**DE GRUYTER** OPEN

Open Math. 2017; 15: 1509–1516

**Open Mathematics**

**Research Article**

R. Chandrashekar*, See Keong Lee, and K.G. Subramanian

# Differential subordination and convexity criteria of integral operators

https://doi.org/10.1515/math-2017-0127
Received March 10, 2017; accepted September 19, 2017.

**Abstract:** A significant connection between certain second-order differential subordination and subordination of $f'(z)$ is obtained. This fundamental result is next applied to investigate the convexity of analytic functions defined in the open unit disk. As a consequence, criteria for convexity of functions defined by integral operators are determined. Connections are also made to earlier known results.

**Keywords:** Convex functions, Differential subordination, Integral operators

**MSC:** 30C45, 30C80

## 1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions $f$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n$ a positive integer, let

$$\mathcal{H}_n(a) = \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\},$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\},$$

with $\mathcal{A}_1 = \mathcal{A}$. For $0 \le \delta < 1$, denote by $\mathcal{CV}(\delta)$ the subclass of $\mathcal{A}$ consisting of convex functions of order $\delta$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, \quad z \in U.$$

The class $\mathcal{CV} := \mathcal{CV}(0)$ is the well-known subclass of convex functions studied widely in geometric function theory.

An analytic function $f$ is subordinate to an analytic function $g$ in $U$, written as $f \prec g$, if there exists an analytic self-map $\omega$ of $U$ with $\omega(0) = 0$ satisfying $f(z) = g(\omega(z))$, $z \in U$.

In geometric function theory, there has been a great interest among authors in determining the starlikeness or convexity of functions based on differential subordination and integral operators, see for example [1–8]. In particular, Kanas and Owa [9] studied the convexity of functions by investigating connections

***Corresponding Author: R. Chandrashekar:** Department of Technology and Management, Faculty of Technology Management and Business, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor, Malaysia, E-mail: chandra@uthm.edu.my
**See Keong Lee:** School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia, E-mail: sklee@usm.my
**K.G. Subramanian:** Department of Mathematics, Madras Christian College, Tambaram, Chennai 600059 India, E-mail: kgsmani1948@yahoo.com

between certain second-order differential subordination and subordination involving expressions of the form $f(z)/z, f'(z)$ and $1 + zf''(z)/f'(z)$. More recently, Supramaniam et al. [10], obtained sufficient conditions to ensure convexity for analytic functions defined by differential inequalities and integral operators of the form

$$f(z) = \int_0^1 \int_0^1 W(r, s, z) dr ds$$

or triple integral operators of the form

$$f(z) = \int_0^1 \int_0^1 \int_0^1 W(r, s, t, z) dr ds dt.$$

In this paper, conditions that would imply convexity of positive order for functions satisfying a second-order and third-order differential subordination are found. As a consequence, conditions on the kernel of certain integral operators are also obtained to ensure functions defined by these operators are convex. The result obtained in this paper presents a more general framework and extend the results of Kanas and Owa [9].

The following results will be required in the sequel.

**Lemma 1.1** ([9, Lemma 1.4, p. 26]). *Let $K, L, N, \eta, \pi$ be nonnegative real, fixed numbers, and let*

$$f(z) \prec 1 + Kz, \;\; g(z) \prec 1 + Lz, \;\; h(z) \prec Nz, \quad z \in U.$$

*Then*

$$\eta f(z) + \pi g(z) \prec \eta + \pi + (\eta K + \pi L)z \tag{1}$$

*and*

$$\eta f(z) + \pi h(z) \prec \eta + (\eta K + \pi N)z.$$

**Lemma 1.2** ([11, Theorem 3.1b, p. 71]). *Let $h$ be convex in $U$, with $h(0) = a, y \neq 0$ and $\operatorname{Re} y \geq 0$. If $p \in \mathcal{H}_n(a)$ and*

$$p(z) + \frac{zp'(z)}{y} \prec h(z),$$

*then*

$$p(z) \prec q(z) \prec h(z),$$

*where*

$$q(z) = \frac{y}{nz^{y/n}} \int_0^z h(t) t^{(y/n)-1} dt.$$

*The function $q$ is convex and is the best $(a, n)$–dominant.*

**Lemma 1.3** ([12, Corollary 1, p. 582]). *Let $\beta > 0$, $\alpha + 2\beta \geq 0$ and $M > 0$. If $f \in \mathcal{A}_n$, and*

$$(1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) \prec 1 + Mz,$$

*then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1 + \alpha n + \beta n(n + 1)},$$

*and the superordinate function is the best dominant.*

## 2 Main results

Analogous to the condition in Lemma 1.3, the following result considers the expression $f'(z)$ in a subordination to obtain a connection with a second-order differential subordination.

**Lemma 2.1.** *Let $\alpha \geq 1$, $\beta > 0$ and $M > 0$. If $f \in \mathcal{A}_n$, and*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) \prec 1 + Mz, \tag{2}$$

*then*

$$f'(z) \prec 1 + \frac{(n+1)Mz}{1+\alpha n+\beta n(n+1)}. \tag{3}$$

*Proof.* First of all, it is known from Lemma 1.3 that

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1+\alpha n+\beta n(n+1)}. \tag{4}$$

Let $P(z) = f'(z)$. Then (2) can be written as

$$\alpha P(z) + \beta z P'(z) + (1-\alpha)\frac{f(z)}{z} \prec 1 + Mz.$$

Now applying the subordination relation (1) of Lemma 1.1, with $\eta = 1$ and $\pi = \alpha - 1$, gives

$$\alpha P(z) + \beta z P'(z) \prec \alpha + \frac{(n+1)(\alpha+\beta n)Mz}{1+\alpha n+\beta n(n+1)}.$$

Lemma 1.2, with $y = \frac{\alpha}{\beta}$, then readily yields

$$P(z) \prec \frac{\alpha}{\beta}\frac{1}{nz^{\alpha/(\beta n)}}\int_0^z \left(1 + \frac{(n+1)(\alpha+\beta n)Mt}{\alpha[1+\alpha n+\beta n(n+1)]}\right) t^{(\alpha/(\beta n))-1}dt$$

which implies

$$f'(z) \prec 1 + \frac{(n+1)Mz}{1+\alpha n+\beta n(n+1)}.$$

This completes the proof. □

An application of Lemma 1.3 and Lemma 2.1 gives the following sufficient condition for convexity of a function satisfying a second-order differential subordination.

**Theorem 2.2.** *Let $\alpha \geq 1$, $\beta > 0$ and $0 \leq \delta < 1$. Further, let $0 < M \leq M_{\alpha,\beta,n}$, where*

$$M_{\alpha,\beta,n} = \frac{\beta(1-\delta)[1+\alpha n+\beta n(n+1)]}{(n+1)\left(\sqrt{(\alpha+\beta n)^2+\alpha^2}+|\alpha-\beta(1-\delta)|\right)}. \tag{5}$$

*If $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) \prec 1 + Mz, \tag{6}$$

*then $f \in \mathcal{CV}(\delta)$.*

*Proof.* Suppose that for $M \leq M_{\alpha,\beta,n}$ given in (5), the subordination (6) holds. Let

$$Q(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

Then (6) can be written as

$$f'(z)[\alpha-\beta+\beta Q(z)]+(1-\alpha)\frac{f(z)}{z}\prec 1+Mz.$$

Similarly as in the proof of Lemma 2.1, an application of Lemma 1.1 by incorporating (4) yields

$$f'(z)[\alpha-\beta+\beta Q(z)]\prec \alpha+\frac{(n+1)(\alpha+\beta n)}{1+\alpha n+\beta n(n+1)}Mz.$$

Recall that a function $f$ is convex of order $\delta$, if $\operatorname{Re}\{Q(z)\}>\delta$ for $z\in U$. Assume, on the contrary, that there exists $z_0\in U$, such that $\operatorname{Re}\left\{Q(z_0)\right\}=\delta$. Then $Q(z_0)=\delta+ix$, for some real number $x$. Hence, a contradiction of the assumption is obtained, if

$$\left|f'(z_0)[\alpha-\beta+\beta(\delta+ix)]-\alpha\right|\geq\frac{(n+1)(\alpha+\beta n)M}{1+\alpha n+\beta n(n+1)}$$

or equivalently

$$\left|\alpha-\beta+\beta(\delta+ix)-\frac{\alpha}{f'(z_0)}\right|\geq\frac{(n+1)(\alpha+\beta n)M}{[1+\alpha n+\beta n(n+1)]|f'(z_0)|}.$$

In view of the fact that

$$\left|\alpha-\beta+\beta(\delta+ix)-\frac{\alpha}{f'(z_0)}\right|\geq\left|\alpha-\beta+\beta\delta-\alpha\operatorname{Re}\frac{1}{f'(z_0)}\right|,$$

it suffices to prove

$$\frac{(n+1)(\alpha+\beta n)M}{[1+\alpha n+\beta n(n+1)]|f'(z_0)|}\leq\left|\alpha-\beta+\beta\delta-\alpha\operatorname{Re}\frac{1}{f'(z_0)}\right|. \tag{7}$$

A computation shows that (7) is equivalent to

$$\frac{(n+1)^2(\alpha+\beta n)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}+\frac{\alpha^2(\operatorname{Im}f'(z_0))^2}{|f'(z_0)|^2}\leq|(\alpha-\beta+\beta\delta)f'(z_0)-\alpha|^2. \tag{8}$$

Taking into account (3), it follows that

$$\begin{aligned}&\frac{(n+1)^2(\alpha+\beta n)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}+\frac{\alpha^2\left(\operatorname{Im}f'(z_0)\right)^2}{|f'(z_0)|^2}\\&\leq\frac{(n+1)^2(\alpha+\beta n)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}+\frac{\alpha^2(n+1)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}\end{aligned} \tag{9}$$

and

$$\left[\beta(1-\delta)-|\alpha-\beta+\beta\delta|\left(\frac{(n+1)M}{1+\alpha n+\beta n(n+1)}\right)\right]^2\leq|(\alpha-\beta+\beta\delta)f'(z_0)-\alpha|^2. \tag{10}$$

For (8) to hold true, using (9) and (10), it is enough to prove

$$\begin{aligned}&\frac{(n+1)^2(\alpha+\beta n)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}+\frac{\alpha^2(n+1)^2M^2}{[1+\alpha n+\beta n(n+1)]^2}\\&\leq\left[\beta(1-\delta)-|\alpha-\beta+\beta\delta|\left(\frac{(n+1)M}{1+\alpha n+\beta n(n+1)}\right)\right]^2\end{aligned}$$

which implies

$$\begin{aligned}&\frac{(n+1)^2\left[2\alpha\beta(n+1-\delta)+\beta^2(n^2-(1-\delta)^2)+\alpha^2\right]}{[1+\alpha n+\beta n(n+1)]^2}M^2\\&\qquad+\frac{2\beta(1-\delta)(n+1)|\alpha-\beta+\beta\delta|}{1+\alpha n+\beta n(n+1)}M-\beta^2(1-\delta)^2\leq 0.\end{aligned} \tag{11}$$

Inequality (11) is fullfilled for $M\leq M_{\alpha,\beta,n}$, where $M_{\alpha,\beta,n}$ is given by (5). This completes the proof. □

**Remark 2.3.**
*When $n = 1$ and $\delta = 0$, various known results are easily obtained as special cases. For instance, the result [9, Theorem 2.2] is easily deduced from Lemma 2.1, while [9, Theorem 2.3] follows from Theorem 2.2.*

The next result gives a convexity criteria for a function defined by a double integral operator associated with Theorem 2.2.

**Theorem 2.4.** *Let $\alpha \geq 1, \beta > 0,\ 0 \leq \delta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| \leq \frac{\beta(1-\delta)[1+\alpha n+\beta n(n+1)]}{(n+1)\left(\sqrt{(\alpha+\beta n)^2+\alpha^2}+|\alpha-\beta(1-\delta)|\right)},$$

*then*

$$f(z) = z + \frac{z^{n+1}}{\beta}\int_0^1\int_0^1 g(rsz)r^{n-1+\frac{1}{\mu}}s^{n-1+\frac{1}{\nu}}\,drds$$

*is in $\mathcal{CV}(\delta)$ where $\mu > 0$ and $\nu > 0$ satisfy $\mu + \nu = \alpha + \beta$ and $\mu\nu = \beta$.*

*Proof.* Let $f \in \mathcal{A}_n$ satisfy

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) - 1 = z^n g(z). \tag{12}$$

From Theorem 2.2, it follows that the solution $f$ of the differential equation (12) is convex of order $\delta$. Let

$$p(z) = \frac{f(z)}{z}.$$

Then (12) simplifies to

$$\beta z^2 p''(z) + (\alpha+2\beta)zp'(z) + p(z) = 1 + z^n g(z). \tag{13}$$

Now let

$$F(z) = p(z) + \nu z p'(z). \tag{14}$$

Then a computation shows that

$$F(z) + \mu z F'(z) = \beta z^2 p''(z) + (\alpha+2\beta)zp'(z) + p(z).$$

Hence (13) simplifies to

$$F(z) + \mu z F'(z) = 1 + z^n g(z),$$

which has a solution

$$F(z) = 1 + \frac{z^n}{\mu}\int_0^1 g(rz)r^{n-1+\frac{1}{\mu}}\,dr. \tag{15}$$

In view of (14), equation (15) can be written as

$$p(z) + \nu z p'(z) = 1 + \frac{z^n}{\mu}\int_0^1 g(rz)r^{n-1+\frac{1}{\mu}}\,dr$$

and has a solution

$$p(z) = 1 + \frac{z^n}{\beta}\int_0^1\int_0^1 g(rsz)r^{n-1+\frac{1}{\mu}}s^{n-1+\frac{1}{\nu}}\,drds,$$

which further implies that

$$f(z) = z + \frac{z^{n+1}}{\beta}\int_0^1\int_0^1 g(rsz)r^{n-1+\frac{1}{\mu}}s^{n-1+\frac{1}{\nu}}\,drds.$$

This completes the proof. □

An application of Theorem 2.2 yields the following sufficient condition for convexity for function defined in terms of a third-order differential subordination.

**Theorem 2.5.** *Let* $\alpha \geq 1, \beta > 0$ *and* $0 \leq \delta < 1$*. Further, let* $\lambda > 0$ *and* $\vartheta \geq 0$ *satisfying*

$$\vartheta + \lambda\left(\frac{\alpha-\lambda}{1-\lambda}\right) = \beta - \gamma, \quad \lambda\vartheta = \gamma, \tag{16}$$

*and* $0 < M \leq M^{\lambda,\vartheta}_{\alpha,\beta,\gamma,n}$*, where*

$$M^{\lambda,\vartheta}_{\alpha,\beta,\gamma,n} = \frac{\vartheta(1-\delta)(1+\lambda n)[1+\alpha n+(\vartheta n(1-\lambda)-\lambda)(n+1)]}{(n+1)\left(\sqrt{(\alpha+\vartheta n(1-\lambda)-\lambda)^2+(\alpha-\lambda)^2}+|(\alpha-\lambda)-\vartheta(1-\delta)(1-\lambda)|\right)}. \tag{17}$$

*If* $f \in \mathcal{A}_n$ *and satisfies the differential subordination*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) + \gamma z^2 f'''(z) \prec 1 + Mz, \tag{18}$$

*then* $f \in \mathcal{CV}(\delta)$.

*Proof.* Let

$$p(z) = \left(\frac{1-\alpha}{1-\lambda}\right)\frac{f(z)}{z} + \left(\frac{\alpha-\lambda}{1-\lambda}\right)f'(z) + \vartheta z f''(z).$$

A brief computation shows that

$$p(z) + \lambda z p'(z) = (1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) + \gamma z^2 f'''(z).$$

Hence (18) can be written as

$$p(z) + \lambda z p'(z) \prec 1 + Mz$$

It follows from Lemma 1.2 that

$$p(z) \prec \frac{1}{\lambda n z^{1/(\lambda n)}}\int_0^z (1+Mt)t^{(-1+1/(\lambda n))}dt = 1 + \frac{M}{1+\lambda n}z,$$

which implies

$$\left(\frac{1-\alpha}{1-\lambda}\right)\frac{f(z)}{z} + \left(\frac{\alpha-\lambda}{1-\lambda}\right)f'(z) + \vartheta z f''(z) \prec 1 + \frac{M}{1+\lambda n}z.$$

By using Theorem 2.2, $f \in \mathcal{CV}(\delta)$ for $M \leq M^{\lambda,\vartheta}_{\alpha,\beta,\gamma,n}$, where $M^{\lambda,\vartheta}_{\alpha,\beta,\gamma,n}$ is given by (17). This completes the proof. □

Corresponding to Theorem 2.5, a sufficient condition for convexity of order $\delta$ for functions defined by a triple integral operator is obtained in the following result.

**Theorem 2.6.** *Let* $\alpha \geq 1, \beta > 0$ *and* $0 \leq \delta < 1$ *and* $g \in \mathcal{H}$*. If*

$$|g(z)| < \frac{\vartheta(1-\delta)(1+\lambda n)[1+\alpha n+(\vartheta n(1-\lambda)-\lambda)(n+1)]}{(n+1)\left(\sqrt{(\alpha+\vartheta n(1-\lambda)-\lambda)^2+(\alpha-\lambda)^2}+|(\alpha-\lambda)-\vartheta(1-\delta)(1-\lambda)|\right)},$$

*where* $\lambda$ *and* $\vartheta$ *are given by* (16)*, then*

$$f(z) = z + \frac{z^{n+1}}{\gamma}\int_0^1\int_0^1\int_0^1 g(rstz)r^{n-1+\frac{1}{\lambda}}s^{n-1+\frac{1}{\mu}}t^{n-1+\frac{1}{\nu}}\,drdsdt$$

*is in* $\mathcal{CV}(\delta)$ *where* $\mu > 0$ *and* $\nu > 0$ *satisfy* $\nu + \mu = (\alpha-\lambda)/(1-\lambda) + \vartheta$ *and* $\mu\nu = \vartheta$.

*Proof.* Let $f \in \mathcal{A}_n$ satisfying

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta z f''(z) + \gamma z^2 f'''(z) - 1 = z^n g(z). \tag{19}$$

By Theorem 2.5, the solution $f$ of the differential equation (19) is a convex function of order $\delta$. Let

$$p(z) = \left(\frac{1-\alpha}{1-\lambda}\right)\frac{f(z)}{z} + \left(\frac{\alpha-\lambda}{1-\lambda}\right) f'(z) + \vartheta z f''(z) - 1.$$

Then (19) reduces to

$$p(z) + \lambda z p'(z) = z^n g(z),$$

which has the solution

$$p(z) = \frac{z^n}{\lambda}\int_0^1 g(rz) r^{n-1+\frac{1}{\lambda}}\, dr. \tag{20}$$

By writing $\phi(z) = \frac{1}{\lambda}\int_0^1 g(rz) r^{n-1+\frac{1}{\lambda}}\, dr$, equation (20) becomes

$$\left(\frac{1-\alpha}{1-\lambda}\right)\frac{f(z)}{z} + \left(\frac{\alpha-\lambda}{1-\lambda}\right) f'(z) + \vartheta z f''(z) - 1 = z^n \phi(z).$$

Comparing this with equation (12) in the proof of Theorem 2.4, the solution $f$ is given by

$$f(z) = z + \frac{z^{n+1}}{\vartheta}\int_0^1\int_0^1 \phi(stz) s^{n-1+\frac{1}{\mu}} t^{n-1+\frac{1}{\nu}}\, ds dt. \tag{21}$$

Substituting for $\phi(stz)$ into (21) yields

$$\begin{aligned} f(z) &= z + \frac{z^{n+1}}{\vartheta}\int_0^1\int_0^1 \left[\frac{1}{\lambda}\int_0^1 g(rstz) r^{n-1+\frac{1}{\lambda}}\, dr\right] s^{n-1+\frac{1}{\mu}} t^{n-1+\frac{1}{\nu}}\, ds dt \\ &= z + \frac{z^{n+1}}{\gamma}\int_0^1\int_0^1\int_0^1 g(rstz) r^{n-1+\frac{1}{\lambda}} s^{n-1+\frac{1}{\mu}} t^{n-1+\frac{1}{\nu}}\, dr ds dt. \end{aligned}$$

This completes the proof. □

**Acknowledgement:** The research of the first and second author is supported by a contract grant from Universiti Tun Hussein Onn Malaysia (U428) and a Fundamental Research Grant Scheme (203/PMATHS/6711568) respectively. The authors are grateful to Prof. S. Ponnusamy, Indian Statistical Institute (ISI), Chennai, India for his valuable comments which improved the presentation of the paper.

# References

[1] Ali R. M., Nargesi M. M., Ravichandran V., Convexity of integral transforms and duality, Complex Var. Elliptic Equ., 2013, 58(11), 1569–1590.

[2] Ali R. M., Lee S.K., Subramanian K. G., Swaminathan A., A third-order differential equation and starlikeness of a double integral operator, Abstr. Appl. Anal., 2011, Art. ID 901235, 10 pp.

[3] Ibrahim R. W., On certain univalent class associated with first order differential subordinations, Tamkang J. Math., 2011, 42(4), 445–451.

[4] Kuroki K., Owa S., Double integral operators concerning starlike of order $\beta$, Int. J. Differential Equations, 2009,Article ID 737129, 13 pages.

[5] Miller S. S.,Mocanu P. T., Double integral starlike operators, Integral Transforms Spec. Funct., 2008, 19(7-8),591–597.

[6] Obradović M., Ponnusamy S., Singh V., Vasundhra P., Differential inequalities and criteria for starlike and univalent functions, Rocky Mountain J. Math., 2006, 36(1), 303–317.

[7] Ponnusamy S., Differential subordination and starlike functions, Complex Variables Theory Appl., 1992, 19(3), 185–194.

[8] Singh S., Gupta S., Singh S., On starlikeness and convexity of analytic functions satisfying a differential inequality, JIPAM. J. Inequal. Pure Appl. Math., 2008, 9(3), Article 81, 7 pp.

[9] Kanas S., Owa S., Second order differential subordinations and certain subordination relations, Sūrikaisekikenkyūsho Kōkyūroku, 1998, 1062, 25–33.

[10] Supramaniam S., Chandrashekar R., Lee S. K., Subramanian K. G., Convexity of functions defined by differential inequalities and integral operators, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 2017, 111(1), 147–157.

[11] Miller S. S.,Mocanu P. T.,Differential subordinations, Dekker, New York, 2000.

[12] Chandrashekar R.,Ali R. M.,Subramanian K. G., Dominant of Functions Satisfying a Differential Subordination and Applications, AIP Conf. Proc., 2014,1605, 580–585.