



Shixing Li\*

# Enumeration of spanning trees in the sequence of Dürer graphs

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**Abstract:** In this paper, we calculate the number of spanning trees in the sequence of Dürer graphs with a special feature that it has two alternate states. Using the electrically equivalent transformations, we obtain the weights of corresponding equivalent graphs and further derive relationships for spanning trees between the Dürer graphs and transformed graphs. By algebraic calculations, we obtain a closed-form formula for the number of spanning trees with regard to iteration step. Finally we compare the entropy of our graph with other studied graphs and see that its value of entropy lies in the interval of those of graphs with average degree being 3 and 4.

**Keywords:** Spanning trees, Electrically equivalent transformation, Entropy

**MSC:** 05C30, 05C50, 05C63

## 1 Introduction

In the study of networks, spanning trees are related to many aspects of networks, such as reliability [1, 2], consensus [3], random walks [4] and nonlinear dynamics [5, 6]. A spanning tree of a connected graph is defined as a minimal set of edges that connect all its nodes. On the other hand, enumeration of spanning trees has been applied in mathematics [7], computer science [8, 9], physics [10, 11], and chemistry [12], to name just a few.

Recently, counting the number of spanning trees has attracted increasing attention [13–16]. It is known that the number of spanning trees can be obtained by matrix-tree theorem [13]. Due to the complexity and diversity of networks, analytically enumerating spanning trees is challenging. For fractal networks, the existing works [17–20] applied the self-similarity, spanning forests, and Laplacian spectrum to obtain exact formulae of spanning trees, while for some self-similar graphs, e.g., generalized Petersen graphs, those methods are not effective. Recently, the enumeration of spanning trees has been investigated by the electrically equivalent transformations in Refs. [21–23].

A sequence of Dürer graphs is the skeleton of Dürer's solid, which belongs to generalized Petersen graphs. This graph has a special feature that it has two alternate states. Calculating the number of spanning trees in this family of graph by its laplacian spectrum does not work. To the best of our knowledge, few analytical results involve the derivation of enumeration of spanning trees. Here we employ the knowledge of electrical networks, where an edge-weighted graph is regarded as an electrical network with the weights equalling the conductances of the corresponding edges. Using the electrically equivalent transformations provided in Refs. [24, 25], we obtain some relationships for spanning trees between the original graph and transformed graphs.

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\*Corresponding Author: Shixing Li: School of Data Sciences, Zhejiang University of Finance & Economics, Hangzhou 310018, China, E-mail: sxlizufe@163.com

We then obtain a closed-form formula for enumeration of spanning trees. Finally we calculate the entropy of spanning trees and compare it with those of other studied graphs.

In Section 2, the construction of the sequence of Dürer graphs is presented. Section 3 shows the electrically equivalent transformations of graphs. The detailed calculations of spanning trees are provided in Section 4. Finally, the conclusions are included in the last section.

## 2 Model presentation

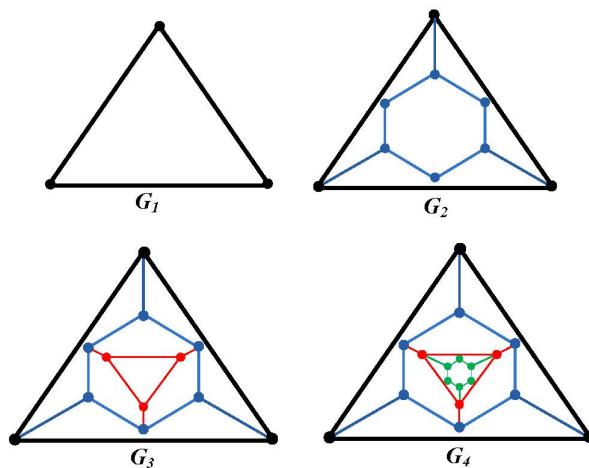
The sequence of Dürer graphs is denoted by  $G_n$  ( $n \geq 1$ ) after  $n$  steps, which is constructed as follows:

For  $n = 1$ ,  $G_1$  is a triangle.

For  $n = 2$ ,  $G_2$  is obtained from  $G_1$ , where a hexagon is put into the triangle and three new adjacent nodes in the hexagon are connected to the nodes in the triangle.

For  $n \geq 3$ ,  $G_n$  is produced by  $G_{n-1}$ . If  $n$  is an even number, a hexagon is inserted into  $G_{n-1}$ ; if  $n$  is an odd number, a triangle is embedded into  $G_{n-1}$ . Figure 1 shows this construction process.

**Fig. 1.** The Dürer graphs  $G_n$  produced at steps  $n = 1, 2, 3$  and  $4$ .



According to the construction, the number of total vertices  $V_n$  and edges  $E_n$  are,

$$V_n = \frac{9n}{2} - \frac{3}{2}, \quad E_n = \frac{15n}{2} - \frac{9}{2}, \quad n = 1, 3, \dots,$$

and

$$V_n = \frac{9n}{2}, \quad E_n = \frac{15n}{2} - 3, \quad n = 2, 4, \dots.$$

Then the average degree is  $\langle k \rangle_n = \frac{2E_n}{V_n}$ , which is approximately  $\frac{10}{3}$  for a large  $n$ .

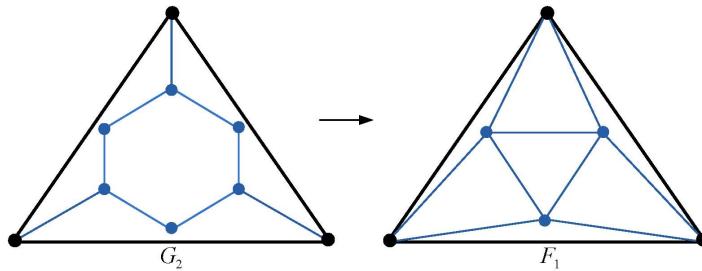
## 3 Electrically equivalent transformations

According to the structures of  $G_n$ , the final state is either a triangle or a hexagon. When the final state is a hexagon, using the star-triangle electrically equivalent transformation, we obtain a new graph, denoted by  $F_t$  and the weights in the new triangles become  $\frac{1}{3}$ , other edge weights remain unchanged. Figure 2 shows the transformation between  $G_2$  and  $F_1$ . Using the results in Ref. [24], we transform a star graph with weights  $a$ ,  $b$  and  $c$  to a triangle graph  $H^*$ , where the weights are  $\frac{bc}{a+b+c}$ ,  $\frac{ac}{a+b+c}$  and  $\frac{ab}{a+b+c}$ . Then we obtain the relationship

of weighted spanning trees between  $H$  and  $H^*$ , i.e.,  $\tau(H) = (a + b + c)\tau(H^*)$ . Since the graph  $G_n$  contains  $\frac{3}{2}n$  identical star graphs, we obtain

$$\tau(G_n) = 3^{\frac{3}{2}n}\tau(F_t). \quad (1)$$

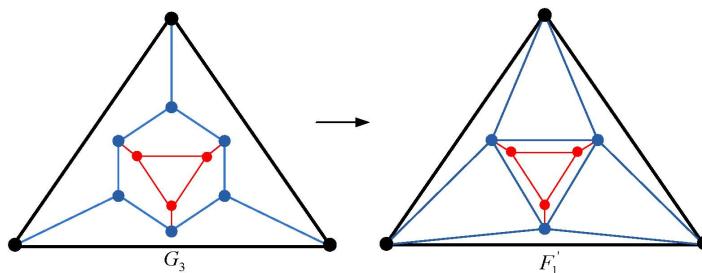
**Fig. 2.** The transformation from  $G_2$  to  $F_1$ .



In the same way, when the final state is a triangle, we transform  $G_n$  into  $F_t^*$ . Figure 3 gives the transformation between  $G_3$  and  $F_1^*$ . The weighted spanning trees reads as

$$\tau(G_n) = 3^{\frac{3}{2}(n-1)}\tau(F_t^*). \quad (2)$$

**Fig. 3.** The transformation from  $G_3$  to  $F_1^*$ .



In order to calculate the functions  $\tau(F_t)$  and  $\tau(F_t^*)$ , we provide the following two Lemmas and three corollaries.

**Lemma 3.1.** *For the edge-weighted graph  $F_1$  (see Fig. 4), suppose the weights of the innermost triangle, outermost triangle and the linked edges are  $a$ ,  $c$  and  $b$ , respectively. Then, we have*

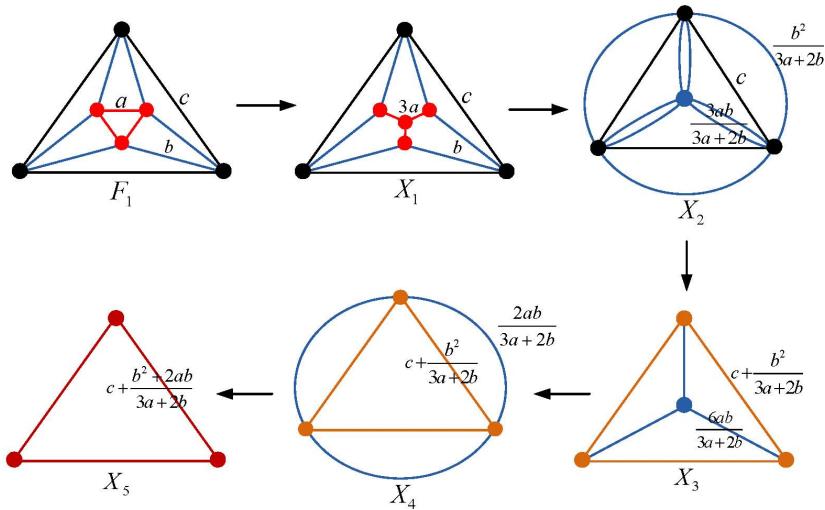
$$\tau(F_1) = 6b(2ab + 2bc + 3ac + b^2)^2.$$

*Proof.* We firstly transform the innermost triangle with weights  $a$  into an electrically equivalent star graph denoted by  $X_1$  with weights  $3a$ , then  $\tau(X_1) = 9a\tau(F_1)$ .

Using the star-triangle transformation, three star graphs are changed into three connected curved edge triangles, where the weights of innermost six curved edges are  $\frac{3ab}{3a+2b}$ , and the weights of outer three curved edges are  $\frac{b^2}{3a+2b}$ . The transformed graph is denoted by  $X_2$ , then  $\tau(X_2) = (\frac{1}{3a+2b})^3\tau(X_1)$ .

Merging six pairs of parallel edges into single edges produces a new graph denoted by  $X_3$ , which includes a triangle with weights  $c + \frac{b^2}{3a+2b}$  and a star graph with weights  $\frac{6ab}{3a+2b}$ . Then  $\tau(X_3) = \tau(X_2)$ . Using the star-triangle transformation again, we obtain a curved edge triangle with weights  $\frac{2ab}{3a+2b}$ , then  $\tau(X_4) = \frac{3a+2b}{18ab}\tau(X_3)$ .

Finally, merging two parallel edges into a single edge to form a new triangle with weights  $c + \frac{b^2+2ab}{3a+2b}$ , we obtain  $\tau(X_5) = \tau(X_4)$ . Through the above five transformations, we obtain  $\tau(F_1) = 6b(2ab + 2bc + 3ac + b^2)^2$ . The whole electrically equivalent transformations from  $F_1$  to  $X_5$  are shown in Fig. 4.

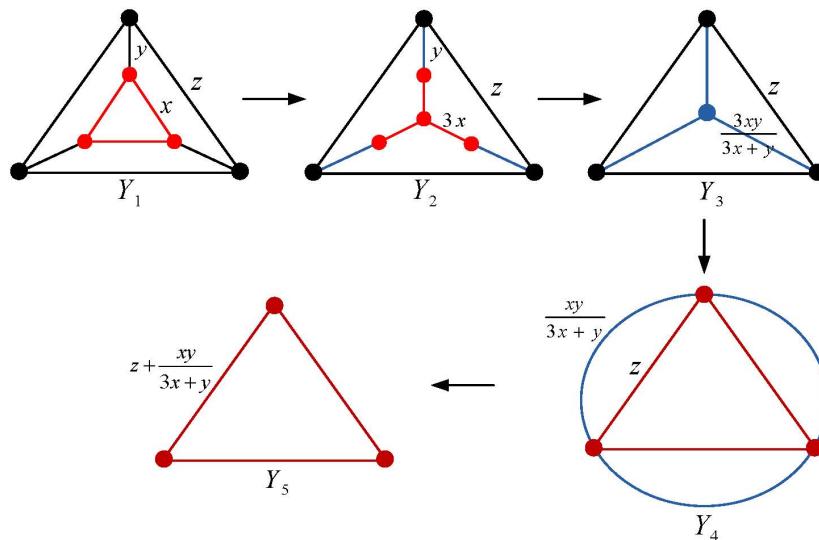
**Fig. 4.** The transformations from  $F_1$  to  $X_5$ .

**Lemma 3.2.** For the edge-weighted graph  $Y_1$  (see Fig. 5), the weights of the innermost triangle, outermost triangle and the linked edges are  $x, z$  and  $y$ . Then,

$$\tau(Y_1) = 3y(xy + 3xz + yz)^2.$$

*Proof.* Using the same transformation between  $F_1$  and  $X_1$ , we change  $Y_1$  into  $Y_2$ , then  $\tau(Y_2) = 9x\tau(Y_1)$ . Merging two serial edges with the weights  $3x$  and  $y$  into a single edge with the weight  $\frac{3xy}{3x+y}$ , we obtain  $\tau(Y_3) = \frac{1}{(3x+y)^3}\tau(Y_2)$ . Then implementing the star-triangle transformation and the curved edge triangle with weights  $\frac{xy}{3x+y}$ . Thus,  $\tau(Y_4) = \frac{3xy}{9xy}\tau(Y_3)$ .

Finally merging parallel edges into a single edge with weight  $z + \frac{xy}{3x+y}$  gives  $\tau(Y_5) = \tau(Y_4)$ . Combining the above-mentioned transformation, we obtain  $\tau(Y_1) = 3y(xy + 3xz + yz)^2$ . Figure 5 gives the electrically equivalent transformations between  $Y_1$  and  $Y_5$ . According to Lemmas 3.1 and 3.2, we obtain the following corollaries.

**Fig. 5.** The transformations from  $Y_1$  to  $Y_5$ .

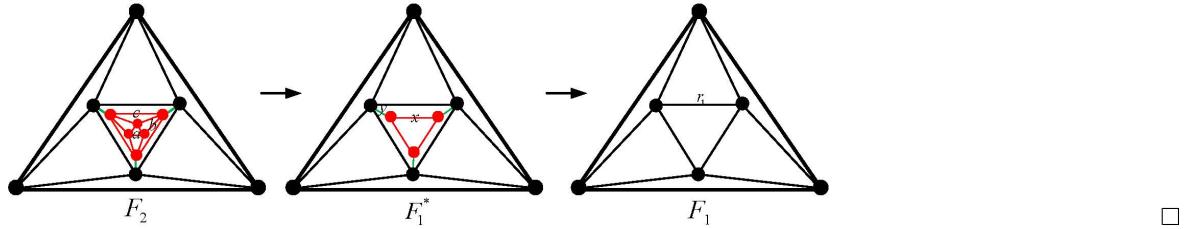
**Corollary 3.3.** Considering a new graph  $F_2$  formed by connecting two graphs  $F_1$  through three edges with the weight  $y$ , we obtain

$$\tau(F_2) = 2by(2b+3a)^2(3x+y)^2\tau(F_1),$$

where  $x = c + \frac{b^2+2ab}{3a+2b}$ .

*Proof.* Based on the transformations in Fig. 6, we have  $\tau(F_2) = 2b(2b+3a)^2\tau(F_1^*)$  and  $\tau(F_1^*) = y(3x+y)^2\tau(F_1)$ .

**Fig. 6.** The transformations from  $F_2$  to  $F_1$ .



**Corollary 3.4.** Based on the transformations of  $F_2$ , we obtain a relationship of enumerating spanning trees between  $F_t$  and  $F_1$ , i.e.,

$$\tau(F_t) = \left(\frac{2}{3}\right)^{t-1} \prod_{i=2}^t (3 + 14r_i)^2 \cdot \tau(F_1), \quad (3)$$

where  $r_i = \frac{16+75r_{i+1}}{27+126r_{i+1}}$ , and  $r_i$  is the edge weights in the innermost triangle of  $F_i$ .

*Proof.* According to the transformations ( $F_1 \rightarrow X_5$  and  $Y_1 \rightarrow Y_5$ ) in Figs. 4 and 5, we obtain  $x = c + \frac{b^2+2ab}{3a+2b}$  and  $r_1 = z + \frac{xy}{3x+y}$ . Let  $z = b = \frac{1}{3}$ ,  $c = y = 1$ ,  $a = r_2$ , it gives  $x = \frac{33r_2+7}{27r_2+6}$  and  $r_1 = \frac{16+75r_2}{27+126r_2}$ . Further,  $\tau(F_2) = \frac{2}{3}(3+14r_2)^2 \cdot \tau(F_1)$ . Through the transformations between  $F_t$  and  $F_1$  and by induction, Equation (3) holds.  $\square$

**Corollary 3.5.** The graphs  $F_t^*$  are produced by inserting a triangle with the weights 1 into  $F_t$ , and connecting them by linked-edges with the weights 1. Then,

$$\tau(F_t^*) = 16\tau(F_t). \quad (4)$$

*Proof.* Let  $x = y = 1$ ,  $z = \frac{1}{3}$  and from Lemma 3.2, it gives  $\tau(Y_1) = 16\tau(Y_5)$ . Using the transformations from  $F_t^*$  to  $F_t$ , Corollary 3.5 is established.  $\square$

## 4 Calculating the number of spanning trees

Using the expression  $r_{t-1} = \frac{16+75r_t}{27+126r_t}$  and denoting the coefficients of  $27 + 126r_t$  and  $16 + 75r_t$  as  $A_t$  and  $B_t$ , we obtain

$$3 + 14r_t = A_0(27 + 126r_t) + B_0(16 + 75r_t),$$

$$3 + 14r_{t-1} = \frac{A_1(27 + 126r_t) + B_1(16 + 75r_t)}{9[A_0(27 + 126r_t) + B_0(16 + 75r_t)]},$$

$$\vdots \qquad \vdots$$

$$3 + 14r_{t-i} = \frac{A_i(27 + 126r_t) + B_i(16 + 75r_t)}{9[A_{i-1}(27 + 126r_t) + B_{i-1}(16 + 75r_t)]}, \quad (5)$$

$$3 + 14r_{t-(i+1)} = \frac{A_{i+1}(27 + 126r_t) + B_{i+1}(16 + 75r_t)}{9[A_i(27 + 126r_t) + B_i(16 + 75r_t)]}, \quad (6)$$

$$\vdots \quad \vdots \\ 3 + 14r_2 = \frac{A_{t-2}(27 + 126r_t) + B_{t-2}(16 + 75r_t)}{9[A_{t-3}(27 + 126r_t) + B_{t-3}(16 + 75r_t)]},$$

where  $A_0 = \frac{1}{9}$ ,  $B_0 = 0$ ;  $A_1 = 3$ ,  $B_1 = 14$ . Substituting Eq. (5) into Eq. (3) yields

$$\tau(F_t) = 2^{t-1} \cdot 3^{-5t+9} [A_{t-2}(27 + 126r_t) + B_{t-2}(16 + 75r_t)]^2 \cdot \tau(F_1). \quad (7)$$

Let  $a = r_1$ ,  $b = \frac{1}{3}$ ,  $c = 1$  and from Lemma 3.1, we obtain

$$\tau(F_1) = \frac{2}{81} \cdot (7 + 33r_1)^2. \quad (8)$$

By the relationship between  $r_t$  and  $r_{t-1}$  and Eqs. (5) and (6), we obtain

$$A_{t+1} = 102A_t - 9A_{t-1}; B_{t+1} = 102B_t - 9B_{t-1}.$$

Their characteristic equation is

$$\lambda^2 - 102\lambda + 9 = 0,$$

with two roots being  $\lambda_1 = 51 + 36\sqrt{2}$  and  $\lambda_2 = 51 - 36\sqrt{2}$ . Then the general solutions are

$$A_t = a_1\lambda_1^t + a_2\lambda_2^t; B_t = b_1\lambda_1^t + b_2\lambda_2^t.$$

Using the initial conditions  $A_0 = \frac{1}{9}$ ,  $B_0 = 0$ ;  $A_1 = 3$ ,  $B_1 = 14$  gives

$$A_t = \frac{3 - \sqrt{2}}{54} \lambda_1^t + \frac{3 + \sqrt{2}}{54} \lambda_2^t; B_t = \frac{7\sqrt{2}}{72} \lambda_1^t - \frac{7\sqrt{2}}{72} \lambda_2^t. \quad (9)$$

In the sequel, we calculate the values of  $r_1$ . By  $r_{t-1} = \frac{16+75r_t}{27+126r_t}$ , its characteristic equation is  $63x^2 - 24x - 8 = 0$ , which has two roots  $x_1 = \frac{4+6\sqrt{2}}{21}$  and  $x_2 = \frac{4-6\sqrt{2}}{21}$ .

Subtracting these two roots from both sides of  $r_{t-1} = \frac{16+75r_t}{27+126r_t}$  yields

$$\begin{aligned} r_{t-1} - \frac{4+6\sqrt{2}}{21} &= \frac{16+75r_t}{27+126r_t} - \frac{4+6\sqrt{2}}{21} \\ &= \frac{51-36\sqrt{2}}{27+126r_t} (r_t - \frac{4+6\sqrt{2}}{21}), \\ r_{t-1} - \frac{4-6\sqrt{2}}{21} &= \frac{16+75r_t}{27+126r_t} - \frac{4-6\sqrt{2}}{21} \\ &= \frac{51+36\sqrt{2}}{27+126r_t} (r_t - \frac{4-6\sqrt{2}}{21}). \end{aligned}$$

Let  $a_t = \frac{r_t - \frac{4+6\sqrt{2}}{21}}{r_t - \frac{4-6\sqrt{2}}{21}}$ , then,

$$a_{t-1} = \frac{17-12\sqrt{2}}{17+12\sqrt{2}} a_t,$$

where

$$a_1 = \frac{r_1 - \frac{4+6\sqrt{2}}{21}}{r_1 - \frac{4-6\sqrt{2}}{21}} = \left( \frac{17-12\sqrt{2}}{17+12\sqrt{2}} \right)^{t-1} a_t.$$

Hence, the expression of  $r_1$  reads as

$$r_1 = \frac{(6\sqrt{2}-4)(577-408\sqrt{2})^{t-1}a_t + 4+6\sqrt{2}}{21-21(577-408\sqrt{2})^{t-1}a_t}. \quad (10)$$

If  $r_t = \frac{1}{3}$ , then  $a_t = \frac{3-6\sqrt{2}}{3+6\sqrt{2}}$ . Plugging Eqs. (7)-(10) into Eq. (1) gives

$$\tau(G_n) = 2^{\frac{n}{2}} \cdot 3^{-n+5} \left( \frac{92+65\sqrt{2}}{24} \lambda_1^{\frac{n-4}{2}} + \frac{92-65\sqrt{2}}{24} \lambda_2^{\frac{n-4}{2}} \right)^2 (7+33r_1)^2,$$

where  $r_1 = \frac{(2\sqrt{2}-4)(577-408\sqrt{2})^{\frac{n}{2}-1}+2\sqrt{2}+4}{(6\sqrt{2}-3)(577-408\sqrt{2})^{\frac{n}{2}-1}+6\sqrt{2}+3}$ ,  $\lambda_1 = 51 + 36\sqrt{2}$  and  $\lambda_2 = 51 - 36\sqrt{2}$ .

If  $r_t = \frac{7}{12}$ , then  $a_t = \frac{11-8\sqrt{2}}{11+8\sqrt{2}}$ . Inserting Eqs. (4) and (7)-(10) into Eq. (2) yields

$$\tau(G_n) = 2^{\frac{n+7}{2}} \cdot 3^{-n+6} \left( \frac{536 + 379\sqrt{2}}{96} \lambda_1^{\frac{n-5}{2}} + \frac{536 - 379\sqrt{2}}{96} \lambda_2^{\frac{n-5}{2}} \right)^2 (7 + 33r_1)^2,$$

where  $r_1 = \frac{(14\sqrt{2}-20)(577-408\sqrt{2})^{\frac{n-3}{2}}+14\sqrt{2}+20}{(24\sqrt{2}-33)(577-408\sqrt{2})^{\frac{n-3}{2}}+24\sqrt{2}+33}$ . Then, we have the following theorem.

**Theorem 4.1.** *The enumeration of spanning trees in the sequence of Dürer graphs is as follows:*

$$\tau(G_n) = \begin{cases} 2^{\frac{n+7}{2}} \cdot 3^{6-n} \left( \frac{536+379\sqrt{2}}{96} \lambda_1^{\frac{n-5}{2}} + \frac{536-379\sqrt{2}}{96} \lambda_2^{\frac{n-5}{2}} \right)^2 \cdot \Phi_1, & n = 1, 3, \dots, \\ 2^{\frac{n}{2}} \cdot 3^{5-n} \left( \frac{92+65\sqrt{2}}{24} \lambda_1^{\frac{n-4}{2}} + \frac{92-65\sqrt{2}}{24} \lambda_2^{\frac{n-4}{2}} \right)^2 \cdot \Phi_2, & n = 2, 4, \dots, \end{cases}$$

where  $\Phi_1 = (7 + 33r_1)^2$  with  $r_1 = \frac{(14\sqrt{2}-20)(577-408\sqrt{2})^{\frac{n-3}{2}}+14\sqrt{2}+20}{(24\sqrt{2}-33)(577-408\sqrt{2})^{\frac{n-3}{2}}+24\sqrt{2}+33}$  and  $\Phi_2 = (7 + 33r_1)^2$  with  $r_1 = \frac{(2\sqrt{2}-4)(577-408\sqrt{2})^{\frac{n}{2}-1}+2\sqrt{2}+4}{(6\sqrt{2}-3)(577-408\sqrt{2})^{\frac{n}{2}-1}+6\sqrt{2}+3}$ .

## 5 Entropy of spanning trees

Using the obtained results for enumeration of spanning trees, we calculate the entropy of spanning trees, denoted by  $E(G)$ , which is given by,

$$\begin{aligned} E(G) &= \lim_{n \rightarrow \infty} \frac{\ln \tau(G_n)}{V_n}, \\ &= \frac{\ln 2 - 2 \ln 3 + 2 \ln(51 + 36\sqrt{2})}{9} \approx 0.860. \end{aligned}$$

Now we compare the value of entropy in our graph with other graphs. For the graphs with average degree 3, the entropy of infinite outerplanar small-world graphs [26] is 0.657, the values of entropy in 3-12-12 and 4-8-8 lattices [27] are 0.721 and 0.787, and the honeycomb lattice [28] is 0.807. While for the graphs with average degree 4, the entropy of the pseudofractal fractal web [29] is 0.896, the fractal scale-free lattice [20] is 1.040, the values of the two-dimensional Sierpinski gasket [15] and the square lattice [28] are 1.049 and 1.166. The entropy of spanning trees in our graph is 0.860, which is larger than those of graphs with average degree 3, but smaller than those of graphs with average degree 4.

## 6 Conclusions

In the present study, we have used the electrically equivalent transformations to solve the number of spanning trees in the sequence of Dürer graphs. Compared to existing methods on enumeration of spanning trees, this method is effective and simple. Applying the transformations, we have converted this graph into a triangle, and obtained the relationships of corresponding edge weights. Using the obtained method, we could calculate the spanning trees of Dürer-like graphs, e.g., the cylinders width being an even number. In addition, our results have shown that the entropy is related to the average degree, whether this conclusion holds for other graphs needs further study.

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