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Deficiency of forests

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Abstract: An edge-magic total labeling of an (n, m) -graph $G = (V, E)$ is a one to one map λ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \dots, n + m\}$ with the property that there exists an integer constant c such that $\lambda(x) + \lambda(y) + \lambda(xy) = c$ for any $xy \in E(G)$. It is called super edge-magic total labeling if $\lambda(V(G)) = \{1, 2, \dots, n\}$. Furthermore, if G has no super edge-magic total labeling, then the minimum number of vertices added to G to have a super edge-magic total labeling, called super edge-magic deficiency of a graph G , is denoted by $\mu_s(G)$ [4]. If such vertices do not exist, then deficiency of G will be $+\infty$. In this paper we study the super edge-magic total labeling and deficiency of forests comprising of combs, 2-sided generalized combs and bistar. The evidence provided by these facts supports the conjecture proposed by Figueroa-Centeno, Ichishima and Muntaner-Bartle [2].

Keywords: Forests, Super edge magic total labeling, Comb, 2-sided generalized comb, Bistar, Deficiency of graph

MSC: 05C78

1 Basic definitions, notations and preliminary results

Let $G = (V, E)$ be a finite, simple, undirected graph having $|V(G)| = n$ and $|E(G)| = m$, where $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. A general orientation for graph theoretic concepts can be seen in [10]. A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive integers). A labeling that uses the vertex set only (or the edge set only), is known as vertex labeling (or the edge labeling). If the domain of the labeling includes all vertices and edges, then such a labeling is called total labeling. Cordial, graceful, harmonious and anti-magic are few types of labeling. A bijective labeling is called an edge-magic total if it satisfies the following property, given any edge $xy \in E(G)$,

$$\lambda(x) + \lambda(y) + \lambda(xy) = c, \quad (1)$$

for some constant c . In other words, an edge-magic total labeling of a graph G is a bijective map λ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \dots, n + m\}$ satisfying (1). The constant c is known as the magic constant and a graph that admits an edge-magic total labeling is called an edge-magic total graph. In [8, 9], Kotzig and Rosa have given the origin of edge-magic total labeling of graphs. Recently, Enomoto *et al.* [1] brought in the name, super edge-magic

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labeling in the sense of Kotzig and Rosa, with the additional property that the vertices receive the smallest labels. In [1] they put forward the following conjecture:

Conjecture 1.1 ([1]). “Every tree is super edge-magic total.”

In this paper we are focused on super edge-magic total labeling. A number of classification problems on edge-magic total graphs have been extensively investigated. For further details see recent survey of graph labelings [6]. Kotzig and Rosa in [9] show that there exists an edge-magic total graph H for any graph G such that $H \cong G \cup nK_1$ for some non-negative integer n . This verity provides the base for the concept of edge-magic total deficiency of a graph G [9], denoted by $\mu(G)$, which is the minimum non-negative integer n such that $G \cup nK_1$ is edge-magic total i.e.,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic total}\}.$$

In the same paper Kotzig and Rosa provide an upper bound of the edge-magic deficiency of a graph G having order n ,

$$\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1),$$

where F_n denotes the n -th Fibonacci number.

The super edge-magic deficiency of a graph G , denoted by $\mu_s(G)$ [4], is mathematically expressed as if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is super edge-magic graph}\},$$

Then

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset \\ \infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is easy to see that $\mu(G) \leq \mu_s(G)$. In [2], Figueroa-Centeno *et al.* conjectured, “Every forest with two components has the super edge-magic deficiency at most 1”. Moreover, in the same paper they showed that $\mu_s(P_m \cup K_{1,n})$ is 1 if $m = 2$ and n is odd or $m = 3$ and $n \not\equiv 0 \pmod{3}$ and 0 otherwise. In [7], S. Javed *et al.* gave the upper bound of deficiencies of disjoint union of graphs consisting of comb, generalized comb and star. In this paper, we frequently use the following two Lemmas.

Lemma 1.2 ([5]). “A graph G with n -vertices and m -edges is super edge-magic total if and only if there exists a bijection $\lambda : V(G) \rightarrow \{1, 2, \dots, n\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of m consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G .”

The above condition is often easier to use than the original one. The following lemma was found first in [1].

Lemma 1.3 ([1]). “If a graph G with n vertices and m edges is super edge-magic total then $m \leq 2n - 3$.”

Definition 1.4. A comb is a graph derived from the path $P_n : u_1, u_2, \dots, u_n$, $n \geq 3$, by adding $n - 1$ new edges $u_{i+1}w_i$; $1 \leq i \leq n - 1$ and this is denoted by Cb_n .

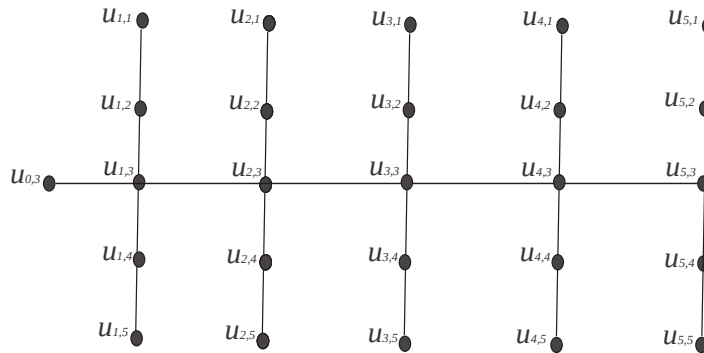
Definition 1.5. A two-sided generalized comb, denoted by $Cb_{n,m}^2$, consists of the vertex set,

$$V(Cb_{n,m}^2) = \{u_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{0, \frac{m+1}{2}}\}$$

and the edge set,

$$E(Cb_{n,m}^2) = \{u_{i,j}u_{i,j+1}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{u_{i, \frac{m+1}{2}}u_{i+1, \frac{m+1}{2}}; 0 \leq i \leq n-1\},$$

i.e., $Cb_{n,m}^2$ is deduced from n paths $P_{i,m} : u_{i,1}, u_{i,2}, \dots, u_{i,m}$; $u_{i,j}u_{i,j+1} \in E(Cb_{n,m}^2)$; $1 \leq i \leq n$; $1 \leq j \leq m-1$ and $n \geq 2$ of length $m-1$, where m is odd, by adding one new vertex $u_{0, \frac{m+1}{2}}$ and n new edges $u_{i, \frac{m+1}{2}}u_{i+1, \frac{m+1}{2}}$; $0 \leq i \leq n-1$. $Cb_{n,m}^2$ for $n = m = 5$ is shown in Fig.1.

Fig. 1. $Cb_{5,5}^2$ 

The graph obtained by $Cb_{n,m}^2$ by deleting the set of vertices $\{u_{i,j}; 1 \leq i \leq n, \frac{m+3}{2} \leq j \leq m\}$ and their adjacent edges is referred to as generalized comb, denoted by $Cb_n(\underbrace{l, \dots, l}_{n\text{-times}})$. The labeling of $Cb_n(\underbrace{l, \dots, l}_{n\text{-times}})$ is discussed in [7].

Definition 1.6. A bistar on n vertices, denoted by $BS(p, q)$; $p, q \geq 1$, $p + q + 2 = n$, is obtained from two stars $K_{1,p}$ and $K_{1,q}$ by joining their central vertices by an edge.

In this paper we formulate the super edge-magic total labeling of two sided generalized comb. Moreover, we determine an upper bound for super edge-magic total deficiency of forests containing comb, bistar and 2-sided generalized comb.

2 Super edge-magic deficiencies of forests of combs and bistar

In this section, we will provide precise value for super edge-magic deficiency of some specific number of copies of the comb Cb_n , we will also give an upper bound for super edge-magic deficiency for disjoint union of bistar $BS(k, k)$ and Cb_n with some restrictions on the parameters k and n .

Theorem 2.1. For n -odd, $n \geq 3$, m -even and $m \equiv 2(\text{mod } 4)$, the graph $G \cong mCb_n$ is super edge magic total.

Proof. Consider the graph $G \cong mCb_n$. Then $|V(G)| = m(2n - 1)$ and $|E(G)| = m(2n - 2)$, where

$$V(G) = \{u_i^k; 1 \leq i \leq n, 1 \leq k \leq m\} \cup \{w_j^k; 1 \leq j \leq n - 1, 1 \leq k \leq m\}$$

and

$$E(G) = \{u_i^k u_{i+1}^k; 1 \leq i \leq n - 1, 1 \leq k \leq m\} \cup \{u_{i+1}^k w_i^k; 1 \leq i \leq n - 1, 1 \leq k \leq m\}.$$

Define a labeling $f : V(G) \rightarrow \{1, 2, \dots, m(2n - 1)\}$ as follows:

$$f(w_{n-1}^k) = \begin{cases} \frac{3m-4k+6}{4} & ; \frac{m}{2} + 1 \leq k \leq \frac{3m+2}{4} \\ \frac{5m-4k+6}{4} & ; \frac{3m+6}{4} \leq k \leq m \end{cases}$$

For $1 \leq k \leq \frac{m}{2}$, k -odd

$$f(u_i^k) = \begin{cases} \frac{m(i-1)+k+1}{2} & ; i \equiv 0(\text{mod } 2) \\ m(n-1) - k + 1 + \frac{mi}{2} & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$f(w_j^k) = \begin{cases} \frac{mj+k+1}{2} & ; j \equiv 0(\text{mod } 2) \\ mn - k + 1 + \frac{m(j-1)}{2} & ; j \equiv 1(\text{mod } 2) \end{cases}$$

For $2 \leq k \leq \frac{m}{2} - 1$, k -even

$$f(u_i^k) = \begin{cases} \frac{m(2i-1)+2(k+1)}{4} & ; i \equiv 0(\text{mod } 2) \\ m(n-1) - k + 1 + \frac{mi}{2} & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$f(w_j^k) = \begin{cases} \frac{m(2j+1)+2(k+1)}{4} & ; j \equiv 0(\text{mod } 2) \\ mn - k + 1 + \frac{m(j-1)}{2} & ; j \equiv 1(\text{mod } 2) \end{cases}$$

For $\frac{m}{2} + 1 \leq k \leq m$, k -even

$$f(u_i^k) = \begin{cases} \frac{m(2n+2i-5)+2(k+1)}{4} & ; i \equiv 0(\text{mod } 2) \\ \frac{m(3n+i-1)}{2} - k + 1 & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$f(w_j^k) = \begin{cases} \frac{m(2n+2j-3)+2(k+1)}{4} & ; j \equiv 0(\text{mod } 2), j \neq n-1 \\ \frac{m(3n+j)}{2} - k + 1 & ; j \equiv 1(\text{mod } 2) \end{cases}$$

For $\frac{m}{2} + 2 \leq k \leq m-1$, k -odd

$$f(u_i^k) = \begin{cases} \frac{m(n+i-2)+k+1}{2} & ; i \equiv 0(\text{mod } 2) \\ \frac{m(3n+i-1)}{2} - k + 1 & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$f(w_j^k) = \begin{cases} \frac{m(n+j-1)+k+1}{2} & ; j \equiv 0(\text{mod } 2), j \neq n-1 \\ \frac{m(3n+j)}{2} - k + 1 & ; j \equiv 1(\text{mod } 2) \end{cases}$$

The labeling f gives the following set of consecutive integers $\{\frac{4mn-m+6}{4}, \frac{4mn-m+10}{4}, \dots, \frac{12mn-9m+2}{4}\}$ that appears as the weights of the edges in the graph. $\square$

Theorem 2.2. For n, m -even, $n \geq 4$ and $m \equiv 2(\text{mod } 4)$, $\mu_s(mCb_n) \leq \frac{m}{2}$.

Proof. Consider the graph $G \cong mCb_n \cup (\frac{m}{2})K_1$. Then $|V(G)| = 2mn - \frac{m}{2}$ and $|E(G)| = m(2n-2)$, where $V(G) = V(mCb_n) \cup \{z_l; 1 \leq l \leq \frac{m}{2}\}$ and $E(G) = E(mCb_n)$.

Define a labeling $g: V(G) \rightarrow \{1, 2, \dots, 2mn - \frac{m}{2}\}$ as follows:

$$g(u_n^k) = \begin{cases} \frac{3m-4k+6}{4} & ; \frac{m}{2} + 1 \leq k \leq \frac{3m+2}{4} \\ \frac{5m-4k+6}{4} & ; \frac{3m+6}{4} \leq k \leq m \end{cases}$$

$$g(z_l) = \frac{3mn}{2} - m + l; 1 \leq l \leq \frac{m}{2}$$

For $1 \leq k \leq \frac{m}{2}$,

$$g(u_i^k) = f(u_i^k) \forall i; 1 \leq i \leq n$$

$$g(w_j^k) = f(w_j^k) \forall j; 1 \leq j \leq n-1,$$

where the labeling f is defined in Theorem 2.1. For $\frac{m}{2} + 1 \leq k \leq m$, k -even

$$g(u_i^k) = \begin{cases} \frac{m(2n+2i-5)+2(k+1)}{4} & ; i \equiv 0(\text{mod } 2), i \neq n \\ \frac{m(3n+i)}{2} - k + 1 & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$g(w_j^k) = \begin{cases} \frac{m(2n+2j-3)+2(k+1)}{4} & ; j \equiv 0(\text{mod } 2) \\ \frac{m(3n+j+1)}{2} - k + 1 & ; j \equiv 1(\text{mod } 2) \end{cases}$$

For $\frac{m}{2} + 2 \leq k \leq m - 1$, k -odd

$$g(u_i^k) = \begin{cases} \frac{m(n+i-2)+k+1}{2} & ; i \equiv 0(\text{mod } 2), i \neq n \\ \frac{m(3n+i)}{2} - k + 1 & ; i \equiv 1(\text{mod } 2) \end{cases}$$

$$g(w_j^k) = \begin{cases} \frac{m(n+j-1)+k+1}{2} & ; j \equiv 0(\text{mod } 2) \\ \frac{m(3n+j+1)}{2} - k + 1 & ; j \equiv 1(\text{mod } 2) \end{cases}$$

The labeling g constitutes the following set of edge weights $\{\frac{4mn-m+6}{4}, \frac{4mn-m+10}{4}, \dots, \frac{12mn-9m+2}{4}\}$. $\square$

In the next Theorem we will compute an upper bound for the edge-magic deficiency of a forest consisting of bistar $BS(k, k)$ and comb Cb_n .

Theorem 2.3. For $k \geq 2$, consider the graph $G \cong BS(k, k) \cup Cb_n$. Then

1. The graph G is super edge-magic total for $n \geq k + 2$ and k -odd.
2. $\mu_s(G) \leq 1$ for $n \geq k + 3$ and k -even.

Proof. Consider the graph $G \cong BS(k, k) \cup Cb_n$. We have

$$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{w_j : 1 \leq j \leq n - 1\} \cup \{z_1, z_2\} \cup \{z_{i,t} : 1 \leq i \leq 2, 1 \leq t \leq k\}$$

and $E(G) = \{z_1 z_{1,t}, z_2 z_{2,t} : 1 \leq t \leq k\} \cup \{u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_{i+1} w_i : 1 \leq i \leq n - 1\} \cup \{z_1 z_2\}$, which give $|V(G)| = 2(n + k) + 1$ and $E(G) = 2(n + k) - 1$.

1. Define a labeling $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$ in the following way:

$$\begin{aligned} f(z_1) &= n + k + 1 \\ f(z_2) &= k + 2 \\ f(z_{1t}) &= t + 1; 1 \leq t \leq k \\ f(z_{2t}) &= n + k + t + 1; 1 \leq t \leq k \\ f(u_i) &= \begin{cases} k + i + 1 & ; 2 \leq i \leq k + 1, i \equiv 0(\text{mod } 2) \\ k + i & ; k + 3 \leq i \leq n, i \equiv 0(\text{mod } 2) \\ n + 2k + i + 1 & ; 1 \leq i \leq n, i \equiv 1(\text{mod } 2) \end{cases} \\ f(w_j) &= \begin{cases} k + j + 2 & ; 2 \leq j \leq k - 1, j \equiv 0(\text{mod } 2) \\ k + j + 1 & ; k + 3 \leq j \leq n - 1, j \equiv 0(\text{mod } 2) \\ n + 2k + j + 2 & ; 1 \leq j \leq n - 1, j \equiv 1(\text{mod } 2) \\ 1 & ; j = k + 1 \end{cases} \end{aligned}$$

The set of edge weights formed under the labeling f consists of the following consecutive integers $\{n + k + 3, n + k + 4, \dots, 3(n + k) + 1\}$.

2. Consider the graph $H \cong G \cup K_1$, where $V(K_1) = \{u\}$.

Define a labeling $g : V(H) \rightarrow \{1, 2, \dots, 2(n + k + 1)\}$ as follows:

$$\begin{aligned} g(u) &= 2 \\ g(z_1) &= n + k + 2 \\ g(z_2) &= k + 3 \end{aligned}$$

$$\begin{aligned}
g(z_{1t}) &= t + 2; 1 \leq t \leq k \\
g(z_{2t}) &= n + k + t + 2; 1 \leq t \leq k \\
g(u_i) &= \begin{cases} k + i + 2 & ; 2 \leq i \leq k + 2, i \equiv 0(\text{mod } 2) \\ k + i + 1 & ; k + 4 \leq i \leq n, i \equiv 0(\text{mod } 2) \\ n + 2k + i + 2 & ; 1 \leq i \leq n, i \equiv 1(\text{mod } 2) \end{cases} \\
g(w_j) &= \begin{cases} k + j + 3 & ; 2 \leq j \leq n - 1, j \equiv 0(\text{mod } 2), j \neq k + 2 \\ n + 2k + j + 3 & ; 1 \leq j \leq n - 1, j \equiv 1(\text{mod } 2) \\ 1 & ; j = k + 2 \end{cases}
\end{aligned}$$

The labeling g gives the following set of consecutive integers $\{n + k + 5, n + k + 6, \dots, 3(n + k + 1)\}$ as the edge weights. $\square$

3 Super edge-magic total labeling of two-sided comb

Theorem 3.1. For $n \geq 2, m \geq 3$ and m -odd, the graph $G \cong Cb_{n,m}^2$ is super edge-magic total.

Proof. Consider the graph $G \cong Cb_{n,m}^2$. Then $|V(G)| = mn + 1$ and $|E(G)| = mn$, where

$$V(G) = \{u_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{0, \frac{m+1}{2}}\}$$

and

$$E(G) = \{u_{i,j}u_{i,j+1}; 1 \leq i \leq n, 1 \leq j \leq m - 1\} \cup \{u_{i, \frac{m+1}{2}}u_{i+1, \frac{m+1}{2}}; 0 \leq i \leq n - 1\}.$$

To show that G is super edge-magic total, we will define a labeling $f : V(G) \rightarrow \{1, 2, \dots, mn + 1\}$ as follows:

For $\frac{m-1}{2}$ -odd,

$$f(u_{0, \frac{m+1}{2}}) = \frac{m+5}{4}$$

For $j \equiv 1(\text{mod } 2)$,

$$\begin{aligned}
f(u_{1,j}) &= \begin{cases} \frac{j+1}{2} & \text{if } 1 \leq j \leq \frac{m-1}{2}; \\ \frac{j+3}{2} & \text{if } \frac{m+3}{2} \leq j \leq m. \end{cases} \\
f(u_{i,j}) &= \begin{cases} \frac{m(i-1)+j+3}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 3 \leq i \leq n \text{ and } 1 \leq j \leq m; \\ \frac{mi-j+4}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 2 \leq j \leq m-1; \\ \lceil \frac{mn}{2} \rceil + \frac{m(i-1)+j+2}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 1 \leq i \leq n \text{ and } 2 \leq j \leq m-1; \\ \lceil \frac{mn}{2} \rceil + \frac{im-j+3}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq m. \end{cases}
\end{aligned}$$

The set of edge weights given by the labeling f consists of the following mn consecutive integers $\{\lceil \frac{mn}{2} \rceil + 3, \lceil \frac{mn}{2} \rceil + 4, \dots, \lceil \frac{3mn}{2} \rceil + 2\}$.

For $\frac{m-1}{2}$ -even,

$$f(u_{0, \frac{m+1}{2}}) = \frac{m+3}{4}$$

For $j \equiv 0(\text{mod } 2)$,

$$f(u_{1,j}) = \begin{cases} \frac{j}{2} & \text{if } 2 \leq j \leq \frac{m-1}{2}; \\ \frac{j+2}{2} & \text{if } \frac{m+3}{2} \leq j \leq m-1. \end{cases}$$

$$f(u_{i,j}) = \begin{cases} \frac{mi-j+3}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq m; \\ \frac{m(i-1)+j+2}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 3 \leq i \leq n \text{ and } 2 \leq j \leq m-1; \\ \lfloor \frac{mn}{2} \rfloor + \frac{m(i-1)+j+3}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m; \\ \lfloor \frac{mn}{2} \rfloor + \frac{mi-j+4}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 2 \leq j \leq m-1. \end{cases}$$

The set of edge weights formed under the labeling f is $\{\lfloor \frac{mn}{2} \rfloor + 3, \lfloor \frac{mn}{2} \rfloor + 4, \dots, \lfloor \frac{3mn}{2} \rfloor + 2\}$. $\square$

4 Super edge-magic deficiency of copies of two-sided comb

Theorem 4.1. For n -even, $n \geq 2$, m -odd and $m \geq 3$, the graph $G \cong 2Cb_{n,m}^2$ is super edge-magic total.

Proof. Define the graph $G \cong 2Cb_{n,m}^2$ in the following way:

$$V(G) = \{u_{i,j}^k; 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq 2\} \cup \{u_{0, \frac{m+1}{2}}^k; 1 \leq k \leq 2\}$$

and

$$E(G) = \{u_{i,j}^k u_{i,j+1}^k; 1 \leq i \leq n, 1 \leq j \leq m-1, 1 \leq k \leq 2\} \\ \cup \{u_{i, \frac{m+1}{2}}^k u_{i+1, \frac{m+1}{2}}^k; 0 \leq i \leq n-1, 1 \leq k \leq 2\}.$$

Then $|V(G)| = 2(mn + 1)$ and $|E(G)| = 2mn$.

For $\frac{m-1}{2}$ -even,

$$f(u_{1,1}^1) = 2(mn + 1)$$

For $1 \leq k \leq 2$,

$$f(u_{0, \frac{m+1}{2}}^k) = (\frac{mn}{2} + 1)(k - 1) + \frac{m + 3}{4}$$

For $j \equiv 0(\text{mod } 2)$,

$$f(u_{1,j}^k) = \begin{cases} (\frac{mn}{2} + 1)(k - 1) + \frac{j}{2} & \text{if } 2 \leq j \leq \frac{m-1}{2}; \\ (\frac{mn}{2} + 1)(k - 1) + \frac{j+2}{2} & \text{if } \frac{m+3}{2} \leq j \leq m-1. \end{cases}$$

$$f(u_{i,j}^k) = \begin{cases} (\frac{mn}{2} + 1)(k - 1) + \frac{mi-j+3}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq m; \\ (\frac{mn}{2} + 1)(k - 1) + \frac{m(i-1)+j+2}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 3 \leq i \leq n-1 \text{ and } 2 \leq j \leq m-1; \\ \frac{mn}{2}(k + 1) + \frac{m(i-1)+j+3}{2} & \text{if } i \equiv 1(\text{mod } 2) \text{ and } j \equiv 1(\text{mod } 2) \\ & \text{for } 1 \leq i \leq n-1, 1 \leq j \leq m \\ & \text{and } (i, j, k) \neq (1, 1, 1); \\ \frac{mn}{2}(k + 1) + \frac{mi-j+4}{2} & \text{if } i \equiv 0(\text{mod } 2) \text{ and } j \equiv 0(\text{mod } 2) \\ & \text{for } 2 \leq i \leq n \text{ and } 2 \leq j \leq m-1. \end{cases}$$

The set of edge weights of G , formed by the labeling defined by f , is $\{mn + 4, mn + 5, \dots, 3(mn + 1)\}$.

For $\frac{m-1}{2}$ -odd,

$$f(u_{n,1}^2) = 1$$

For $1 \leq k \leq 2$,

$$f(u_{0, \frac{m+1}{2}}^k) = \frac{mn}{2}(k + 1) + k + \frac{m + 1}{4}$$

For $j \equiv 1 \pmod{2}$,

$$f(u_{1,j}^k) = \begin{cases} \frac{mn(k+1)+j-1}{2} + k & \text{if } 1 \leq j \leq \frac{m-1}{2}; \\ \frac{mn(k+1)+j+1}{2} + k & \text{if } \frac{m+3}{2} \leq j \leq m. \end{cases}$$

$$f(u_{i,j}^k) = \begin{cases} \frac{mn}{2}(k-1) + \frac{m(i-1)+j+2}{2} & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 1 \leq i \leq n-1 \text{ and } 2 \leq j \leq m-1; \\ \frac{mn}{2}(k-1) + \frac{mi-j+3}{2} & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 2 \leq i \leq n, 1 \leq j \leq m \\ & \text{and } (i, j, k) \neq (n, 1, 2); \\ \frac{mn(k+1)+im-j}{2} + k + 1 & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 2 \leq i \leq n \text{ and } 2 \leq j \leq m-1; \\ \frac{mn(k+1)+m(i-1)+j+1}{2} + k & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 3 \leq i \leq n-1 \text{ and } 1 \leq j \leq m. \end{cases}$$

The labeling f gives the following set of edge weights $\{mn+3, mn+4, \dots, 3mn+2\}$. $\square$

Theorem 4.2. For n, m -odd, $n \geq 3$ and $m > 3$, the graph $G \cong 2Cb_{n,m}^2$ has the super edge-magic deficiency at most 1.

Proof. Consider the graph $H \cong G \cup K_1$, where $V(K_1) = \{z\}$. To show that H is super edge-magic total, define the labeling $g : V(H) \rightarrow \{1, 2, \dots, 2mn+3\}$ as follows:

For $\frac{m-1}{2}$ -even,

$$g(z) = \frac{3mn+5}{2}$$

$$g(u_{1,1}^1) = mn+1$$

$$g(u_{n,m}^2) = 1$$

For $1 \leq k \leq 2$,

$$g(u_{0, \frac{m+1}{2}}^k) = \frac{2mn(k+1) + 6k + m + 1}{4}$$

For $j \equiv 0 \pmod{2}$,

$$g(u_{1,j}^k) = \begin{cases} \frac{mn(k+1)+3k+j-1}{2} & \text{if } 2 \leq j \leq \frac{m-1}{2}; \\ \frac{mn(k+1)+3k+j+1}{2} & \text{if } \frac{m+3}{2} \leq j \leq m-1. \end{cases}$$

$$g(u_{i,j}^k) = \begin{cases} \frac{(mn+1)(k-1)+m(i-1)+j+1}{2} & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m, \\ & (i, j, k) \neq (1, 1, 1) \text{ and } (i, j, k) \neq (n, m, 2); \\ \frac{(mn+1)(k-1)+mi-j+2}{2} & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 2 \leq i \leq n-1, 2 \leq j \leq m-1; \\ \frac{mn(k+1)+3k+im-j+2}{2} & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 2 \leq i \leq n-1 \text{ and } 1 \leq j \leq m; \\ \frac{mn(k+1)+m(i-1)+3k+j+1}{2} & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 3 \leq i \leq n \text{ and } 2 \leq j \leq m-1. \end{cases}$$

The set of edge weights produced by the labeling g consists of the following set of $2mn$ consecutive integers $\{mn+4, mn+5, \dots, 3(mn+1)\}$.

For $\frac{m-1}{2}$ -odd,

$$g(z) = \frac{3mn+5}{2}$$

$$g(u_{1,2}^1) = 2mn+3$$

For $1 \leq k \leq 2$,

$$g(u_{0, \frac{m+1}{2}}^k) = \frac{(mn+3)(k-1)}{2} + \frac{m+5}{4}$$

For $j \equiv 1 \pmod{2}$,

$$g(u_{1,j}^k) = \begin{cases} \frac{mn(k-1)+3k+j-2}{2} & \text{if } 1 \leq j \leq \frac{m-1}{2}; \\ \frac{mn(k-1)+3k+j}{2} & \text{if } \frac{m+3}{2} \leq j \leq m. \end{cases}$$

$$g(u_{i,j}^k) = \begin{cases} \frac{(mn+3)(k-1)-j+mi+4}{2} & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 2 \leq i \leq n-1 \text{ and } 2 \leq j \leq m-1; \\ \frac{mn(k-1)+m(i-1)+3k+j}{2} & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 3 \leq i \leq n, 1 \leq j \leq m; \\ \frac{mn(k+1)+m(i-1)+k+j+3}{2} & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 0 \pmod{2} \\ & \text{for } 1 \leq i \leq n \text{ and } 2 \leq j \leq m-1 \\ & \text{and } (i, j, k) \neq (1, 2, 1); \\ \frac{mn(k+1)+mi-j+k+4}{2} & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 1 \pmod{2} \\ & \text{for } 2 \leq i \leq n-1 \text{ and } 1 \leq j \leq m. \end{cases}$$

The set of edge weights under the labeling g is $\{mn+6, mn+7, \dots, 3mn+5\}$. $\square$

Concluding remarks

In [3], Figueroa-Centeno *et al.* discovers that if a graph is super edge-magic, then an odd number of copies of the graph is also super edge-magic. In this paper, we extend this concept for an even number of copies of comb, so the result in [3] significantly generalizes our results. It is also shown that the two-sided generalized comb, denoted by $Cb_{n,m}^2$ is super edge-magic total. Moreover we have found upper bounds for the super edge-magic deficiency of forests mCb_n , $Cb_n \cup BS(k, k)$ and $2Cb_{n,m}^2$ for different values of the parameters k , m and n . In this context we formulate some open problems:

1. Let n -odd, $n \geq 3$ and $m \geq 3$. Determine the exact value of the super edge-magic deficiency of $2Cb_{n,m}^2$.
2. For $k_1, k_2 \geq 2$, $k_1 \neq k_2$ and $n \geq 3$, find an upper bound of the super edge-magic deficiency of $BS(k_1, k_2) \cup Cb_n$.
3. For $n \geq 3$ and $m \equiv 0 \pmod{4}$. Calculate the upper bound of the super edge-magic deficiency of mCb_n .

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