183(1), 167-185.

[3] Auslander M., Reiten I., *k-Gorenstein algebras and syzygy modules*, J. Pure Appl. Algebra, 1994, 92(1), 1-27.

[4] Huang Z. Y., *Extension closure of k-torsionfree modules*, Comm. Algebra, 1999, 27(3), 1457-1464 .

[5] Huang Z. Y., *k-Gorenstein modules. Acta Math. Sinica (English Series)*,2007, 23(8), 1463-1474.

[6] Iwanaga Y., Sato H., *On Auslander's n-Gorenstein rings*, J. Pure and Applied Alg., 1996, 106, 61-76 .

[7] Iwanaga Y., *Duality Over Auslander-Gorenstein rings*, Math Scand., 1997,81(2), 184-190.

[8] Iyama O., *Symmetry and duality on n-Gorenstein rings*, J. Algebra, 2003, 269(2), 528-535.

[9] Iyama O., *The relationship between homological properties and representantion theoretic realization of artin algebras*, Trans. Amer. Math. Society, 2005, 357(2), 709-734.

[10] Simson D., *Linear representations of partially ordered sets and vector space categories*, Algebra, Logic and Applications, 4, Gordon and Breach Science Publishers, Montreux, 1992, MR95g:16013.

[11] Dao H. L., Iyama O., Takahashi R., Wemyss M., *Gorenstein modifications and $\mathbb{Q}$-Gorenstein rings*, 2016, arxiv: 1611.04137.

[12] Iwanaga Y., *On rings with finite self-injective dimension*, Comm Alg., 1979, 7(4),393-414.

[13] M. Auslander and M. Bridger, *Stabe Module Theory*, Memoirs Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.