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Perfect codes in power graphs of finite groups

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Abstract: The power graph of a finite group is the graph whose vertex set is the group, two distinct elements being adjacent if one is a power of the other. The enhanced power graph of a finite group is the graph whose vertex set consists of all elements of the group, in which two vertices are adjacent if they generate a cyclic subgroup. In this paper, we give a complete description of finite groups with enhanced power graphs admitting a perfect code. In addition, we describe all groups in the following two classes of finite groups: the class of groups with power graphs admitting a total perfect code, and the class of groups with enhanced power graphs admitting a total perfect code. Furthermore, we characterize several families of finite groups with power graphs admitting a perfect code, and several other families of finite groups with power graphs which do not admit perfect codes.

Keywords: Power graph, Enhanced power graph, Finite group, Perfect code, Total perfect code

MSC: 05C25, 05C69, 94B99

1 Introduction

Every graph Γ considered in this paper is finite, simple, and undirected with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A code in Γ is simply a subset of $V(\Gamma)$. A code C of Γ is called a *perfect code* [1] if C is an independent set such that every vertex in $V(\Gamma) \setminus C$ is adjacent to exactly one vertex in C . A code C is said to be a *total perfect code* [2] in Γ if every vertex of Γ is adjacent to exactly one vertex in C .

Since the beginning of coding theory in the late 1940s, perfect codes have been important objects of study in information theory; see the surveys [3, 4] on perfect codes and related definitions in the classical setting. It is known in [5] that deciding whether a graph has a total perfect code is NP-complete. Beginning with [1], perfect codes in general graphs have also attracted considerable attention in the community of graph theory (see [6–9]). In particular, perfect codes in Cayley graphs of groups are especially charming objects of study (see [10–12]). For more information on coding applications of algebraic constructions, the readers are referred to [13, §9.1 and §9.2].

Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications (cf. [14, 15]) and are related to automata theory (cf. [16, 17]). The *undirected power graph* Γ_G of a finite group G has the vertex set G and two distinct elements are adjacent if one is a power of the other. The *enhanced power graph* Δ_G of a finite group G is the graph whose vertex set consists of G , in which two distinct vertices are adjacent if they generate a cyclic subgroup. The concepts of a power graph and an undirected power

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graph were first introduced by Kelarev and Quinn [18] and Chakrabarty *et al.* [19], respectively. Since the paper deals only with undirected graphs, we use the term “power graph” to refer to an undirected power graph. In recent years, the study of power graphs has been growing (see [19–30]). Also, see [31] for a survey of results and open questions on power graphs. In order to measure how close the power graph is to the commuting graph [32], Aalipour *et al.* [33] introduced the enhanced power graph which lies in between. See [34] for some properties of the enhanced power graphs.

In this paper, we always use G to denote a finite group with the identity e . Denote by G^* the set $G \setminus \{e\}$. For a subset S of G , let $\Gamma_{G[S]}$ (resp. $\Delta_{G[S]}$) denote the induced subgraph of Γ_G (resp. Δ_G) by S . If the situation is unambiguous, then we denote $\Gamma_{G[S]}$ (resp. $\Delta_{G[S]}$) simply by Γ_S (resp. Δ_S).

The paper is devoted to studying the perfect codes of the power graph of a finite group. We first give sharp lower and upper bounds for the size of a subset of G to be a perfect code in Γ_{G^*} and characterize the groups achieving the bounds (see Theorem 2.2). We also give several families of groups G such that Γ_{G^*} admits a perfect code, and several other families of groups G such that Γ_{G^*} does not admit perfect codes. Furthermore, we obtain a complete characterization of finite groups whose enhanced power graphs admit a perfect code (see Theorem 2.10). In particular, we characterize the groups G such that Δ_{G^*} admits a perfect code with size 1 (see Theorem 2.11), which answers a question posed by Bera and Bhuniya [34]. Also, we classify all nilpotent groups G such that Δ_{G^*} admits a perfect code with size 1 (see Proposition 2.12), which extends [34, Theorems 3.2 and 3.3]. In Sect. 3, we describe all groups in the following two classes of finite groups: the class of groups with power graphs admitting a total perfect code (see Corollary 3.4), and the class of groups with enhanced power graphs admitting a total perfect code (see Theorem 3.5).

2 Perfect codes

We first remark both Δ_G and Γ_G admit a perfect code $\{e\}$. If G is cyclic, then a generator g of G is adjacent to every element of $G^* \setminus \{g\}$ in Γ_{G^*} and Δ_{G^*} , and so in this case, both Γ_{G^*} and Δ_{G^*} admit a perfect code $\{g\}$.

In the rest of this section, therefore, we always assume that G is noncyclic. We focus on studying the perfect codes of Γ_{G^*} and Δ_{G^*} . We first give sharp lower and upper bounds for the size of a subset of G to be a perfect code in Γ_{G^*} , and characterize the groups achieving the bounds. Next, we give some families of finite groups G such that Γ_{G^*} admits a perfect code, and give several other families of groups G such that Γ_{G^*} does not admit perfect codes. Finally, we give a complete characterization of finite groups G such that Δ_{G^*} admits a perfect code.

2.1 Power graphs

The *neighborhood* of a vertex x in a graph Γ , denoted by $N_\Gamma(x)$, is the set of vertices which have distance one from x . If the situation is unambiguous, then we denote $N_\Gamma(x)$ simply by $N(x)$. A *maximal cyclic subgroup* of G is a cyclic subgroup, which is not a proper subgroup of some proper cyclic subgroup of G . We remark that a finite group has a unique maximal cyclic subgroup if and only if the group is cyclic of prime power order. Denote by $\mathcal{M}_G$ the set of all maximal cyclic subgroups of G . Note that G is always noncyclic. Write

$$\mathcal{M}_G = \{M_1, M_2, \dots, M_t\} \quad (1)$$

where t is a positive integer at least 2.

Lemma 2.1. *With reference to (1), if C is a perfect code of Γ_{G^*} , then for any $1 \leq i \leq t$, there exists precisely one vertex in C such that it belongs to M_i .*

Proof. Let x be a generator of M_i for some $1 \leq i \leq t$. Suppose that $x \notin C$. Since C is a perfect code, there exists an element a in C such that a is adjacent to x . If $\langle x \rangle \subseteq \langle a \rangle$, since $\langle x \rangle$ is maximal cyclic, we have $\langle x \rangle = \langle a \rangle$, and so $a \in M_i$. If $\langle a \rangle \subseteq \langle x \rangle$, then it is clear that $a \in M_i$. It follows that there exists at least one vertex in C such that it belongs to M_i .

Now assume, to the contrary, that C contains two distinct elements y and z with $y, z \in M_i$. Since C is an independent set, $x \notin C$. Therefore, x is adjacent to exactly one vertex of C . Since $y, z \in M_i$ and $\langle x \rangle = M_i$, it follows that $y, z \in N(x)$, a contradiction. $\square$

With reference to (1), let $M_G = \bigcup_{1 \leq i < j \leq t} (M_i \cap M_j)$. For each $1 \leq i \leq t$, write

$$M_i^* = M_i \setminus M_G. \quad (2)$$

The generalized quaternion group of order $4n$ is defined by

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle, \quad n \geq 2. \quad (3)$$

By verifying (3), $y^{-1} = x^n y$, $|x^i y| = 4$ and $(x^i y)^{-1} = x^{2n-i} y$ for $i \in \{1, \dots, n-1\}$. Now we have the following theorem.

Theorem 2.2. Suppose that Γ_{G^*} admits a perfect code C . With reference to (1),

$$1 \leq |C| \leq t, \quad (4)$$

the lower bound holds if and only if $G \cong Q_{2^m}$ where Q_{2^m} is the generalized quaternion group of order 2^m and $m \geq 3$, and the upper bound holds if and only if there exists a set $\{x_1, \dots, x_t\} \subseteq G^*$ satisfying the conditions:

- (i) With reference to (2), $x_i \in M_i^*$ for each $1 \leq i \leq t$;
- (ii) $N(x_i) \cap N(x_j) = \emptyset$ for each two distinct i, j in $\{1, \dots, t\}$;
- (iii) $\bigcup_{i=1}^t N(x_i) = G^* \setminus \{x_1, \dots, x_t\}$.

Proof. By Lemma 2.1, we have $1 \leq |C| \leq t$. Clearly, $|C| = 1$ if and only if there exists an element a in G^* such that $N(a) = G^* \setminus \{a\}$. Since we are assuming that G is not cyclic, it follows from [21, Proposition 4] that $|C| = 1$ if and only if $G \cong Q_{2^m}$, as required.

Now write $D = \{x_1, \dots, x_t\}$. Suppose that D satisfies conditions (i)–(iii). By (iii), D is an independent set of Γ_{G^*} . Also, by (ii) and (iii), it is easy to see that every vertex in $G^* \setminus D$ is adjacent to exactly one vertex of D . This implies that D is a perfect code of Γ_{G^*} of size t , as required.

Let $C = \{x_1, \dots, x_t\}$ be a perfect code of Γ_{G^*} of size t . With reference to (1), if $x_i \in M_j \cap M_l$ and $j \neq l$ for some i, j, l in $\{1, \dots, t\}$, by the Dirichlet principle, there exist two elements in C such that they both belong to same maximal cyclic subgroup of G , contrary to Lemma 2.1. We conclude that (i) holds. Also, since C is a perfect code, it is clear that C satisfies (ii) and (iii). $\square$

Now we give a family of groups G such that Γ_{G^*} admits a perfect code C , where $|C|$ satisfies the upper bound of (4).

Example 2.3. Let $G = \mathbb{Z}_p \times \mathbb{Z}_q^n$, where p and q are two distinct primes and n is a positive integer at least 2. Then Γ_{G^*} admits a perfect code of size $\frac{q^n-1}{q-1}$. In particular, every perfect code of Γ_{G^*} has size $|\mathcal{M}_G|$.

Proof. Clearly, G has a unique subgroup of order p , say P . Let $t = \frac{q^n-1}{q-1}$. Then it is not hard to see that G has t maximal cyclic subgroups $M_1, \dots, M_t$ and each of them is isomorphic to $\mathbb{Z}_{pq}$. Furthermore, the intersection of each two distinct maximal cyclic subgroups is P . Suppose that C is a perfect code of Γ_{G^*} . If there exists x in P^* such that $x \in C$, then by Lemma 2.1 $|C| = 1$, and hence we have a contradiction by Theorem 2.2, since G is neither cyclic nor generalized quaternion. It follows that if C has an element $y \in M_i$ for some $i \in \{1, \dots, t\}$, then $y \in M_i^*$, which implies that C has size t by Lemma 2.1.

Now take elements $x_1, \dots, x_t$ in $\bigcup_{i=1}^t M_i^*$ with $x_i \in M_i^*$ for each $i \in \{1, \dots, t\}$, $|x_t| = pq$, and $|x_j| = q$ for each $1 \leq j \leq t-1$. By Theorem 2.2, it is easy to check that $\{x_1, \dots, x_t\}$ is a perfect code, as required. $\square$

The following example shows that, although Γ_{G^*} admits a perfect code, the size of each perfect code of Γ_{G^*} does not achieve the upper bound of (4).

Example 2.4. Let $G = \mathbb{Z}_2^n \times \mathbb{Z}_{12}$, where n is a positive integer at least 2. Then Γ_{G^*} admits a perfect code of size $2^{n+1} - 1$. In particular, Γ_{G^*} does not admit a perfect code of size $|\mathcal{M}_G|$.

Proof. It is evident that G has a unique subgroup P of order 3, and has 2^n and $2^{n+1} - 2$ maximal cyclic subgroups of order 12 and 6, respectively. Moreover, the intersection of each two maximal cyclic subgroups of order 12 has order 6. Write

$$\mathcal{M}_G = \{M_1, \dots, M_{2^n}, K_1, \dots, K_{2^{n+1}-2}\},$$

where $|M_i| = 12$ and $|K_j| = 6$ for all $1 \leq i \leq 2^n$ and $1 \leq j \leq 2^{n+1} - 2$, $\bigcap_{i=1}^{2^n} M_i \cong \mathbb{Z}_6$ and $\bigcap_{i=1}^{2^{n+1}-2} K_i = P$.

Suppose that C is a perfect code of Γ_{G^*} of size $|\mathcal{M}_G|$. By Theorem 2.2, there exist $x \in M_1 \setminus M_2$ and $y \in M_2 \setminus M_1$ such that $x, y \in C$. Take $a \in M_1 \cap M_2$ with $|a| = 2$. It follows that $a \notin C$ and $a \in N(x) \cap N(y)$, a contradiction. We conclude that Γ_{G^*} does not admit perfect codes of size $|\mathcal{M}_G|$.

Take x_1 in M_1 , $x_0 \in K_1$ and $x_i \in K_i$ for all $i \in \{2, \dots, 2^{n+1} - 2\}$, so that $|x_0| = 6$ and $|x_j| = 2$ for each $1 \leq j \leq 2^{n+1} - 2$. Then it is easy to check that $\{x_0, x_1, \dots, x_{2^{n+1}-2}\}$ is a perfect code of Γ_{G^*} , as required. $\square$

Finally, we remark that many power graphs do not admit perfect codes. In the following we give two families of examples.

Proposition 2.5. *With reference to (1), suppose that G satisfies*

$$M_i \cong \mathbb{Z}_{4p}, \quad \bigcap_{j=1}^t M_j \cong \mathbb{Z}_{2p}$$

for some odd prime p and all $1 \leq i \leq t$. Then Γ_{G^} does not admit perfect codes.*

Proof. Let $\bigcap_{j=1}^t M_j = P$. Suppose for a contradiction that C is a perfect code of Γ_{G^*} . Note that G is not isomorphic to a generalized quaternion 2-group. It follows from Theorem 2.2 that $|C| \geq 2$. Also, by Lemma 2.1 we deduce $C \cap P^* = \emptyset$, and so there exist $x \in M_1 \setminus P$ and $y \in M_2 \setminus P$ such that $x, y \in C$. We conclude that $|x|, |y| \in \{4p, 4\}$. Take z in P with $|z| = 2$. Then $z \in N(x) \cap N(y)$, which is a contradiction since C is a perfect code. $\square$

The following result follows from Proposition 3.3.

Example 2.6. *Let $G = Q_8 \times \mathbb{Z}_p$ for some odd prime p . Then Γ_{G^*} does not admit perfect codes.*

Proposition 2.7. *Suppose that G satisfies*

$$\mathcal{M}_G = \{M_{11}, M_{12}, M_{21}, M_{22}, \dots, M_{k1}, M_{k2}\},$$

where k is a positive integer at least 2, $\bigcap_{i=1}^k M_{i1} \cong \mathbb{Z}_p$, $M_{i1} \cap M_{i2} \cong \mathbb{Z}_{2p}$ and $M_{ij} \cong \mathbb{Z}_{4p}$ for some odd prime p , all $1 \leq i \leq k$ and all $1 \leq j \leq 2$. Then Γ_{G^} does not admit perfect codes.*

Proof. It is clear that G has a unique subgroup of order p , say P . Suppose that C is a perfect code of Γ_{G^*} , we work to obtain a contradiction. Clearly, G is not a generalized quaternion 2-group. By Theorem 2.2, $|C| \geq 2$, and it follows from Lemma 2.1 that $P^* \cap C = \emptyset$. Also, Lemma 2.1 implies that there exist x in M_{11} and y in M_{12} such that $x, y \in C$. If $x \neq y$, then $x \in M_{11} \setminus M_{12}$ and $y \in M_{12} \setminus M_{11}$, and so $|x|, |y| \in \{4p, 4\}$, which implies that for the involution z of $M_{11} \cap M_{12}$, we have $z \in N(x) \cap N(y)$, a contradiction. We deduce $x = y$, and hence $x \in (M_{11} \cap M_{12}) \setminus P$, which implies that $|x| = 2$ or $2p$. Since C is a perfect code, the element of order 4 of M_{11} is adjacent to x in Γ_{G^*} , and thus $|x| = 2$. Similarly, considering M_{i1} and M_{i2} for each $2 \leq i \leq k$, we conclude that every element of C is an involution. It follows that there does not exist an element in C such that it is adjacent to a generator of P , which is a contradiction. $\square$

By Proposition 2.7, we have the following example.

Example 2.8. $\Gamma_{(\mathbb{Z}_4 \times \mathbb{Z}_{12})^*}$ does not admit perfect codes.

2.2 Enhanced power graphs

In this subsection, we give a complete characterization of finite groups whose enhanced power graphs admit a perfect code. We begin with the following lemma. Since Γ_G^* is a subgraph of Δ_G^* , the proof of Lemma 2.9 is similar to the proof of Lemma 2.1.

Lemma 2.9. *With reference to (1), if C is a perfect code of Δ_G^* , then for any $1 \leq i \leq t$, there exists precisely one element in C such that it belongs to M_i .*

Suppose that G is a group with the property that each two maximal cyclic subgroups of G have trivial intersection, or that if

$$\bigcap_{i=1}^l M_i \neq \{e\}$$

for some maximal cyclic subgroups $M_1, \dots, M_l$ of G (here l may be 1), and there exists M_k in $\mathcal{M}_G \setminus \{M_1, \dots, M_l\}$ such that $M_k \cap M_m$ is nontrivial for some $m \in \{1, 2, \dots, l\}$, then

$$\left(\bigcap_{i=1}^l M_i \right) \cap M_k \neq \{e\}.$$

Then G is said to satisfy the *intersection property*. For example, both the elementary abelian p -group $\mathbb{Z}_p^n$ and $\mathbb{Z}_p \times \mathbb{Z}_q^n$ satisfy the intersection property, where p and q are distinct primes.

Theorem 2.10. *Δ_G^* admits a perfect code if and only if G satisfies the intersection property.*

Proof. First, assume that G satisfies the intersection property. Write

$$\mathcal{M}_G = \{M_{11}, M_{12}, \dots, M_{1l_1}, M_{21}, M_{22}, \dots, M_{2l_2}, \dots, M_{m1}, M_{m2}, \dots, M_{ml_m}, \\ M_{m+1}, M_{m+2}, \dots, M_{m+n}\},$$

where $l_i \geq 2$ and $\bigcap_{j=1}^{l_i} M_{ij}$ is nontrivial for each $1 \leq i \leq m$, and every two of the rest have trivial intersection. For each $1 \leq i \leq m$ and $1 \leq k \leq n$, take $a_i \in \bigcap_{j=1}^{l_i} M_{ij}^*$ and $b_k \in M_{m+k}^*$. Let $C = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$.

Suppose that a_1 and a_2 are adjacent in Δ_G^* . Then $\langle a_1, a_2 \rangle$ is cyclic, and so there exists a maximal cyclic subgroup M of G such that $\langle a_1, a_2 \rangle \subseteq M$. Since G satisfies the intersection property, a_1 must belong to one of $\{M_{11}, M_{12}, \dots, M_{1l_1}\}$ and a_2 must belong to one of $\{M_{21}, M_{22}, \dots, M_{2l_2}\}$, which implies that

$$M \in \{M_{11}, M_{12}, \dots, M_{1l_1}\} \cap \{M_{21}, M_{22}, \dots, M_{2l_2}\},$$

a contradiction.

We conclude that a_1 and a_2 are nonadjacent in Δ_G^* . Similarly, we can obtain that C is an independent set of Δ_G^* . Let y be an arbitrary element of $G^* \setminus C$. Without loss of generality, let $y \in M_{11}^* \setminus C$. Clearly, a_1 and y are adjacent in Δ_G^* . Suppose that there exists $z \in C \setminus \{a_1\}$ such that y is adjacent to z . If $z = a_i$ for some $2 \leq i \leq m$, then $\langle z, y \rangle$ is contained in one of $\{M_{i1}, M_{i2}, \dots, M_{il_i}\}$, and by the intersection property of G , it follows that $M_{11} \cap M_{i1}$ is nontrivial, contrary to the hypotheses of $\mathcal{M}_G$. Similarly, we can show that $z \neq b_i$ for some $1 \leq i \leq n$. We conclude, therefore, that y is adjacent to exactly one vertex in C . It follows that C is a perfect code, as required.

Conversely, assume that Δ_G^* admits a perfect code D . Suppose for a contradiction that G does not satisfy the intersection property. In other words, there exist 3 distinct maximal cyclic subgroups M_1, M_2, M_3 of G such that

$$|M_1 \cap M_2| \neq 1, |M_1 \cap M_3| \neq 1, |M_1 \cap M_2 \cap M_3| = 1.$$

If $D \cap M_1 \cap M_2$ is trivial, then by Lemma 2.9, there exist $m_1 \in M_1 \setminus M_2$ and $m_2 \in M_2 \setminus M_1$ such that $m_1, m_2 \in D$, and hence $x \in N(m_1) \cap N(m_2)$ for some $x \in M_1^* \cap M_2^*$, a contradiction. We conclude that there exists $u \in M_1^* \cap M_2^*$ such that $u \in D$. By Lemma 2.9 again, there exists $v \in M_3 \cap D$ such that both u and v belong to same maximal cyclic subgroup. Take $w \in M_1^* \cap M_3^*$. Clearly, $w \neq v$, so we deduce $w \in N(v) \cap N(u)$, and this contradiction completes the proof. $\square$

A graph Γ is said to satisfy the *cone property* if Γ has a vertex which is adjacent to every vertex except itself. Bera and Bhuniya [34] posed the question: Characterize all finite non-abelian groups G such that Δ_{G^*} satisfies the cone property. Now we give an answer to the question.

Theorem 2.11. *The following are equivalent for any group G with*

$$\mathcal{M}_G = \{M_1, M_2, \dots, M_t\}.$$

- (i) Δ_{G^*} satisfies the cone property.
- (ii) $\bigcap_{i=1}^t M_i$ is nontrivial.
- (iii) Δ_{G^*} admits a perfect code of size 1.

Proof. First, assume (i), and let x be a vertex with $|N(x)| = |G| - 2$ in Δ_{G^*} . Without loss of generality, let $x \in M_1$. Choose a generator x_2 of M_2 . Since x is adjacent to x_2 , $\langle x, x_2 \rangle = \langle x_2 \rangle$, which implies $x \in M_2$. Similarly, we deduce that x belongs to M_i for each $3 \leq i \leq t$. It follows that $x \in \bigcap_{i=1}^t M_i$, proving (ii).

Now assume (ii), so that $y \in \bigcap_{i=1}^t M_i^*$. For some $z \in G^* \setminus \{y\}$, let $z \in M_i$ for some $1 \leq i \leq t$. Then $y, z \in M_i$, and thus $\langle y, z \rangle$ is cyclic, and it follows that y and z are adjacent in Δ_{G^*} . This implies that $\{y\}$ is a perfect code of Δ_{G^*} , and (iii) follows.

Finally, since it is obvious that (iii) implies (i), the proof is complete. $\square$

Next, we classify all finite nilpotent groups G such that Δ_{G^*} admits a perfect code of size 1, which extends [34, Theorems 3.2 and 3.3].

Proposition 2.12. *Let G be a nilpotent group. Then Δ_{G^*} admits a perfect code of size 1 if and only if*

$$G \cong Q_{2^n} \times H \text{ or } \mathbb{Z}_{p^m} \times K,$$

where $n \geq 3$, $m \geq 1$, p is a prime, and both H and K are nilpotent with $2 \nmid |H|$ and $p \nmid |K|$.

Proof. First, assume that Δ_{G^*} admits a perfect code $\{x\}$. Let q be a prime divisor of $|x|$ and let y be an element of G with $|y| = p$. If $y \notin \langle x \rangle$, since x and y are adjacent, $\langle x, y \rangle$ is cyclic, and so $\langle x, y \rangle$ has two distinct subgroups of order p , a contradiction. It follows that G has a unique subgroup of order p . Now let P be the unique Sylow p -subgroup of G . By [35, Theorem 5.4.10], a p -group having a unique subgroup of order p is either cyclic or a generalized quaternion, so it follows that P is either cyclic or a generalized quaternion. Since G is nilpotent, the desired result follows.

Conversely, assume that $G \cong Q_{2^n} \times H$ or $\mathbb{Z}_{p^m} \times K$, where $n \geq 3$, $m \geq 1$, p is a prime, and both H and K are nilpotent with $2 \nmid |H|$ and $p \nmid |K|$. Then G has a unique subgroup P of order 2 or p , and so we conclude that every maximal cyclic subgroup of G contains P . It follows that the intersection of all maximal cyclic subgroups is nontrivial. Now the desired result follows from Theorem 2.11. $\square$

2.3 Examples

In this subsection, we give some families of finite groups whose power graphs or enhanced power graphs admit a perfect code.

Proposition 2.13. *If every two maximal cyclic subgroups of G have trivial intersection, then both Δ_{G^*} and Γ_{G^*} admit a perfect code.*

Proof. By Theorem 2.10, Δ_{G^*} admits a perfect code. In the following we prove that Γ_{G^*} admits a perfect code. With reference to (1), let $\langle x_i \rangle = M_i$ for all $1 \leq i \leq t$ and let $C = \{x_1, x_2, \dots, x_t\}$. Since M_i is maximal cyclic, C is an independent set of Γ_{G^*} . Let x be an arbitrary element of $G^* \setminus C$. Without loss of generality, say $x \in M_1$. Clearly, x_1 and x are adjacent. Since $|M_1 \cap M_j| = 1$ for each $2 \leq j \leq t$, x is not adjacent to each of $C \setminus \{x_1\}$. So C is a perfect code of Γ_{G^*} , as required. $\square$

A finite group is called a *P-group* [36] if every nontrivial element of the group has prime order. For example, $\mathbb{Z}_p^m$ is a *P-group* for some prime p . Also, it is clear that every two maximal cyclic subgroups of a *P-group* have trivial intersection. A finite group is called a *CP-group* [37] if every nontrivial element of the group has prime power order. For example, every *p-group* is a *CP-group*. Certainly, a *P-group* is also a *CP-group*.

By Proposition 2.13 we see that both Δ_{H^*} and Γ_{H^*} admit a perfect code for each *P-group* H . What is more, here we prove that both Δ_{G^*} and Γ_{G^*} admit a perfect code for each *CP-group* G .

Theorem 2.14. *Let G be a CP-group. Then both Δ_{G^*} and Γ_{G^*} admit a perfect code.*

Proof. For two distinct elements x, y of G^* , if x is a power of y , or y is a power of x , then $\langle x, y \rangle$ is cyclic. Also, if $\langle x, y \rangle$ is cyclic, since G is a *CP-group*, $\langle x, y \rangle$ is isomorphic to a cyclic group of prime power order, which implies that one of $\{x, y\}$ is a power of the other. It follows that Δ_{G^*} and Γ_{G^*} are the same, and hence it suffices to prove Δ_{G^*} admits a perfect code.

As refer to (1), assume that there exist two distinct indices i, j in $\{1, \dots, t\}$ such that $M_i \cap M_j$ is nontrivial. Let $x \in M_i \cap M_j$ with $|x| = p$, where p is a prime. If there exists M_t in $\mathcal{M}_G \setminus \{M_i, M_j\}$ such that $M_t \cap M_i$ is nontrivial, then $x \in M_t$, and so we deduce $M_i \cap M_j \cap M_t$ is nontrivial. This implies that G satisfies the intersection property. It follows from Theorem 2.10 that Δ_{G^*} admits a perfect code. $\square$

For $n \geq 3$, denote by D_{2n} the dihedral group of order $2n$, where

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.$$

It is not hard to see that $\mathcal{M}_{D_{2n}} = \{\langle a \rangle, \langle ab \rangle, \langle a^2b \rangle, \dots, \langle a^{n-1}b \rangle\}$ and $|a^i b| = 2$ for each $1 \leq i \leq n$. It follows that every two maximal cyclic subgroups of D_{2n} have trivial intersection. The following result is immediate by Proposition 2.13.

Example 2.15. *Both $\Gamma_{D_{2n}^*}$ and $\Delta_{D_{2n}^*}$ admit a perfect code.*

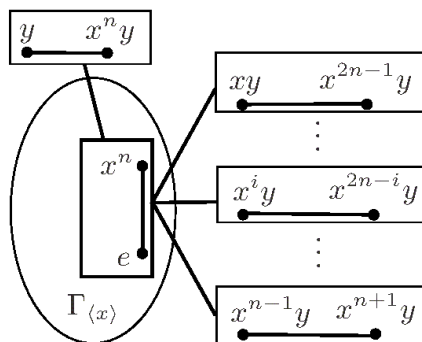
For the generalized quaternion group Q_{4n} , by (3) we have

$$V(\Gamma_{Q_{4n}}) = \{e, x, \dots, x^{2n-1}\} \cup \left(\bigcup_{i=0}^{n-1} \{x^i y, (x^i y)^{-1}\} \right),$$

$$E(\Gamma_{Q_{4n}}) = E(\Gamma_{\langle x \rangle}) \cup \bigcup_{i=0}^{n-1} E(\Gamma_{\langle x^i y \rangle}).$$

The structure of $\Gamma_{Q_{4n}}$ is shown in Figure 1. Now we study the perfect codes of $\Gamma_{Q_{4n}^*}$ and $\Delta_{Q_{4n}^*}$.

Fig. 1. $\Gamma_{Q_{4n}}$



Example 2.16. (i) *For any $n \geq 2$, $\Gamma_{Q_{4n}^*}$ admits a perfect code if and only if n is a power of 2.*
(ii) *For any $n \geq 2$, $\Delta_{Q_{4n}^*}$ admits a perfect code.*

Proof. (i) If n is a power of 2, then Q_{4n} is a CP -group, and so $\Gamma_{Q_{4n}^*}$ admits a perfect code by Theorem 2.14. Conversely, let C be a perfect code of $\Gamma_{Q_{4n}^*}$. Suppose that n is not a power of 2, we work to obtain a contradiction. With reference to (3), we deduce that

$$\mathcal{M}_{Q_{4n}} = \{\langle x \rangle, \langle xy \rangle, \langle x^2y \rangle, \dots, \langle x^n y \rangle\}, \quad x^n \in \bigcap_{i=1}^n \langle x^i y \rangle.$$

If $x^n \in C$, since $x^n \in \langle x \rangle$, $C = \{x^n\}$ by Lemma 2.1, contrary to Theorem 2.2. We conclude that $x^n \notin C$. By Lemma 2.1 again, there exist $b, c \in C$ such that $b \in \langle xy \rangle$ and $c \in \langle x^2y \rangle$. Since $|xy| = |x^2y| = 4$ and $(xy)^2 = (x^2y)^2 = x^n \notin C$, it follows that x^n is adjacent to both b and c . This is a contradiction since C is a perfect code.

(ii) With reference to (3), x^n belongs to each maximal cyclic subgroup of Q_{4n} . The desired result follows from Theorem 2.11. $\square$

3 Total perfect codes

In this section we characterize all finite groups whose power graphs or enhanced power graphs admit a total perfect code. The following observation follows easily from the definition of a total perfect code.

Observation 3.1. *Let Γ be a graph. A code C of Γ is a total perfect code if and only if the subgraph of Γ induced by C is a matching and $\{N(u) \setminus C : u \in C\}$ is a partition of $V(\Gamma) \setminus C$.*

Theorem 3.2. *Suppose that Γ is a graph containing a vertex x of degree $n - 1$, where $n = |V(\Gamma)|$. Then Γ admits a total perfect code if and only if Γ has a leaf. In particular, C is a total perfect code of Γ if and only if $C = \{a, b\}$ for some $a, b \in V(\Gamma)$ with $|N(a)| = n - 1$ and $|N(b)| = 1$.*

Proof. If Γ has a leaf y , then by Observation 3.1 $\{x, y\}$ is a total perfect code, as desired. Now suppose that Γ admits a total perfect code C . Since $|N(x)| = n - 1$, $x \in C$. Also, since the subgraph of Γ induced by C is a matching, we may assume that $C = \{x, z\}$ for some $z \in V(\Gamma)$. It follows that $\{N(x) \setminus \{z\}, N(z) \setminus \{x\}\}$ is a partition of $V(\Gamma) \setminus C$, and hence $N(z) \setminus \{x\} = \emptyset$. This implies that z is a leaf, as required. $\square$

An involution x of G is *maximal* if the only cyclic subgroup containing x is $\langle x \rangle$. For example, each involution of D_{2m} is maximal for some odd number m . We remark that $\mathbb{Z}_n$ has a maximal involution if and only if $n = 2$. The proof of the following result is straightforward.

Proposition 3.3. *The following are equivalent.*

- (i) Γ_G has a leaf.
- (ii) Δ_G has a leaf.
- (iii) G has a maximal involution.

Since in Γ_G and Δ_G , e has degree $|G| - 1$. The following result is immediate by Theorem 3.2 and Proposition 3.3.

Corollary 3.4. *The following are equivalent.*

- (i) Γ_G admits a total perfect code.
- (ii) Δ_G admits a total perfect code.
- (iii) G has a maximal involution.

The *exponent* of G is the least common multiple of the orders of the elements of G . Next, we characterize all finite groups G such that Γ_{G^*} or Δ_{G^*} admits a total perfect code.

Theorem 3.5. *The following are equivalent.*

- (i) Γ_{G^*} admits a total perfect code.

- (ii) G is a finite group of exponent 3.
- (iii) Δ_{G^*} admits a total perfect code.

Proof. First, suppose that Γ_{G^*} admits a total perfect code C . Let $x, y \in C$ with $\{x, y\} \in E(\Gamma_{G^*})$. Assume that one of $\{x, y\}$ has order at least 4, without loss of generality, let $|x| \geq 4$. If $y \neq x^{-1}$, by Observation 3.1 $x^{-1} \in V(\Gamma_{G^*}) \setminus C$, and thus we deduce $x^{-1} \in N(x) \cap N(y)$, a contradiction. It follows that $y = x^{-1}$. Also, it is clear $x^2 \notin C$, and so $x^2 \in N(x) \cap N(y)$, a contradiction.

We conclude that both x and y have order at most 3. Observe that $|x| = 3$ and $y = x^{-1}$. It follows that every element of C has order 3 and C is inverse-closed. If $V(\Gamma_{G^*}) \setminus C$ has an element u , then there exists $z \in C$ such that u and z are adjacent, however, u^{-1} and z are adjacent and $u^{-1} \in C$, a contradiction. It follows that $V(\Gamma_{G^*}) = C$, and so every nontrivial element of G has order 3, hence (ii) follows.

Now if G is a finite group of exponent 3, then G^* is a total perfect code of Γ_{G^*} . It follows that (i) and (ii) are equivalent. Similarly, we can conclude that (ii) and (iii) are equivalent. $\square$

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