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Extended Riemann-Liouville type fractional derivative operator with applications

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Abstract: The main purpose of this paper is to introduce a class of new extended forms of the beta function, Gauss hypergeometric function and Appell-Lauricella hypergeometric functions by means of the modified Bessel function of the third kind. Some typical generating relations for these extended hypergeometric functions are obtained by defining the extension of the Riemann-Liouville fractional derivative operator. Their connections with elementary functions and Fox's H-function are also presented.

Keywords: Gamma function, Extended beta function, Riemann-Liouville fractional derivative, Hypergeometric functions, Fox H-function, Generating functions, Mellin transform, Integral representations

MSC: 26A33, 33B15, 33B20, 33C05, 33C20, 33C65

1 Introduction

Extensions and generalizations of some known special functions are important both from the theoretical and applied point of view. Also many extensions of fractional derivative operators have been developed and applied by many authors (see [2–6, 11, 12, 19–21] and [17, 18]). These new extensions have proved to be very useful in various fields such as physics, engineering, statistics, actuarial sciences, economics, finance, survival analysis, life testing and telecommunications. The above-mentioned applications have largely motivated our present study.

The extended incomplete gamma functions constructed by using the exponential function are defined by

$$(\alpha, z; p) = \int_0^z t^{\alpha-1} \exp\left(-t - \frac{p}{t}\right) \mathrm{d}t \qquad (\Re(p) > 0;\ \ p = 0, \Re(\alpha) > 0) \tag{1}$$

and

$$\Gamma(\alpha, z; p) = \int_z^\infty t^{\alpha-1} \exp\left(-t - \frac{p}{t}\right) \mathrm{d}t \qquad (\Re(p) \geq 0) \tag{2}$$

with $|\arg z| < \pi$, which have been studied in detail by Chaudhry and Zubair (see, for example, [2] and [4]). The extended incomplete gamma functions $\gamma(\alpha, z; p)$ and $\Gamma(\alpha, z; p)$ satisfy the following decomposition formula

$$(\alpha, z; p) + \Gamma(\alpha, z; p) = \Gamma_p(\alpha) = 2p^{\alpha/2} K_\alpha\left(2\sqrt{p}\right) \qquad (\Re(p) > 0)\,, \tag{3}$$

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where $\Gamma_p(\alpha)$ is called extended gamma function, and $K_\alpha(z)$ is the modified Bessel function of the third kind, or the Macdonald function with its integral representation given by (see [7])

$$K_\alpha(z) = \frac{1}{2}\int_0^\infty \exp\left[-\kappa(z|t)\right]\frac{\mathrm{d}t}{t^{\alpha+1}}, \tag{4}$$

where $\Re(z) > 0$ and

$$\kappa(z|t) = \frac{z}{2}\left(t + \frac{1}{t}\right). \tag{5}$$

For $\alpha = \frac{1}{2}$, we have

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{6}$$

Instead of using the exponential function, Chaudhry and Zubair extended (1) and (2) in the following form (see [3], see also [4])

$$\gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}}\int_0^x t^{\alpha-\frac{3}{2}}\exp(-t)\,K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right)\mathrm{d}t \tag{7}$$

and

$$\Gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}\exp(-t)\,K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right)\mathrm{d}t, \tag{8}$$

where $\Re(x) > 0$, $\Re(p) > 0$, $-\infty < \alpha < \infty$.

Inspired by their construction of (7) and (8), we aim to introduce a class of new special functions and fractional derivative operator by suitably using the modified Bessel function $K_\alpha(z)$.

The present paper is organized as follows: In Section 2, we first define the extended beta function and study some of its properties such as different integral representations and its Mellin transform. Then some extended hypergeometric functions are introduced by using the extended beta function. The extended Riemann-Liouville type fractional derivative operator and its properties are given in Section 3. In Section 4, the linear and bilinear generating relations for the extended hypergeometric functions are derived. Finally, the Mellin transforms of the extended fractional derivative operator are determined in Section 5.

2 Extended beta and hypergeometric functions

This section is divided into two subsections. In subsection-1, we define the extended beta function $B_\mu(x, y; p; m)$ and study some of its properties. In subsection-2, we introduce the extended Gauss hypergeometric function $F_\mu(a, b; c; z; p; m)$, the Appell hypergeometric functions $F_{1,\mu}, F_{2,\mu}$ and the Lauricella hypergeometric function $F_{3,\mu}^D$ and then obtain their integral representations. Throughout the present study, we shall assume that $\Re(p) > 0$ and $m > 0$.

2.1 Extended beta function

Definition 2.1. *The extended beta function $B_\mu(x, y; p; m)$ with $\Re(p) > 0$ is defined by*

$$B_\mu(x, y; p; m) := \sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\,K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)\mathrm{d}t, \tag{9}$$

where $x, y \in \mathbb{C}$, $m > 0$ and $\Re(\mu) \geq 0$.

Remark 2.2. *Taking $m = 1$, $\mu = 0$ and making use of* (6), (9) *reduces to the extended beta function $B_\mu(x, y; p)$ defined by Chaudhry* et al. *[5, Eq. (1.7)]*

$$B(x,y;p) := B_0(x,y;p;1) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)\mathrm{d}t, \tag{10}$$

where $\Re(p) > 0$ and $x, y \in \mathbb{C}$.

Theorem 2.3. *The following integral representations for the extended beta functions $B_\mu(x, y; p; m)$ with $\Re(p) > 0$ are valid*

$$B_\mu(x,y;p;m) = 2\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}} \cos^{2(x-1)}\theta\,\sin^{2(y-1)}\theta\,K_{\mu+\frac{1}{2}}\left(p\sec^{2m}\theta\,\csc^{2m}\theta\right)\mathrm{d}\theta \tag{11}$$

$$= \sqrt{\frac{2p}{\pi}}\int_0^{\infty}\frac{u^{x-\frac{3}{2}}}{(1+u)^{x+y-1}}K_{\mu+\frac{1}{2}}\left(p\frac{(1+u)^{2m}}{u^m}\right)\mathrm{d}u \tag{12}$$

$$= 2^{2-x-y}\sqrt{\frac{2p}{\pi}}\int_{-1}^{1}(1+u)^{x-\frac{3}{2}}(1-u)^{y-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{2^{2m}p}{(1-u^2)^m}\right)\mathrm{d}u. \tag{13}$$

Proof. These formulas can be obtained by using the transformations $t = \cos^2\theta$, $t = \frac{u}{1+u}$ and $t = \frac{1+u}{2}$ in (9), respectively. □

Theorem 2.4. *The following expression holds true*

$$B_\mu(x,y;p,m) = \frac{1}{2}\sqrt{\frac{2p}{\pi}}\int_0^{\infty} B_0\left(x-\frac{m}{2}, y-\frac{m}{2}; \kappa(p|u); m\right)\frac{\mathrm{d}u}{u^{\mu+\frac{3}{2}}}, \tag{14}$$

$$(x, y \in \mathbb{C},\ m > 0,\ \Re(p) > 0,\ \Re(\mu) \geq 0)$$

where $\kappa(p|u)$ is given by (5).

Proof. Expressing $B_\mu(x, y; p, m)$ in its integral form with the help of (9), and taking (4) into account, we obtain

$$\begin{aligned}
B_\mu(x,y;p,m) &= \sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)\mathrm{d}t \\
&= \frac{1}{2}\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\left\{\int_0^{\infty}\exp\left[-\frac{\kappa(p|u)}{t^m(1-t)^m}\right]\frac{\mathrm{d}u}{u^{\mu+\frac{3}{2}}}\right\}\mathrm{d}t \\
&= \frac{1}{2}\sqrt{\frac{2p}{\pi}}\int_0^{\infty}\left\{\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\exp\left[-\frac{\kappa(p|u)}{t^m(1-t)^m}\right]\mathrm{d}t\right\}\frac{\mathrm{d}u}{u^{\mu+\frac{3}{2}}},
\end{aligned} \tag{15}$$

where $\kappa(p|u)$ is given by (5).

In order to write the inner integral as our extended beta function, we need the following variant of (6), that is,

$$e^{-z} = \sqrt{\frac{2}{\pi}z}\,K_{\frac{1}{2}}(z).$$

Then, we have

$$\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\exp\left[-\frac{\kappa(p|u)}{t^m(1-t)^m}\right]\mathrm{d}t$$

$$= \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} \sqrt{\frac{2}{\pi} \frac{\kappa(p|u)}{t^m (1-t)^m}} K_{\frac{1}{2}} \left(\frac{\kappa(p|u)}{t^m (1-t)^m} \right) \mathrm{d}t$$

$$= \sqrt{\frac{2\kappa(p|u)}{\pi}} \int_0^1 t^{x-\frac{m}{2}-\frac{3}{2}} (1-t)^{y-\frac{m}{2}-\frac{3}{2}} K_{\frac{1}{2}} \left(\frac{\kappa(p|u)}{t^m (1-t)^m} \right) \mathrm{d}t$$

$$= B_0 \left(x - \frac{m}{2}, y - \frac{m}{2}; \kappa(p|u); m \right). \tag{16}$$

Substituting (16) into (15) we obtain the required result (14). □

Remark 2.5. *It is interesting to note that using the definition of the extended beta function (see, [11, 12]) we can get the following expression for* (9)

$$B_\mu(x, y; p, m) = \frac{1}{2} \sqrt{\frac{2p}{\pi}} \int_0^\infty B_{\kappa(p|u);m,m} \left(x - \frac{1}{2}, y - \frac{1}{2} \right) \frac{\mathrm{d}u}{u^{\mu+\frac{3}{2}}}, \tag{17}$$

where the function $B_{b;\rho,\lambda}(x, y)$ *is given by [12, p. 631, Eq. (2)]*

$$B_{b;\rho,\lambda}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{b}{t^\rho (1-t)^\lambda} \right) \mathrm{d}t \quad (x, y \in \mathbb{C}).$$

The following theorem establishes the relation between the Mellin transform

$$\mathfrak{M}\{f(x); x \to s\} = \int_0^\infty x^{s-1} f(x)\, \mathrm{d}x$$

and the extended beta function.

Theorem 2.6. *Let* $x, y \in \mathbb{C}$, $m > 0$, $\Re(\mu) \geq 0$ *and*

$$\Re(s) > \max\left\{ \Re(\mu), -\frac{1}{2} + \frac{1}{2m} - \frac{\Re(x)}{m}, -\frac{1}{2} + \frac{1}{2m} - \frac{\Re(y)}{m} \right\}.$$

Then we have the following relation

$$\mathfrak{M}\{B_\mu(x, y; p; m) : p \to s\}$$
$$= \frac{1}{2^\mu} \frac{\Gamma(s+\mu)\, \Gamma\left(\frac{s}{2} - \frac{\mu}{2}\right) \Gamma\left(x + ms + \frac{m-1}{2}\right) \Gamma\left(y + ms + \frac{m-1}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{\mu}{2}\right) \Gamma(x + y + 2ms + m - 1)}$$
$$= \frac{1}{2^\mu} \frac{\Gamma(s+\mu)\, \Gamma\left(\frac{s}{2} - \frac{\mu}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{\mu}{2}\right)} B\left(x + ms + \frac{m-1}{2}, y + ms + \frac{m-1}{2} \right). \tag{18}$$

Proof. First, we have

$$\mathfrak{M}\{B_\mu(x, y; p; m) : p \to s\}$$
$$= \int_0^\infty p^{s-1} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \mathrm{d}t\mathrm{d}p$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} \left[\int_0^\infty p^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \mathrm{d}p \right] \mathrm{d}t$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}} (1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}}\, \mathrm{d}t \int_0^\infty u^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}}(u)\, \mathrm{d}u$$
$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(x + ms + \frac{m-1}{2}\right) \Gamma\left(y + ms + \frac{m-1}{2}\right)}{\Gamma(x + y + 2ms + m - 1)} \int_0^\infty u^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}}(u)\, \mathrm{d}u. \tag{19}$$

Since the Mellin transform of the Macdonald function $K_\nu(z)$ is given by [9, p. 37, Eq.(1.7.41)]:

$$\mathfrak{M}\{K_\nu(z): z \to s\} = 2^{s-2}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right), \tag{20}$$

the last integral in (19) can be evaluated as

$$\int_0^\infty u^{s+\frac{1}{2}-1}K_{\mu+\frac{1}{2}}(u)\,\mathrm{d}u = 2^{s-\frac{3}{2}}\Gamma\left(\frac{s}{2}+\frac{\mu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right) = 2^{-\frac{1}{2}-\mu}\sqrt{\pi}\frac{\Gamma(s+\mu)\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)}, \tag{21}$$

where we have used

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Finally, we get

$$\mathfrak{M}\{B_\mu(x,y;p;m): p \to s\} = \frac{1}{2^\mu}\frac{\Gamma(s+\mu)\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)\Gamma\left(x+ms+\frac{m-1}{2}\right)\Gamma\left(y+ms+\frac{m-1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)\Gamma(x+y+2ms+m-1)}. \qquad \square$$

Now we can derive the Fox H-function representation of the extended beta function defined in (9).

Let m, n, p, q be integers such that $0 \le m \le q$, $0 \le n \le p$, and for parameters $a_i, b_i \in \mathbb{C}$ and for parameters $\alpha_i, \beta_j \in \mathbb{R}^+$ $(i = 1, \dots, p;\ j = 1, \dots, q)$, the H-function is defined in terms of a Mellin-Barnes integral in the following manner ([8, pp. 1–2]; see also [10, p. 343, Definition E.1.] and [13, p. 2, Definition 1.1.]):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}}\Theta(s)z^{-s}\mathrm{d}s, \tag{22}$$

where

$$\Theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j+\beta_j s)\prod_{i=1}^n \Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i+\alpha_i s)\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j s)}, \tag{23}$$

with the contour $\mathfrak{L}$ suitably chosen. As convention, the empty product is equal to one. The theory of the H-function is well explained in the books of Mathai [14], Mathai and Saxena ([15], Ch.2), Srivastava, Gupta and Goyal ([22], Ch.1) and Kilbas and Saigo ([8], Ch.1 and Ch.2). Note that (18) and (20) mean $H_{p,q}^{m,n}(z)$ in (22) is the inverse Mellin transform of $\Theta(s)$ in (23).

Theorem 2.7. *Let $\Re(p) > 0$, $x, y \in \mathbb{C}$, $m > 0$ and $\Re(\mu) \ge 0$, then*

$$B_\mu(x,y;p;m) = \frac{1}{2^\mu}H_{2,4}^{4,0}\left[p\left|\begin{matrix}\left(\frac{\mu}{2},\frac{1}{2}\right),(x+y+m-1,2m)\\ \\ (\mu,1),\left(-\frac{\mu}{2},\frac{1}{2}\right),\left(x+\frac{m-1}{2},m\right),\left(y+\frac{m-1}{2},m\right)\end{matrix}\right.\right],$$

where $B_\mu(x,y;p;m)$ is as defined in (9).

Proof. The result is obtained by taking the inverse Mellin transform of (18) in Theorem 2.6 and using (22) and (23). □

2.2 Extended hypergeometric functions

Definition 2.8. *The extended Gauss hypergeometric function $F_\mu(a,b;c;z;p;m)$ is defined by*

$$F_\mu(a,b;c;z;p;m) := \sum_{n=0}^\infty (a)_n\frac{B_\mu(b+n,c-b;p;m)}{B(b,c-b)}\frac{z^n}{n!}, \tag{24}$$

where $\Re(p) > 0$, $\Re(\mu) \ge 0$, $0 < \Re(b) < \Re(c)$ and $|z| < 1$.

Definition 2.9. *The extended Appell hypergeometric function $F_{1,\mu}$ is defined by*

$$F_{1,\mu}(a,b,c;d;x,y;p;m) := \sum_{n,k=0}^{\infty} (b)_n (c)_k \frac{B_\mu(a+n+k,d-a;p;m)}{B(a,d-a)} \frac{x^n}{n!} \frac{y^k}{k!}, \tag{25}$$

where $\Re(p) > 0$, $\Re(\mu) \geq 0$, $0 < \Re(a) < \Re(d)$ *and* $|x| < 1$, $|y| < 1$.

Definition 2.10. *The extended Appell hypergeometric function $F_{2,\mu}$ is defined by*

$$F_{2,\mu}(a,b,c;d,e;x,y;p;m) := \sum_{n,k=0}^{\infty} (a)_{n+k} \frac{B_\mu(b+n,d-b;p;m)}{B(b,d-b)} \frac{B_\mu(c+k,e-c;p;m)}{B(c,e-c)} \frac{x^n}{n!} \frac{y^k}{k!}, \tag{26}$$

where $\Re(p) > 0$, $\Re(\mu) \geq 0$, $0 < \Re(b) < \Re(d)$, $0 < \Re(c) < \Re(e)$ *and* $|x| + |y| < 1$.

Definition 2.11. *The extended Lauricella hypergeometric function $F_{D,\mu}^3$ is*

$$F_{D,\mu}^3(a,b,c,d;e;x,y,z;p;m) := \sum_{n,k,r=0}^{\infty} (b)_n (c)_k (d)_r \frac{B_\mu(a+n+k+r,e-a;p;m)}{B(a,e-a)} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^r}{r!}, \tag{27}$$

where $\Re(p) > 0$, $\Re(\mu) \geq 0$, $0 < \Re(a) < \Re(e)$ *and* $|x| < 1$, $|y| < 1$, $|z| < 1$.

Here, it is important to mention that when we take $m = 1$, $\mu = 0$ and then letting $p \to 0$, function (24) reduces to the ordinary Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;z) \equiv {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{28}$$

where $(x)_n$ denotes the Pochhammer symbol defined, in terms of the familiar gamma function, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0;\ x \in \mathbb{C} \setminus \{0\}) \\ x(x+1)\cdots(x+n-1) & (n \in \mathbb{N};\ x \in \mathbb{C}). \end{cases}$$

For conditions of convergence and other related details of this function, see [1], [9] and [16]. Similarly, we can reduce the functions (25), (26) and (27) to the well-known Appell functions F_1, F_2 and Lauricella function F_D^3, respectively (see [16] and [23]).

Now, we establish the integral representations of the extended hypergeometric functions given by (24), (25), (26) and (27) as follows.

Theorem 2.12. *The following integral representation for the extended Gauss hypergeometric function $F_\mu(a,b;c;z;p;m)$ is valid*

$$F_\mu(a,b;c;z;p;m) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b,c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) \mathrm{d}t, \tag{29}$$

where $|\arg(1-z)| < \pi$, $\Re(p) > 0$, $m > 0$ *and* $\Re(\mu) \geq 0$.

Proof. By using (9) and employing the binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} \qquad (|zt| < 1), \tag{30}$$

we get the above integral representation. □

Theorem 2.13. *The following integral representation for the extended hypergeometric function $F_{1,\mu}$ is valid*

$$F_{1,\mu}(a,b,c;d;x,y;p;m) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(a,d-a)} \times \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} (1-xt)^{-b} (1-yt)^{-c} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) dt, \tag{31}$$

Proof. For simplicity, let $\mathfrak{I}$ denote the left-hand side of (31). Then, using (25) yields

$$\mathfrak{I} = \sum_{n,k=0}^{\infty} (b)_n (c)_k \frac{B_\mu(a+n+k, d-a; p; m)}{B(a, d-a)} \frac{x^n}{n!} \frac{y^k}{k!}, \tag{32}$$

By applying (9) to the integrand of (31), after a little simplification, we have

$$\mathfrak{I} = \sum_{n,k=0}^{\infty} \left\{ \sqrt{\frac{2p}{\pi}} \int_0^1 t^{a+n+k-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \mathrm{d}t \right\} \frac{(b)_n (c)_k}{B(a, d-a)} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{33}$$

By interchanging the order of summation and integration in (33), we get

$$\begin{aligned} \mathfrak{I} &= \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)} \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \\ &\quad \times \left\{ \sum_{n=0}^{\infty} \frac{(b)_n}{n!} (xt)^n \right\} \left\{ \sum_{k=0}^{\infty} \frac{(c)_k}{k!} (yt)^k \right\} \mathrm{d}t \\ &= \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)} \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} \\ &\quad \times (1-xt)^{-b} (1-yt)^{-c} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \mathrm{d}t, \end{aligned} \tag{34}$$

which proves the integral representation (31). □

To establish Theorem 2.13, we need to recall the following elementary series identity involving the bounded sequence of $\{f(N)\}_{N=0}^{\infty}$ stated in the following result.

Lemma 2.14. *For a bounded sequence $\{f(N)\}_{N=0}^{\infty}$ of essentially arbitrary complex numbers, we have*

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n+k) \frac{x^n}{n!} \frac{y^k}{k!}. \tag{35}$$

Theorem 2.15. *The following integral representation for the extended hypergeometric function $F_{2,\mu}$ is valid*

$$\begin{aligned} F_{2,\mu}(a, b, c; d, e; x, y; p; m) &= \frac{2p}{\pi} \frac{1}{B(b, d-b) B(c, e-c)} \\ &\quad \times \int_0^1 \int_0^1 t^{b-\frac{3}{2}} (1-t)^{d-b-\frac{3}{2}} w^{b-\frac{3}{2}} (1-w)^{e-c-\frac{3}{2}} (1-xt-yw)^{-a} \\ &\quad \times K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) K_{\mu+\frac{1}{2}} \left(\frac{p}{w^m (1-w)^m} \right) \mathrm{d}t\mathrm{d}w. \end{aligned} \tag{36}$$

Proof. Let $\mathcal{L}$ denote the left-hand side of (36). Then, using (26) yields

$$\mathcal{L} = \sum_{n,k=0}^{\infty} (a)_{n+k} \frac{B_\mu(b+n, d-b; p; m)}{B(b, d-b)} \frac{B_\mu(c+k, e-c; p; m)}{B(c, e-c)} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{37}$$

By applying (9) to the integrand of (32), we have

$$\begin{aligned} \mathcal{L} &= \frac{2p}{\pi} \sum_{n,k=0}^{\infty} \left\{ \int_0^1 t^{b+n-\frac{3}{2}} (1-t)^{d-b-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) \mathrm{d}t \right\} \\ &\quad \times \left\{ \int_0^1 w^{b+n-\frac{3}{2}} (1-w)^{e-c-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{w^m (1-w)^m} \right) \mathrm{d}w \right\} \end{aligned}$$

$$\times \frac{(a)_{n+k}}{B(b,d-b)\,B(c,e-c)} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{38}$$

Next, interchanging the order of summation and integration in (38), which is guaranteed, yields

$$\begin{aligned} \mathcal{L} = & \frac{2p}{\pi} \frac{1}{B(b,d-b)\,B(c,e-c)} \int_0^1 \int_0^1 t^{b-\frac{3}{2}} (1-t)^{d-b-\frac{3}{2}} w^{b-\frac{3}{2}} (1-w)^{e-c-\frac{3}{2}} \\ & \times K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) K_{\mu+\frac{1}{2}}\left(\frac{p}{w^m(1-w)^m}\right) \\ & \times \left(\sum_{n,k=0}^{\infty} (a)_{n+k} \frac{(xt)^n}{n!} \frac{(yw)^k}{k!}\right) \mathrm{d}t\mathrm{d}w. \end{aligned} \tag{39}$$

Finally, applying (35) to the double series in (39), we obtain the right-hand side of (36). □

Theorem 2.16. *The following integral representation for the extended hypergeometric function $F_{D,\mu}^3$ is valid*

$$\begin{aligned} F_{D,\mu}^3(a,b,c,d;e;x,y,z;p;m) = & \frac{1}{B(a,e-a)} \sqrt{\frac{2p}{\pi}} \\ & \times \int_0^1 \frac{t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}}{(1-xt)^b\,(1-yt)^c\,(1-zt)^d} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) \mathrm{d}t \end{aligned} \tag{40}$$

Proof. A similar argument in the proof of Theorem 2.15 will be able to establish the integral representation in (40). Therefore, details of the proof are omitted. □

3 Extended Riemann-Liouville fractional derivative operator

We first recall that the classical Riemann-Liouville fractional derivative is defined by (see [23, p. 286])

$$D_z^{\nu} f(z) := \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{-\nu-1} f(t)\,\mathrm{d}t,$$

where $\Re(\nu) < 0$ and the integration path is a line from 0 to z in the complex t-plane. It coincides with the fractional integral of order $-\nu$. In case $m-1 < \Re(\nu) < m$, $m \in \mathbb{N}$, it is customary to write

$$D_z^{\nu} f(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m} D_z^{\nu-m} f(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m} \left\{ \frac{1}{\Gamma(m-\nu)} \int_0^z (z-t)^{m-\nu-1} f(t)\,\mathrm{d}t \right\}.$$

We present the following new extended Riemann-Liouville-type fractional derivative operator.

Definition 3.1. *The extended Riemann-Liouville fractional derivative is defined as*

$$D_z^{\nu,\mu;p;m} f(z) := \frac{1}{\Gamma(-\nu)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-\nu-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) \mathrm{d}t, \tag{41}$$

where $\Re(\nu) < 0$, $\Re(p) > 0$, $\Re(m) > 0$ and $\Re(\mu) \geq 0$.

For $n-1 < \Re(\nu) < n$, $n \in \mathbb{N}$, we write

$$D_z^{\nu,\mu;p;m} f(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n} D_z^{\nu-n,\mu;p;m} f(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left\{ \frac{1}{\Gamma(n-\nu)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{n-\nu-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) \mathrm{d}t \right\}. \tag{42}$$

Remark 3.2. *If we take $m = 0$, $\mu = 0$, and $p \to 0$, then the above extended Riemann-Liouville fractional derivative operator reduces to the classical Riemann-Liouville fractional derivative operator.*

Now, we begin our investigation by calculating the extended fractional derivatives of some elementary functions. For our purpose, we first establish two results involving the extended Riemann-Liouville fractional derivative operator.

Lemma 3.3. *Let* $\Re(\nu) < 0$, *then we have*

$$D_z^{\nu,\mu;p;m}\left\{z^{\lambda}\right\} = \frac{z^{\lambda-\nu}}{\Gamma(-\nu)} B_{\mu}\left(\lambda + \frac{3}{2}, -\nu + \frac{1}{2}; p; m\right). \tag{43}$$

Proof. Using Definition 3.1 and 1, we have

$$\begin{aligned} D_z^{\nu,\mu;p;m}\left\{z^{\lambda}\right\} &= \frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\nu-1} t^{\lambda} K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) \mathrm{d}t \\ &= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^1 (1-u)^{(-\nu+\frac{1}{2})-\frac{3}{2}} u^{(\lambda+\frac{3}{2})-\frac{3}{2}} K_{\mu+\frac{1}{2}}\left(\frac{p}{u^m(1-u)^m}\right) \mathrm{d}u \\ &= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)} B_{\mu}\left(\lambda + \frac{3}{2}, -\nu + \frac{1}{2}; p; m\right). \end{aligned}$$ □

Next, we apply the extended Riemann-Liouville fractional derivative to a function $f(z)$ analytic at the origin.

Lemma 3.4. *Let* $\Re(\nu) < 0$ *and suppose that a function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < \rho)$ *for some* $\rho \in \mathbb{R}_+$. *Then we have*

$$D_z^{\nu,\mu;p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu;p;m}\left\{z^n\right\}.$$

Proof. Using Definition 3.1 to the function $f(z)$ with its series expansion, we have

$$D_z^{\nu,\mu;p;m}\{f(z)\} = \frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\nu-1} K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)\sum_{n=0}^{\infty} a_n t^n \, \mathrm{d}t.$$

Since the power series converges uniformly on any closed disk centered at the origin with its radius smaller than ρ, so does the series on the line segment from 0 to a fixed z for $|z| < \rho$. This fact guarantees term-by-term integration as follows:

$$D_z^{\nu,\mu;p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n \left\{\frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\nu-1} K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) t^n \, \mathrm{d}t\right\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu;p;m}\left\{z^n\right\}.$$ □

As a consequence we have the following result.

Theorem 3.5. *Let* $\Re(\nu) < 0$ *and suppose that a function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < \rho)$ *for some* $\rho \in \mathbb{R}_+$. *Then we have*

$$D_z^{\nu,\mu;p;m}\left\{z^{\lambda-1} f(z)\right\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu;p;m}\left\{z^{\lambda+n-1}\right\} = \frac{z^{\lambda-\nu-1}}{\Gamma(-\nu)}\sum_{n=0}^{\infty} a_n B_{\mu}\left(\lambda + n + \frac{1}{2}, -\nu + \frac{1}{2}; p; m\right) z^n.$$

We present two subsequent theorems which may be useful to find certain generating function.

Theorem 3.6. *For* $\Re(\nu) > \Re(\lambda) > -\frac{1}{2}$, *we have*

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)} B\left(\lambda + \frac{1}{2}, \nu - \lambda + \frac{1}{2}\right) F_{\mu}\left(\alpha, \lambda + \frac{1}{2}; \nu + 1; z; p; m\right) \quad (|z| < 1;\ \alpha \in \mathbb{C}). \tag{44}$$

Proof. Using (30) and applying Lemmas 3.3 and 3.4, we obtain

$$D_z^{\lambda-\nu,\mu;p;m}\{z^{\lambda-1}(1-z)^{-\alpha}\} = D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}\sum_{l=0}^{\infty}(\alpha)_l\frac{z^l}{l!}\right\}$$
$$= \sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!}D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda+l-1}\right\}$$
$$= \sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!}\frac{B_\mu(\lambda+l+\frac{1}{2},\nu-\lambda+\frac{1}{2};p;m)}{\Gamma(\nu-\lambda)}z^{\nu+l-1}.$$

By using (24), we can get

$$D_z^{\lambda-\nu,\mu;p;m}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)F_\mu\left(\alpha,\lambda+\frac{1}{2};\nu+1;z;p;m\right).$$ □

Theorem 3.7. *Let* $\Re(\nu)>\Re(\lambda)>-\frac{1}{2},\Re(\alpha)>0,\Re(\beta)>0;|az|<1$ *and* $|bz|<1$. *Then we have*

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$
$$= \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)F_{1,\mu}\left(\lambda+\frac{1}{2},\alpha,\beta;\nu+1;az,bz;p;m\right). \tag{45}$$

Proof. Use the binomial theorem for $(1-az)^{-\alpha}$ and $(1-bz)^{-\beta}$. Apply Lemmas 3.3 and 3.4 to obtain

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$
$$= D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_l(\beta)_k\frac{(az)^l}{l!}\frac{(bz)^k}{k!}\right\}$$
$$= \sum_{l,k=0}^{\infty}(\alpha)_l(\beta)_k D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda+l+k-1}\right\}\frac{(a)^l}{l!}\frac{(b)^k}{k!}$$
$$= z^{\nu-1}\sum_{l,k=0}^{\infty}(\alpha)_l(\beta)_k\frac{B_\mu\left(\lambda+l+k+\frac{1}{2},\nu-\lambda+\frac{1}{2};p;m\right)}{\Gamma(\nu-\lambda)}\frac{(az)^n}{l!}\frac{(bz)^k}{k!}.$$

By using (25), we get

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$
$$= \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)F_{1,\mu}\left(\lambda+\frac{1}{2},\alpha,\beta;\nu+1;az,bz;p;m\right).$$ □

Theorem 3.8. *Let* $\Re(\nu)>\Re(\lambda)>-\frac{1}{2},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,|az|<1,|bz|<1$ *and* $|cz|<1$. *Then we have*

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}$$
$$\times B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)F_{D,\mu}^3\left(\lambda+\frac{1}{2},\alpha,\beta,\gamma;\nu+1;az,bz,cz;p;m\right). \tag{46}$$

Proof. As in the proof of Theorem 3.7, taking the binomial theorem for $(1-az)^{-\alpha}$, $(1-bz)^{-\beta}$ and $(1-cz)^{-\gamma}$ and applying Lemmas 3.3 and 3.4 and taking Definition 5 into account, one can easily prove Theorem 3.8. □

Theorem 3.9. *Let* $\Re(\nu)>\Re(\lambda)>-\frac{1}{2},\Re(\alpha)>0,\Re(\gamma)>\Re(\beta)>0;|\frac{x}{1-z}|<1$ *and* $|x|+|z|<1$. *Then we have*

$$D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\beta;\gamma;\frac{x}{1-z};p;m\right)\right\}$$
$$= z^{\nu-1}\frac{B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)}{\Gamma(\nu-\lambda)}F_{2,\mu}\left(\alpha,\beta,\lambda+\frac{1}{2};\gamma,\nu+1;x,z;p;m\right) \tag{47}$$

Proof. By using (30) and applying the Definition 2.8 for F_μ, we get

$$\begin{aligned}
&D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\beta;\gamma;\frac{x}{1-z};p;m\right)\right\}\\
&=D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{B_\mu(\beta+n,\gamma-\beta;p;m)}{B(\beta,\gamma-\beta)}\left(\frac{x}{1-z}\right)^n\right\}\\
&=\sum_{n=0}^{\infty}(\alpha)_n\frac{B_\mu(\beta+n,\gamma-\beta;p;m)}{B(\beta,\gamma-\beta)}D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha-n}\right\}\frac{x^n}{n!}.
\end{aligned}$$

Using Theorem 3.6 for $D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha-n}\right\}$ and interpreting the extended hypergeometric function F_μ as its series representation, we get

$$\begin{aligned}
&D_z^{\lambda-\nu,\mu;p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\beta;\gamma;\frac{x}{1-z};p;m\right)\right\}\\
&=\frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)\sum_{n,k=0}^{\infty}\left\{(\alpha)_{n+k}\frac{B_\mu\left(\beta+n,\gamma-\beta;p;m\right)}{B\left(\beta,\gamma-\beta\right)}\right.\\
&\qquad\left.\times\frac{B_\mu\left(\lambda+k+\frac{1}{2},\nu-\lambda+\frac{1}{2};p;m\right)}{B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)}\frac{x^n}{n!}\frac{z^k}{k!}\right\}\\
&=\frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B\left(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2}\right)F_{2,\mu}\left(\alpha,\beta,\lambda+\frac{1}{2};\gamma,\nu+1;x,z;p;m\right).
\end{aligned}$$

This completes the proof. □

4 Generating functions involving the extended Gauss hypergeometric function

Here, we establish some linear and bilinear generating relations for the extended hypergeometric function F_μ by using Theorems 3.6, 3.7 and 3.9.

Theorem 4.1. *Let* $\Re(\lambda)>0$ *and* $\Re(\beta)>\Re(\alpha)>-\frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_\mu\left(\lambda+n,\alpha+\frac{1}{2};\beta+1;z;p;m\right)t^n=(1-t)^{-\lambda}F_\mu\left(\lambda,\alpha+\frac{1}{2};\beta+1;\frac{z}{1-t};p;m\right). \tag{48}$$

$$(|z|<\min\{1,|1-t|\})$$

Proof. We start by recalling the elementary identity

$$[(1-z)-t]^{-\lambda}=(1-t)^{-\lambda}\left(1-\frac{z}{1-t}\right)^{-\lambda}$$

and expand its left-hand side to obtain

$$(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\lambda}\left(1-\frac{z}{1-t}\right)^{-\lambda}\quad(|t|<|1-z|).$$

Multiplying both sides of the above equality by $z^{\alpha-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D_z^{\alpha-\beta;\mu;p;m}$ on both sides, we find

$$D_z^{\alpha-\beta;\mu;p;m}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n t^n}{n!}z^{\alpha-1}(1-z)^{-\lambda-n}\right\}=D_z^{\alpha-\beta;\mu;p;m}\left\{(1-t)^{-\lambda}z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right\}.$$

Uniform convergence of the involved series makes it possible to exchange the summation and the fractional operator to give

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta;\mu;p;m}\left\{z^{\alpha-1}(1-z)^{-\lambda-n}\right\}t^n=(1-t)^{-\lambda}D_z^{\alpha-\beta;\mu;p;m}\left\{z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right\}.$$

The result then follows by applying Theorem 3.6 to both sides of the last identity. □

Theorem 4.2. *Let* $\Re(\lambda) > 0$, $\Re(\gamma) > 0$ *and* $\Re(\beta) > \Re(\alpha) > -\frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_\mu\left(\gamma - n, \alpha + \frac{1}{2}; \beta + 1; z; p; m\right) t^n$$
$$= (1-t)^{-\lambda} F_{1,\mu}\left(\alpha + \frac{1}{2}, \gamma, \lambda; \beta + 1; z; \frac{-zt}{1-t}; p; m\right).$$
$$(|z| < 1;\ \ |t| < |1-z|;\ \ |z||t| < |1-t|)$$

Proof. Considering the following identity

$$[1-(1-z)t]^{-\lambda} = (1-t)^{-\lambda}\left(1 + \frac{zt}{1-t}\right)^{-\lambda}$$

and expanding its left-hand side as a power series, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^n t^n = (1-t)^{-\lambda}\left(1 - \frac{-zt}{1-t}\right)^{-\lambda} \quad (|t| < |1-z|).$$

Multiplying both sides by $z^{\alpha-1}(1-z)^{-\gamma}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D_z^{\alpha-\beta;\mu;p;m}$ on both sides, we find

$$D_z^{\alpha-\beta;\mu;p;m}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1}(1-z)^{-\gamma}(1-z)^n t^n\right\}$$
$$= D_z^{\alpha-\beta;\mu;p;m}\left\{(1-t)^{-\lambda} z^{\alpha-1}(1-z)^{-\gamma}\left(1 - \frac{-zt}{1-t}\right)^{-\lambda}\right\}.$$

The given conditions are found to allow us to exchange the order of the summation and the fractional derivative to yield

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta;\mu;p;m}\left\{z^{\alpha-1}(1-z)^{-\gamma+n}\right\} t^n$$
$$= (1-t)^{-\lambda} D_z^{\alpha-\beta;\mu;p;m}\left\{z^{\alpha-1}(1-z)^{-\gamma}\left(1 - \frac{-zt}{1-t}\right)^{-\lambda}\right\}.$$

Finally the result follows by using Theorems 3.6 and 3.7. □

Theorem 4.3. *Let* $\Re(\xi) > \Re(\delta) > -\frac{1}{2}$, $\Re(\beta) > \Re(\alpha) > -\frac{1}{2}$ *and* $\Re(\lambda) > 0$. *Then we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_\mu\left(\lambda + n, \alpha + \frac{1}{2}; \beta + 1; z; p; m\right) F_\mu\left(-n, \delta + \frac{1}{2}; \xi + 1; u; p; m\right) t^n$$
$$= (1-t)^{-\lambda} F_{2,\mu}\left(\lambda, \alpha + \frac{1}{2}, \delta + \frac{1}{2}; \beta + 1, \xi + 1; \frac{z}{1-t}, \frac{-ut}{1-t}; p; m\right).$$
$$\left(|z| < 1;\ \ \left|\frac{1-u}{1-z}t\right| < 1;\ \ \left|\frac{z}{1-t}\right| + \left|\frac{ut}{1-t}\right| < 1\right)$$

Proof. Replacing t by $(1-u)\,t$ in (48) and multiplying both sides of the resulting identity by $u^{\delta-1}$ gives

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_\mu\left(\lambda + n, \alpha + \frac{1}{2}; \beta + 1; z; p; m\right) u^{\delta-1}(1-u)^n t^n$$
$$= u^{\delta-1}[1-(1-u)\,t]^{-\lambda} F_\mu\left(\lambda, \alpha + \frac{1}{2}; \beta + 1; \frac{z}{1-(1-u)t}; p; m\right).$$
$$\left(\Re(\lambda) > 0,\ \ \Re(\beta) > \Re(\alpha) > -\frac{1}{2}\right)$$

Applying the fractional derivative $D_u^{\delta-\xi,\mu;p;m}$ to both sides of the resulting identity and changing the order of the summation and the fractional derivative yields

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_\mu\left(\lambda+n,\alpha+\frac{1}{2};\beta+1;z;p;m\right)D_u^{\delta-\xi,\mu;p;m}\left\{u^{\delta-1}(1-u)^n\right\}t^n$$
$$=D_u^{\delta-\xi,\mu;p;m}\left\{u^{\delta-1}[1-(1-u)t]^{-\lambda}F_\mu\left(\lambda,\alpha+\frac{1}{2};\beta+1;\frac{z}{1-(1-u)t};p;m\right)\right\}.$$
$$(|(1-u)t|<1;\ \ |ut|<|1-t|)$$

The last identity can be written as follows:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_\mu\left(\lambda+n,\alpha+\frac{1}{2};\beta+1;z;p;m\right)D_u^{\delta-\xi,\mu;p;m}\left\{u^{\delta-1}(1-u)^n\right\}t^n$$
$$=(1-t)^{-\lambda}\ D_u^{\delta-\xi,\mu;p;m}\left\{u^{\delta-1}\left[1-\frac{-ut}{1-t}\right]^{-\lambda}F_\mu\left(\lambda,\alpha+\frac{1}{2};\beta+1;\frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}};p;m\right)\right\}.$$

Finally the use of Theorems 3.6 and 3.9 in the resulting identity is seen to give the desired result. □

5 Mellin transforms and further results

In this section, we first obtain the Mellin transform of the extended Riemann-Liouville fractional derivative operator.

Theorem 5.1. *Let* $\Re(\nu)<0$, $m>0$, $\Re(\mu)\geq 0$ *and*

$$\Re(s)>\max\left\{\Re(\mu),-\frac{1}{2}-\frac{1}{m}-\frac{\Re(\lambda)}{m},-\frac{1}{2}+\frac{\Re(\nu)}{m}\right\}.$$

Then we have the following relation

$$\mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{z^\lambda\right\}:s\right]=\frac{z^{\lambda-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\,\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)\Gamma\left(\lambda+ms+\frac{m}{2}+1\right)\Gamma\left(-\nu+ms+\frac{m}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)\Gamma(\lambda-\nu+2ms+m+1)}$$
$$=\frac{z^{\lambda-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\,\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)}B\left(\lambda+ms+\frac{m}{2}+1,-\nu+ms+\frac{m}{2}\right).\tag{49}$$

Proof. Taking the Mellin transform and using Lemma 3.3, we have

$$\mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{z^\lambda\right\}:s\right]=\int_0^\infty p^{s-1}D_z^{\nu,\mu;p;m}\left\{z^\lambda\right\}\mathrm{d}p=\frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\int_0^\infty p^{s-1}B_\mu\left(\lambda+\frac{3}{2},-\nu+\frac{1}{2};p;m\right)\mathrm{d}p.$$

Applying Theorem 2.6 to the last integral yields the desired result. □

Theorem 5.2. *Let* $\Re(\nu)<0$, $m>0$, $\Re(\mu)\geq 0$, $|z|<1$ *and*

$$\Re(s)>\max\left\{\Re(\mu),-\frac{1}{2}-\frac{1}{m},-\frac{1}{2}+\frac{\Re(\nu)}{m}\right\}.$$

Then we have the following relation

$$\mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{(1-z)^{-\alpha}\right\}:s\right]=\frac{z^{-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\,\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)}B\left(ms+\frac{m}{2}+1,-\nu+ms+\frac{m}{2}\right)$$
$$\times\,{}_2F_1\left(\alpha,ms+\frac{m}{2}+1;-\nu+2ms+m+1;z\right),\tag{50}$$

where ${}_2F_1$ *is a well known Gauss hypergeometric function given by* (28).

Proof. Using the binomial series for $(1-z)^{-\lambda}$ and Theorem 5.1 with $\lambda = n$ yields

$$\mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{(1-z)^{-\alpha}\right\}:s\right] = \mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n\right\}:s\right]$$
$$= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathfrak{M}\left[D_z^{\nu,\mu;p;m}\left\{z^n\right\}:s\right]$$
$$= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{z^{n-\nu}}{2^{\mu}\Gamma(-\nu)}\frac{\Gamma(s+\mu)\Gamma\left(\frac{s}{2}-\frac{\mu}{2}\right)\Gamma\left(n+ms+\frac{m}{2}+1\right)\Gamma\left(-\nu+ms+\frac{m}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\mu}{2}\right)\Gamma\left(n-\nu+2ms+m+1\right)}.$$

Then the last expression is easily seen to be equal to the desired one. □

Now we present the extended Riemann-Liouville fractional derivative of z^{λ} in terms of the Fox H-function.

Theorem 5.3. *Let* $\Re(p) > 0$, $\Re(\mu) \geq 0$, $\Re(\nu) < 0$ *and* $m > 0$. *Then we have*

$$D_z^{\nu,\mu;p;m}\left\{z^{\lambda}\right\} = \frac{z^{\lambda-\nu}}{2^{\mu}\Gamma(-\nu)}H_{2,4}^{4,0}\left[p\left|\begin{matrix}\left(\frac{\mu}{2},\frac{1}{2}\right),(\lambda-\nu+m+1,2m)\\(\mu,1),\left(-\frac{\mu}{2},\frac{1}{2}\right),\left(\lambda+\frac{m}{2}+1,m\right),\left(-\nu+\frac{m}{2},m\right)\end{matrix}\right.\right].$$

Proof. The result can be obtained by taking the inverse Mellin transform of the result in Lemma 3.3 with the aid of Theorem 2.7. □

Applying the result in Theorem 3.3 to the Maclaurin series of e^z and the series expression of the Gauss hypergeometric function ${}_2F_1$ gives the extended Riemann-Liouville fractional derivatives of e^z and ${}_2F_1$ asserted by the following theorems.

Theorem 5.4. *If* $\Re(\nu) < 0$, *then we have*

$$D_z^{\nu,\mu;p;m}\{e^z\} = \frac{z^{-\nu}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}B_{\mu}\left(n+\frac{3}{2},-\nu+\frac{1}{2};p;m\right)\frac{z^n}{n!}.$$

Theorem 5.5. *If* $\Re(\nu) < 0$, *then we have*

$$D_z^{\nu,\mu;p;m}\left\{{}_2F_1(a,b;c;z)\right\} = \frac{z^{-\nu}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}B_{\mu}\left(n+\frac{3}{2},-\nu+\frac{1}{2};p;m\right)\frac{z^n}{n!}\qquad(|z|<1).$$

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