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Research Article

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Enumeration of spanning trees in the sequence of Dürer graphs

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Abstract: In this paper, we calculate the number of spanning trees in the sequence of Dürer graphs with a special feature that it has two alternate states. Using the electrically equivalent transformations, we obtain the weights of corresponding equivalent graphs and further derive relationships for spanning trees between the Dürer graphs and transformed graphs. By algebraic calculations, we obtain a closed-form formula for the number of spanning trees with regard to iteration step. Finally we compare the entropy of our graph with other studied graphs and see that its value of entropy lies in the interval of those of graphs with average degree being 3 and 4.

Keywords: Spanning trees, Electrically equivalent transformation, Entropy

MSC: 05C30, 05C50, 05C63

1 Introduction

In the study of networks, spanning trees are related to many aspects of networks, such as reliability [1, 2], consensus [3], random walks [4] and nonlinear dynamics [5, 6]. A spanning tree of a connected graph is defined as a minimal set of edges that connect all its nodes. On the other hand, enumeration of spanning trees has been applied in mathematics [7], computer science [8, 9], physics [10, 11], and chemistry [12], to name just a few.

Recently, counting the number of spanning trees has attracted increasing attention [13–16]. It is known that the number of spanning trees can be obtained by matrix-tree theorem [13]. Due to the complexity and diversity of networks, analytically enumerating spanning trees is challenging. For fractal networks, the existing works [17–20] applied the self-similarity, spanning forests, and Laplacian spectrum to obtain exact formulae of spanning trees, while for some self-similar graphs, e.g., generalized Petersen graphs, those methods are not effective. Recently, the enumeration of spanning trees has been investigated by the electrically equivalent transformations in Refs. [21–23].

A sequence of Dürer graphs is the skeleton of Dürer's solid, which belongs to generalized Petersen graphs. This graph has a special feature that it has two alternate states. Calculating the number of spanning trees in this family of graph by its laplacian spectrum does not work. To the best of our knowledge, few analytical results involve the derivation of enumeration of spanning trees. Here we employ the knowledge of electrical networks, where an edge-weighted graph is regarded as an electrical network with the weights equalling the conductances of the corresponding edges. Using the electrically equivalent transformations provided in Refs. [24, 25], we obtain some relationships for spanning trees between the original graph and transformed graphs.

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We then obtain a closed-form formula for enumeration of spanning trees. Finally we calculate the entropy of spanning trees and compare it with those of other studied graphs.

In Section 2, the construction of the sequence of Dürer graphs is presented. Section 3 shows the electrically equivalent transformations of graphs. The detailed calculations of spanning trees are provided in Section 4. Finally, the conclusions are included in the last section.

2 Model presentation

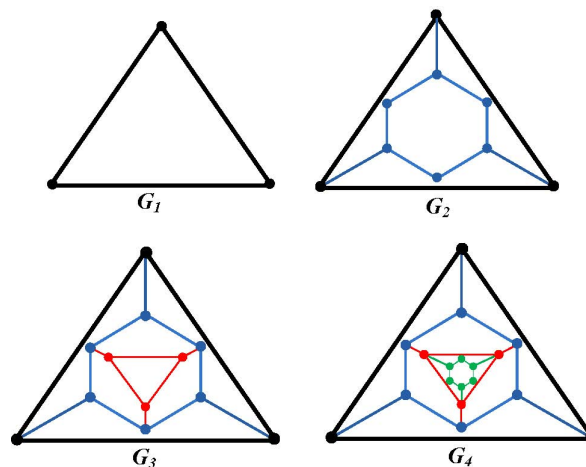
The sequence of Dürer graphs is denoted by $G_n (n \geq 1)$ after n steps, which is constructed as follows:

For $n = 1$, G_1 is a triangle.

For $n = 2$, G_2 is obtained from G_1 , where a hexagon is put into the triangle and three new adjacent nodes in the hexagon are connected to the nodes in the triangle.

For $n \geq 3$, G_n is produced by G_{n-1} . If n is an even number, a hexagon is inserted into G_{n-1} ; if n is an odd number, a triangle is embedded into G_{n-1} . Figure 1 shows this construction process.

Fig. 1. The Dürer graphs G_n produced at steps $n = 1, 2, 3$ and 4.



According to the construction, the number of total vertices V_n and edges E_n are,

$$V_n = \frac{9n}{2} - \frac{3}{2}, E_n = \frac{15n}{2} - \frac{9}{2}, n = 1, 3, \dots,$$

and

$$V_n = \frac{9n}{2}, E_n = \frac{15n}{2} - 3, n = 2, 4, \dots.$$

Then the average degree is $\langle k \rangle_n = \frac{2E_n}{V_n}$, which is approximately $\frac{10}{3}$ for a large t .

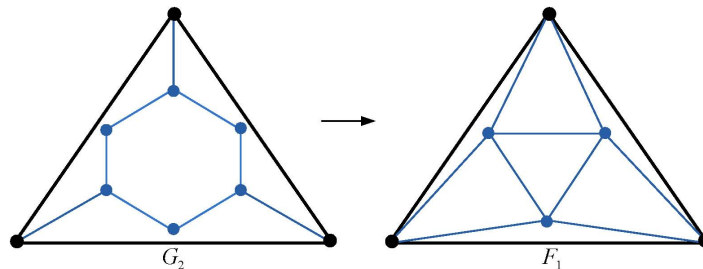
3 Electrically equivalent transformations

According to the structures of G_n , the final state is either a triangle or a hexagon. When the final state is a hexagon, using the star-triangle electrically equivalent transformation, we obtain a new graph, denoted by F_t and the weights in the new triangles become $\frac{1}{3}$, other edge weights remain unchanged. Figure 2 shows the transformation between G_2 and F_1 . Using the results in Ref. [24], we transform a star graph with weights a , b and c to a triangle graph H^* , where the weights are $\frac{bc}{a+b+c}$, $\frac{ac}{a+b+c}$ and $\frac{ab}{a+b+c}$. Then we obtain the relationship

of weighted spanning trees between H and H^* , i.e., $\tau(H) = (a + b + c)\tau(H^*)$. Since the graph G_n contains $\frac{3}{2}n$ identical star graphs, we obtain

$$\tau(G_n) = 3^{\frac{3}{2}n} \tau(F_t). \quad (1)$$

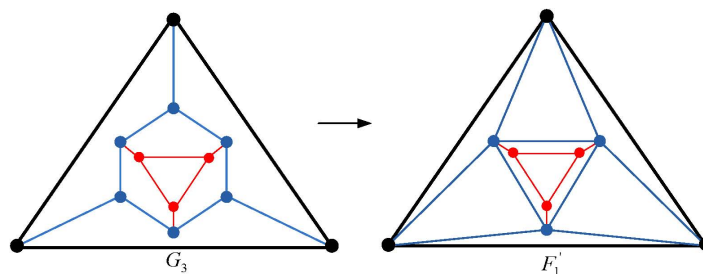
Fig. 2. The transformation from G_2 to F_1 .



In the same way, when the final state is a triangle, we transform G_n into F_t^* . Figure 3 gives the transformation between G_3 and F_1^* . The weighted spanning trees reads as

$$\tau(G_n) = 3^{\frac{3}{2}(n-1)} \tau(F_t^*). \quad (2)$$

Fig. 3. The transformation from G_3 to F_1^* .



In order to calculate the functions $\tau(F_t)$ and $\tau(F_t^*)$, we provide the following two Lemmas and three corollaries.

Lemma 3.1. For the edge-weighted graph F_1 (see Fig. 4), suppose the weights of the innermost triangle, outermost triangle and the linked edges are a , c and b , respectively. Then, we have

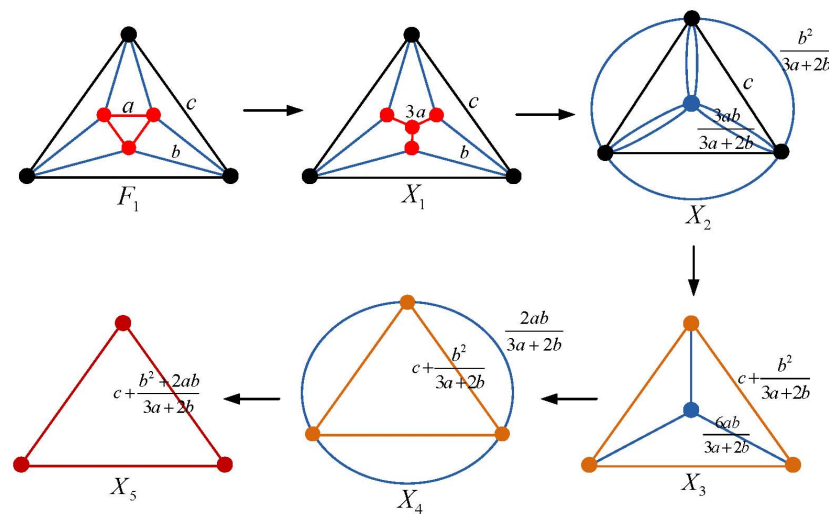
$$\tau(F_1) = 6b(2ab + 2bc + 3ac + b^2)^2.$$

Proof. We firstly transform the innermost triangle with weights a into an electrically equivalent star graph denoted by X_1 with weights $3a$, then $\tau(X_1) = 9a\tau(F_1)$.

Using the star-triangle transformation, three star graphs are changed into three connected curved edge triangles, where the weights of innermost six curved edges are $\frac{3ab}{3a+2b}$, and the weights of outer three curved edges are $\frac{b^2}{3a+2b}$. The transformed graph is denoted by X_2 , then $\tau(X_2) = (\frac{1}{3a+2b})^3 \tau(X_1)$.

Merging six pairs of parallel edges into single edges produces a new graph denoted by X_3 , which includes a triangle with weights $c + \frac{b^2}{3a+2b}$ and a star graph with weights $\frac{6ab}{3a+2b}$. Then $\tau(X_3) = \tau(X_2)$. Using the star-triangle transformation again, we obtain a curved edge triangle with weights $\frac{2ab}{3a+2b}$, then $\tau(X_4) = \frac{3a+2b}{18ab} \tau(X_3)$.

Finally, merging two parallel edges into a single edge to form a new triangle with weights $c + \frac{b^2+2ab}{3a+2b}$, we obtain $\tau(X_5) = \tau(X_4)$. Through the above five transformations, we obtain $\tau(F_1) = 6b(2ab + 2bc + 3ac + b^2)^2$. The whole electrically equivalent transformations from F_1 to X_5 are shown in Fig. 4.

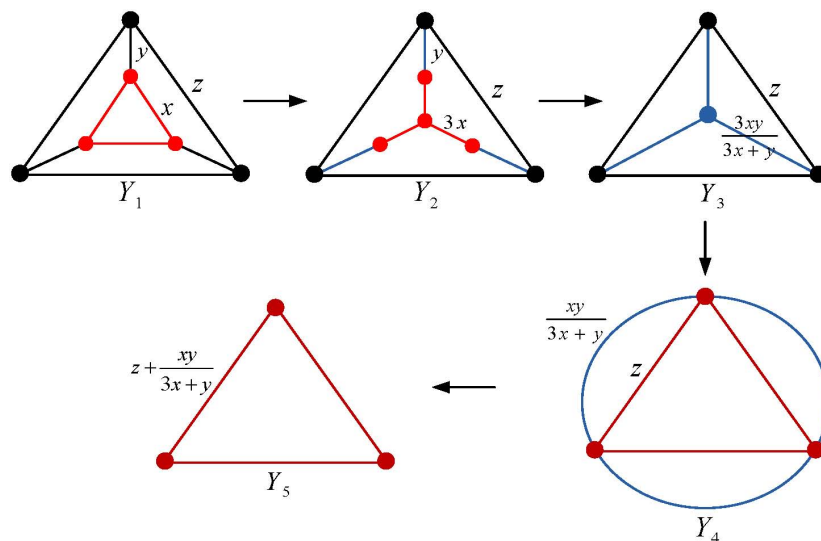
Fig. 4. The transformations from F_1 to X_5 .

Lemma 3.2. For the edge-weighted graph Y_1 (see Fig. 5), the weights of the innermost triangle, outermost triangle and the linked edges are x , z and y . Then,

$$\tau(Y_1) = 3y(xy + 3xz + yz)^2.$$

Proof. Using the same transformation between F_1 and X_1 , we change Y_1 into Y_2 , then $\tau(Y_2) = 9x\tau(Y_1)$. Merging two serial edges with the weights $3x$ and y into a single edge with the weight $\frac{3xy}{3x+y}$, we obtain $\tau(Y_3) = \frac{1}{(3x+y)^3} \tau(Y_2)$. Then implementing the star-triangle transformation and the curved edge triangle with weights $\frac{xy}{3x+y}$. Thus, $\tau(Y_4) = \frac{3x+y}{9xy} \tau(Y_3)$.

Finally merging parallel edges into a single edge with weight $z + \frac{xy}{3x+y}$ gives $\tau(Y_5) = \tau(Y_4)$. Combining the above-mentioned transformation, we obtain $\tau(Y_1) = 3y(xy + 3xz + yz)^2$. Figure 5 gives the electrically equivalent transformations between Y_1 and Y_5 . According to Lemmas 3.1 and 3.2, we obtain the following corollaries.

Fig. 5. The transformations from Y_1 to Y_5 .

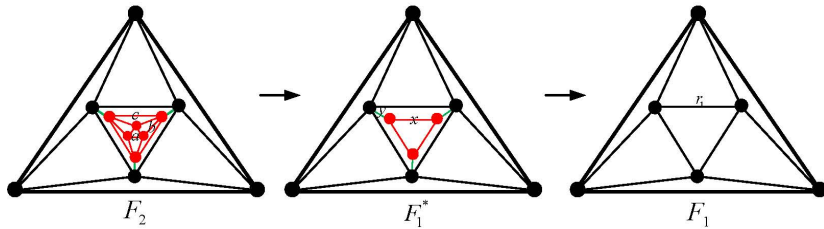
Corollary 3.3. Considering a new graph F_2 formed by connecting two graphs F_1 through three edges with the weight y , we obtain

$$\tau(F_2) = 2by(2b + 3a)^2(3x + y)^2\tau(F_1),$$

where $x = c + \frac{b^2+2ab}{3a+2b}$.

Proof. Based on the transformations in Fig. 6, we have $\tau(F_2) = 2b(2b + 3a)^2\tau(F_1^*)$ and $\tau(F_1^*) = y(3x + y)^2\tau(F_1)$.

Fig. 6. The transformations from F_2 to F_1 .



□

Corollary 3.4. Based on the transformations of F_2 , we obtain a relationship of enumerating spanning trees between F_t and F_1 , i.e.,

$$\tau(F_t) = \left(\frac{2}{3}\right)^{t-1} \prod_{i=2}^t (3 + 14r_i)^2 \cdot \tau(F_1), \quad (3)$$

where $r_i = \frac{16+75r_{i+1}}{27+126r_{i+1}}$, and r_i is the edge weights in the innermost triangle of F_i .

Proof. According to the transformations ($F_1 \rightarrow X_5$ and $Y_1 \rightarrow Y_5$) in Figs. 4 and 5, we obtain $x = c + \frac{b^2+2ab}{3a+2b}$ and $r_1 = z + \frac{xy}{3x+y}$. Let $z = b = \frac{1}{3}$, $c = y = 1$, $a = r_2$, it gives $x = \frac{33r_2+7}{27r_2+6}$ and $r_1 = \frac{16+75r_2}{27+126r_2}$. Further, $\tau(F_2) = \frac{2}{3}(3+14r_2)^2 \cdot \tau(F_1)$. Through the transformations between F_t and F_1 and by induction, Equation (3) holds. □

Corollary 3.5. The graphs F_t^* are produced by inserting a triangle with the weights 1 into F_t , and connecting them by linked-edges with the weights 1. Then,

$$\tau(F_t^*) = 16\tau(F_t). \quad (4)$$

Proof. Let $x = y = 1$, $z = \frac{1}{3}$ and from Lemma 3.2, it gives $\tau(Y_1) = 16\tau(Y_5)$. Using the transformations from F_t^* to F_t , Corollary 3.5 is established. □

4 Calculating the number of spanning trees

Using the expression $r_{t-1} = \frac{16+75r_t}{27+126r_t}$ and denoting the coefficients of $27 + 126r_t$ and $16 + 75r_t$ as A_t and B_t , we obtain

$$\begin{aligned} 3 + 14r_t &= A_0(27 + 126r_t) + B_0(16 + 75r_t), \\ 3 + 14r_{t-1} &= \frac{A_1(27 + 126r_t) + B_1(16 + 75r_t)}{9[A_0(27 + 126r_t) + B_0(16 + 75r_t)]}, \\ &\vdots \\ 3 + 14r_{t-i} &= \frac{A_i(27 + 126r_t) + B_i(16 + 75r_t)}{9[A_{i-1}(27 + 126r_t) + B_{i-1}(16 + 75r_t)]}, \end{aligned} \quad (5)$$

$$3 + 14r_{t-(i+1)} = \frac{A_{i+1}(27 + 126r_t) + B_{i+1}(16 + 75r_t)}{9[A_i(27 + 126r_t) + B_i(16 + 75r_t)]}, \quad (6)$$

$$\begin{aligned} & \vdots \\ 3 + 14r_2 &= \frac{A_{t-2}(27 + 126r_t) + B_{t-2}(16 + 75r_t)}{9[A_{t-3}(27 + 126r_t) + B_{t-3}(16 + 75r_t)]}, \end{aligned}$$

where $A_0 = \frac{1}{9}$, $B_0 = 0$; $A_1 = 3$, $B_1 = 14$. Substituting Eq. (5) into Eq. (3) yields

$$\tau(F_t) = 2^{t-1} \cdot 3^{-5t+9} [A_{t-2}(27 + 126r_t) + B_{t-2}(16 + 75r_t)]^2 \cdot \tau(F_1). \quad (7)$$

Let $a = r_1$, $b = \frac{1}{3}$, $c = 1$ and from Lemma 3.1, we obtain

$$\tau(F_1) = \frac{2}{81} \cdot (7 + 33r_1)^2. \quad (8)$$

By the relationship between r_t and r_{t-1} and Eqs. (5) and (6), we obtain

$$A_{t+1} = 102A_t - 9A_{t-1}; B_{t+1} = 102B_t - 9B_{t-1}.$$

Their characteristic equation is

$$\lambda^2 - 102\lambda + 9 = 0,$$

with two roots being $\lambda_1 = 51 + 36\sqrt{2}$ and $\lambda_2 = 51 - 36\sqrt{2}$. Then the general solutions are

$$A_t = a_1\lambda_1^t + a_2\lambda_2^t; B_t = b_1\lambda_1^t + b_2\lambda_2^t.$$

Using the initial conditions $A_0 = \frac{1}{9}$, $B_0 = 0$; $A_1 = 3$, $B_1 = 14$ gives

$$A_t = \frac{3 - \sqrt{2}}{54}\lambda_1^t + \frac{3 + \sqrt{2}}{54}\lambda_2^t; B_t = \frac{7\sqrt{2}}{72}\lambda_1^t - \frac{7\sqrt{2}}{72}\lambda_2^t. \quad (9)$$

In the sequel, we calculate the values of r_1 . By $r_{t-1} = \frac{16+75r_t}{27+126r_t}$, its characteristic equation is $63x^2 - 24x - 8 = 0$, which has two roots $x_1 = \frac{4+6\sqrt{2}}{21}$ and $x_2 = \frac{4-6\sqrt{2}}{21}$.

Subtracting these two roots from both sides of $r_{t-1} = \frac{16+75r_t}{27+126r_t}$ yields

$$\begin{aligned} r_{t-1} - \frac{4 + 6\sqrt{2}}{21} &= \frac{16 + 75r_t}{27 + 126r_t} - \frac{4 + 6\sqrt{2}}{21} \\ &= \frac{51 - 36\sqrt{2}}{27 + 126r_t} \left(r_t - \frac{4 + 6\sqrt{2}}{21} \right), \\ r_{t-1} - \frac{4 - 6\sqrt{2}}{21} &= \frac{16 + 75r_t}{27 + 126r_t} - \frac{4 - 6\sqrt{2}}{21} \\ &= \frac{51 + 36\sqrt{2}}{27 + 126r_t} \left(r_t - \frac{4 - 6\sqrt{2}}{21} \right). \end{aligned}$$

Let $a_t = \frac{r_t - \frac{4+6\sqrt{2}}{21}}{r_t - \frac{4-6\sqrt{2}}{21}}$, then,

$$a_{t-1} = \frac{17 - 12\sqrt{2}}{17 + 12\sqrt{2}} a_t,$$

where

$$a_1 = \frac{r_1 - \frac{4+6\sqrt{2}}{21}}{r_1 - \frac{4-6\sqrt{2}}{21}} = \left(\frac{17 - 12\sqrt{2}}{17 + 12\sqrt{2}} \right)^{t-1} a_t.$$

Hence, the expression of r_1 reads as

$$r_1 = \frac{(6\sqrt{2} - 4)(577 - 408\sqrt{2})^{t-1} a_t + 4 + 6\sqrt{2}}{21 - 21(577 - 408\sqrt{2})^{t-1} a_t}. \quad (10)$$

If $r_t = \frac{1}{3}$, then $a_t = \frac{3-6\sqrt{2}}{3+6\sqrt{2}}$. Plugging Eqs. (7)-(10) into Eq. (1) gives

$$\tau(G_n) = 2^{\frac{n}{2}} \cdot 3^{-n+5} \left(\frac{92 + 65\sqrt{2}}{24} \lambda_1^{\frac{n-4}{2}} + \frac{92 - 65\sqrt{2}}{24} \lambda_2^{\frac{n-4}{2}} \right)^2 (7 + 33r_1)^2,$$

where $r_1 = \frac{(2\sqrt{2}-4)(577-408\sqrt{2})^{\frac{n}{2}-1}+2\sqrt{2}+4}{(6\sqrt{2}-3)(577-408\sqrt{2})^{\frac{n}{2}-1}+6\sqrt{2}+3}$, $\lambda_1 = 51 + 36\sqrt{2}$ and $\lambda_2 = 51 - 36\sqrt{2}$.

If $r_t = \frac{7}{12}$, then $a_t = \frac{11-8\sqrt{2}}{11+8\sqrt{2}}$. Inserting Eqs. (4) and (7)-(10) into Eq. (2) yields

$$\tau(G_n) = 2^{\frac{n+7}{2}} \cdot 3^{-n+6} \left(\frac{536+379\sqrt{2}}{96} \lambda_1^{\frac{n-5}{2}} + \frac{536-379\sqrt{2}}{96} \lambda_2^{\frac{n-5}{2}} \right)^2 (7+33r_1)^2,$$

where $r_1 = \frac{(14\sqrt{2}-20)(577-408\sqrt{2})^{\frac{n-3}{2}}+14\sqrt{2}+20}{(24\sqrt{2}-33)(577-408\sqrt{2})^{\frac{n-3}{2}}+24\sqrt{2}+33}$. Then, we have the following theorem.

Theorem 4.1. *The enumeration of spanning trees in the sequence of Dürer graphs is as follows:*

$$\tau(G_n) = \begin{cases} 2^{\frac{n+7}{2}} \cdot 3^{6-n} \left(\frac{536+379\sqrt{2}}{96} \lambda_1^{\frac{n-5}{2}} + \frac{536-379\sqrt{2}}{96} \lambda_2^{\frac{n-5}{2}} \right)^2 \cdot \Phi_1, & n = 1, 3, \dots, \\ 2^{\frac{n}{2}} \cdot 3^{5-n} \left(\frac{92+65\sqrt{2}}{24} \lambda_1^{\frac{n-4}{2}} + \frac{92-65\sqrt{2}}{24} \lambda_2^{\frac{n-4}{2}} \right)^2 \cdot \Phi_2, & n = 2, 4, \dots, \end{cases}$$

where $\Phi_1 = (7+33r_1)^2$ with $r_1 = \frac{(14\sqrt{2}-20)(577-408\sqrt{2})^{\frac{n-3}{2}}+14\sqrt{2}+20}{(24\sqrt{2}-33)(577-408\sqrt{2})^{\frac{n-3}{2}}+24\sqrt{2}+33}$ and $\Phi_2 = (7+33r_1)^2$ with $r_1 = \frac{(2\sqrt{2}-4)(577-408\sqrt{2})^{\frac{n}{2}-1}+2\sqrt{2}+4}{(6\sqrt{2}-3)(577-408\sqrt{2})^{\frac{n}{2}-1}+6\sqrt{2}+3}$.

5 Entropy of spanning trees

Using the obtained results for enumeration of spanning trees, we calculate the entropy of spanning trees, denoted by $E(G)$, which is given by,

$$\begin{aligned} E(G) &= \lim_{n \rightarrow \infty} \frac{\ln \tau(G_n)}{V_n}, \\ &= \frac{\ln 2 - 2 \ln 3 + 2 \ln(51 + 36\sqrt{2})}{9} \approx 0.860. \end{aligned}$$

Now we compare the value of entropy in our graph with other graphs. For the graphs with average degree 3, the entropy of infinite outerplanar small-world graphs [26] is 0.657, the values of entropy in 3-12-12 and 4-8-8 lattices [27] are 0.721 and 0.787, and the honeycomb lattice [28] is 0.807. While for the graphs with average degree 4, the entropy of the pseudofractal fractal web [29] is 0.896, the fractal scale-free lattice [20] is 1.040, the values of the two-dimensional Sierpinski gasket [15] and the square lattice [28] are 1.049 and 1.166. The entropy of spanning trees in our graph is 0.860, which is larger than those of graphs with average degree 3, but smaller than those of graphs with average degree 4.

6 Conclusions

In the present study, we have used the electrically equivalent transformations to solve the number of spanning trees in the sequence of Dürer graphs. Compared to existing methods on enumeration of spanning trees, this method is effective and simple. Applying the transformations, we have converted this graph into a triangle, and obtained the relationships of corresponding edge weights. Using the obtained method, we could calculate the spanning trees of Dürer-like graphs, e.g., the cylinders width being an even number. In addition, our results have shown that the entropy is related to the average degree, whether this conclusion holds for other graphs needs further study.

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