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Fourier series of functions involving higher-order ordered Bell polynomials

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Abstract: In 1859, Cayley introduced the ordered Bell numbers which have been used in many problems in number theory and enumerative combinatorics. The ordered Bell polynomials were defined as a natural companion to the ordered Bell numbers (also known as the preferred arrangement numbers). In this paper, we study Fourier series of functions related to higher-order ordered Bell polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

Keywords: Fourier series, Bernoulli functions, Higher-order ordered Bell polynomials

MSC: 11B73, 11B83, 05A19

1 Introduction

The ordered Bell polynomials of order r ($r \in \mathbb{Z}_{>0}$) are defined by the generating function

$$\left(\frac{1}{2-e^t}\right)^r e^{xt} = \sum_{m=0}^{\infty} b_m^{(r)}(x)\frac{t^m}{m!}, \quad (\text{see } [2, 3, 7, 8, 15, 18, 20]). \tag{1}$$

When $x = 0$, $b_m^{(r)} = b_m^{(r)}(0)$ are called the ordered Bell numbers of order r. These higher-order ordered Bell polynomials and numbers are further generalizations of the ordered Bell polynomials and numbers which are respectively given by $b_m(x) = b_m^{(1)}(x)$ and $b_m = b_m^{(1)}$. The ordered Bell polynomials $b_m(x)$ were defined in [11] as a natural companion to the ordered Bell numbers which were introduced in 1859 by Cayley to count certain plane trees with $m + 1$ totally ordered leaves. The ordered Bell numbers have been used in many counting problems in number theory and enumerative combinatorics since its first appearance. They are all positive integers, as we can see from

$$b_m = \sum_{n=0}^{m} n!S_2(m,n) = \sum_{n=0}^{\infty}\frac{n^m}{2^{n+1}}, \quad (m \ge 0). \tag{2}$$

Here we would like to point out that the ordered Bell numbers are also known (or mostly known) as the preferred arrangement numbers (see [8]). The ordered Bell polynomial $b_m(x)$ has degree m and is a monic polynomial with integral coefficients, as we can see from

$$b_0(x) = 1, b_m(x) = x^m + \sum_{l=0}^{m-1}\binom{m}{l} b_l(x), \quad (m \ge 1) \tag{3}$$

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(see [11]). From (1), we can derive the following.

$$\frac{d}{dx}b_m^{(r)} = mb_{m-1}^{(r)}, \quad (m \geq 1), \\ b_m^{(r)}(x+1) - b_m^{(r)}(x) = b_m^{(r)}(x) - b_m^{(r-1)}(x), \quad (m \geq 0). \tag{4}$$

Also, from these we immediately get

$$b_m^{(r)}(1) - b_m^{(r)} = b_m^{(r)} - b_m^{(r-1)}, \quad (m \geq 0), \\ \int_0^1 b_m^{(r)}(x)dx = \frac{1}{m+1}(b_{m+1}^{(r)}(1) - b_{m+1}^{(r)}) = \frac{1}{m+1}(b_{m+1}^{(r)} - b_{m+1}^{(r-1)}). \tag{5}$$

For any real number x, we let $< x >= x - [x] \in [0,1)$ denote the fractional part of x. Let $B_m(x)$ be the Bernoulli polynomials given by the generating function

$$\frac{t}{e^t - 1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}. \tag{6}$$

For later use, we will state the following facts about Bernoulli functions $B_m(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{7}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{8}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. Here we will consider the following three types of functions $\alpha_m(< x >)$, $\beta_m(< x >)$, and $\gamma_m(< x >)$ involving higher order ordered Bell polynomials.

In this paper, we will derive their Fourier series expansions and in addition express each of them in terms of Bernoulli functions:
(1) $\alpha_m(< x >) = \sum_{k=0}^{m} b_k^{(r)}(< x >) < x >^{m-k}, (m \geq 1)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(< x >) < x >^{m-k}, (m \geq 1)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(< x >) < x >^{m-k}, (m \geq 2)$.

The reader may refer to any book for elementary facts about Fourier analysis (for example, see [1,16,21]).

As to $\gamma_m(< x >)$, we note that the polynomial identity (9) follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(x)x^{m-k} = \frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)B_s(x), \tag{9}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers and $\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)}(2b_k^{(r)} - b_k^{(r-1)})$. The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2 , respectively. It is noteworthy that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [6]) and the Miki's identity (see [5,17,19]). Some related works on Fourier series expansions for analogous functions can be found in the recent papers [9,10,13,14]. From now on, we will assume that $r \geq 2$. The case of $r = 1$ has been treated as a special case of the result in [4].

2 Fourier series of functions of the first type

Let $\alpha_m(x) = \sum_{k=0}^{m} b_k^{(r)}(x)x^{m-k}, \quad (m \geq 1)$. Then we will consider the function

$$\alpha_m(<x>) = \sum_{k=0}^{m} b_k^{(r)}(<x>) <x>^{m-k}, \tag{10}$$

defined on $\mathbb{R}$ which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx}, \tag{11}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(<x>)e^{-2\pi inx}dx = \int_0^1 \alpha_m(x)e^{-2\pi inx}dx. \tag{12}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \alpha'_m(x) &= \sum_{k=0}^{m}\{kb_{k-1}^{(r)}(x)x^{m-k} + (m-k)b_k^{(r)}(x)x^{m-k-1}\} \\ &= \sum_{k=1}^{m} kb_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)b_k^{(r)}(x)x^{m-k-1} \\ &= \sum_{k=0}^{m-1}(k+1)b_k^{(r)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}(m-k)b_k^{(r)}(x)x^{m-1-k} \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{13}$$

From this, we get

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{14}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{15}$$

For $m \geq 1$, we put

$$\begin{aligned} \Delta_m &= \alpha_m(1) - \alpha_m(0) = \sum_{k=0}^{m}\left(b_k^{(r)}(1) - b_k^{(r)}\delta_{m,k}\right) \\ &= \sum_{k=0}^{m}\left(2b_k^{(r)} - b_k^{(r-1)} - b_k^{(r)}\delta_{m,k}\right) = \sum_{k=0}^{m}\left(2b_k^{(r)} - b_k^{(r-1)}\right) - b_m^{(r)}. \end{aligned} \tag{16}$$

We note from this that

$$\alpha_m(0) = \alpha_m(1) \iff \Delta_m = 0, \tag{17}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{18}$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.
Case 1 : $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_m^{(m-1)} - \frac{1}{2\pi in}\Delta_m, \end{aligned} \tag{19}$$

from which by induction on m we can deduce that

$$A_n^{(m)} = -\sum_{j=1}^{m} \frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}. \tag{20}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{21}$$

$\alpha_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for those positive integers with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers with $\Delta_m \neq 0$. Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} \alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi in}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >) + \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{22}$$

Now, we can state our first result.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l = \sum_{k=0}^{l}(2b_k^{(r)} - b_k^{(r-1)}) - b_l^{(r)}. \tag{23}$$

Assume that m is a positive integer with $\Delta_m = 0$. Then we have the following.
(a) $\sum_{k=0}^{m} b_k^{(r)}(< x >) < x >^{m-k}$ has the Fourier series expansion

$$\sum_{k=0}^{m} b_k^{(r)}(< x >) < x >^{m-k} = \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}, \tag{24}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.
(b)

$$\sum_{k=0}^{m} b_k^{(r)}(< x >) < x >^{m-k} = \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >), \tag{25}$$

for all $x \in \mathbb{R}$, where $B_j(< x >)$ is the Bernoulli function.

Assume next that $\Delta_m \neq 0$, for a positive integer m. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{26}$$

for $x \in \mathbb{Z}$. We can now state our second result.

Theorem 2.2. *For each positive integer l, we let*

$$\Delta_l = \sum_{k=0}^{l}(2b_k^{(r)} - b_k^{(r-1)}) - b_l^{(r)}. \tag{27}$$

Assume that m is a positive integer with $\Delta_m \neq 0$. Then we have the following.
(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x} = \begin{cases} \sum_{k=0}^{m} b_k^{(r)}(<x>)<x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ b_m^{(r)} + \frac{1}{2}\Delta_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \tag{28}$$

(b)

$$\begin{aligned} &\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) = \sum_{k=0}^{m} b_k^{(r)}(<x>)<x>^{m-k}, \quad \text{for } x \in \mathbb{Z}^c \quad ; \\ &\frac{1}{m+2}\sum_{j=0, j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) = b_m^{(r)} + \frac{1}{2}\Delta_m, \ \text{for } x \in \mathbb{Z}. \end{aligned} \tag{29}$$

3 Fourier series of functions of the second type

Let $\beta_m(x) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k^{(r)}(x)x^{m-k}$, $(m \geq 1)$. Then we will consider the function

$$\beta_m(<x>) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k^{(r)}(<x>)<x>^{m-k}, \tag{30}$$

defined on $\mathbb{R}$ which is periodic with period 1. The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{31}$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(<x>)e^{-2\pi i n x}dx = \int_0^1 \beta_m(x)e^{-2\pi i n x}dx. \tag{32}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}b_{k-1}^{(r)}(x)x^{m-k} + \frac{m-k}{k!(m-k)!}b_k^{(r)}(x)x^{m-k-1}\right\} \\ &= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}b_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k^{(r)}(x)x^{m-k-1} \\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k^{(r)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k^{(r)}(x)x^{m-1-k} \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{33}$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x), \tag{34}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{35}$$

For $m \geq 1$, we have

$$\begin{aligned}\Omega_m = \beta_m(1) - \beta_m(0) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\left(b_k^{(r)}(1) - b_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\left(2b_k^{(r)} - b_k^{(r-1)} - b_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\left(2b_k^{(r)} - b_k^{(r-1)}\right) - \frac{1}{m!}b_m^{(r)}.\end{aligned} \tag{36}$$

From this, we note that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{37}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{38}$$

We are now going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx = -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\beta_m(1) - \beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m,\end{aligned} \tag{39}$$

from which by induction on m we can easily get

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \tag{40}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{41}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer m. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}\beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c,\\ 0, & \text{for} \quad x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{42}$$

Now, we state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{1}{k!(l-k)!}(2b_k^{(r)} - b_k^{(r-1)}) - \frac{1}{l!}b_l^{(r)}. \tag{43}$$

Assume that m is a positive integer with $\Omega_m = 0$. Then we have the following.
(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(<x>)<x>^{m-k}$ has the Fourier series expansion

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(<x>)<x>^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}, \tag{44}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.
(b)

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(<x>)<x>^{m-k} = \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>), \tag{45}$$

for all $x \in \mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m. \tag{46}$$

for $x \in \mathbb{Z}$. Now we state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{1}{k!(l-k)!}(2b_k^{(r)} - b_k^{(r-1)}) - \frac{1}{l!}b_l^{(r)}. \tag{47}$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.
(a)

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(<x>)<x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{m!} b_m^{(r)} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{48}$$

(b)

$$\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k^{(r)}(<x>)<x>^{m-k}, \tag{49}$$

for $x \in \mathbb{Z}^c$;

$$\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \frac{1}{m!} b_m^{(r)} + \frac{1}{2}\Omega_m, \tag{50}$$

for $x \in \mathbb{Z}$.

4 Fourier series of functions of the third type

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(x) x^{m-k}$, $(m \geq 2)$. Then we will consider the function.

$$\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(<x>)<x>^{m-k}, \tag{51}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} C_m^{(r)} e^{2\pi inx}, \tag{52}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(<x>)e^{-2\pi inx}dx = \int_0^1 \gamma_m(x)e^{-2\pi inx}dx. \tag{53}$$

To proceed further, we need to observe the following.

$$\begin{aligned}\gamma_m'(x) &= \sum_{k=1}^{m-1}\frac{1}{m-k}b_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=1}^{m-1}\frac{1}{k}b_k^{(r)}(x)x^{m-k-1}\\ &= \sum_{k=0}^{m-2}\frac{1}{m-1-k}b_k^{(r)}(x)x^{m-1-k} + \sum_{k=1}^{m-1}\frac{1}{k}b_k^{(r)}(x)x^{m-1-k}\\ &= \sum_{k=1}^{m-2}\left(\frac{1}{m-1-k}+\frac{1}{k}\right)b_k^{(r)}(x)x^{m-1-k} + \frac{1}{m-1}x^{m-1} + \frac{1}{m-1}b_{m-1}^{(r)}(x)\\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}x^{m-1} + \frac{1}{m-1}b_{m-1}^{(r)}(x).\end{aligned} \tag{54}$$

From this, we see that

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x) - \frac{1}{m(m+1)}x^{m+1} - \frac{1}{m(m+1)}b_{m+1}^{(r)}(x)\right)\right)' = \gamma_m(x) \tag{55}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)}(b_{m+1}^{(r)}(1) - b_{m+1}^{(r)})\right). \tag{56}$$

For $m \geq 2$, we put

$$\begin{aligned}\Lambda_m = \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(b_k^{(r)}(1) - b_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}(2b_k^{(r)} - b_k^{(r-1)}).\end{aligned} \tag{57}$$

Notice here that,

$$\gamma_m(0) = \gamma_m(1) \iff \Lambda_m = 0, \tag{58}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)})\right). \tag{59}$$

We are now going to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi in}\int_0^1\left\{(m-1)\gamma_{m-1}(x) + \frac{1}{m-1}x^{m-1} + \frac{1}{m-1}b_{m-1}^{(r)}(x)\right\}e^{-2\pi inx}dx\\ &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx + \frac{1}{2\pi in(m-1)}\int_0^1 b_{m-1}^{(r)}(x)e^{-2\pi inx}dx.\end{aligned} \tag{60}$$

We can show that

$$\int_0^1 x^l e^{-2\pi inx}dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}, & \text{for} \quad n \neq 0, \\ \frac{1}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{61}$$

Also, from the paper [12], we have

$$\int_0^1 b_l^{(r)}(x)e^{-2\pi inx}dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}(b_{l-k+1}^{(r)} - b_{l-k+1}^{(r-1)}), & \text{for} \quad n \neq 0, \\ \frac{1}{l+1}(b_{l+1}^{(r)} - b_{l+1}^{(r-1)}), & \text{for} \quad n = 0. \end{cases} \tag{62}$$

From (60), (61), and (62), we have

$$C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m, \tag{63}$$

where

$$\Phi_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}(1 + b_{m-k}^{(r)} - b_{m-k}^{(r-1)}). \tag{64}$$

From (63) by induction on m we can deduce that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}. \tag{65}$$

We note here that

$$\begin{aligned} &\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\ &= \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j} \frac{(m-j)_{k-1}}{(2\pi in)^k}(1 + b_{m-j-k+1}^{(r)} - b_{m-j-k+1}^{(r-1)}) \\ &= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}}(1 + b_{m-j-k+1}^{(r)} - b_{m-j-k+1}^{(r-1)}) \\ &= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \\ &= \sum_{s=2}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \sum_{j=1}^{s-1} \frac{1}{m-j} \\ &= \frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}). \end{aligned} \tag{66}$$

Putting everything altogether, from (65), we finally obtain

$$C_n^{(m)} = -\frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s} \left\{ \Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \right\}. \tag{67}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)}) \right). \tag{68}$$

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for an integer $m \geq 2$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)})\right)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)\right\}e^{2\pi inx}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)})\right)\\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)\left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= \frac{1}{m}\sum_{s=0, s\neq 1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)B_s(< x >)\\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c,\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned} \tag{69}$$

Now, we are going to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}(2b_k^{(r)} - b_k^{(r-1)}), \tag{70}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.
(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k^{(r)}(< x >) < x >^{m-k}$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)})\right)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)\right\}e^{2\pi inx},
\end{aligned} \tag{71}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.
(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\sum_{s=0, s\neq 1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)})\right)B_s(< x >),
\end{aligned} \tag{72}$$

for all $x \in \mathbb{R}$, where $B_s(< x >)$ is the Bernoulli function.

Assume next that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers.Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{73}$$

for $x \in \mathbb{Z}$. Next, we are going to state our second result.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} (2b_k^{(r)} - b_k^{(r-1)}), \tag{74}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.
(a)

$$\begin{aligned} &\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}(1 + b_{m+1}^{(r)} - b_{m+1}^{(r-1)})\right) \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \right) \right\} e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(<x>) <x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{2}\Lambda_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{75}$$

(b)

$$\begin{aligned} &\frac{1}{m} \sum_{s=0}^{m} \binom{m}{s} \left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \right) B_s(<x>) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k^{(r)}(<x>) <x>^{m-k}, \end{aligned} \tag{76}$$

for $x \in \mathbb{Z}^c$ and

$$\frac{1}{m} \sum_{s=0, s\neq 1}^{m} \binom{m}{s} \left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (1 + b_{m-s+1}^{(r)} - b_{m-s+1}^{(r-1)}) \right) B_s(<x>) = \frac{1}{2}\Lambda_m, \tag{77}$$

for $x \in \mathbb{Z}$.

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