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Some new inequalities of Hermite-Hadamard type for s -convex functions with applications

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Abstract: In this paper, we present several new and generalized Hermite-Hadamard type inequalities for s -convex as well as s -concave functions via classical and Riemann-Liouville fractional integrals. As applications, we provide new error estimations for the trapezoidal formula.

Keywords: s -convex function, Hermite-Hadamard inequality, Hölder inequality, Trapezoidal formula

MSC: 26D15, 26A51, 26D20

1 Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \rightarrow \mathbb{R}$ is said to be convex (concave) on I if the inequality

$$f[\lambda x + (1 - \lambda)y] \leq (\geq) \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

A large number of important properties and inequalities have been established for the class of convex (concave) functions since the convexity (concavity) was introduced more than a hundred years ago [1-21]. But one of the most important inequalities for the convex (concave) function is the Hermite-Hadamard inequality [22], which can be stated as follows:

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be a convex function on I . Then the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

holds for all $a, b \in I$ with $a < b$. Both inequalities given in (1) hold in the reversed direction if f is concave on the interval I .

Recently, the improvements, generalizations, refinements and applications for the Hermite-Hadamard inequality have attracted the attention of many researchers [23-41].

Hudzik and Maligranda [42] generalized the convex (concave) function to s -convex (concave) function.

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Let $s \in (0, 1]$. Then the function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex on the interval $[0, \infty)$ if the inequality

$$f[\lambda x + (1 - \lambda)y] \leq \lambda^s f(x) + (1 - \lambda)^s f(y) \tag{2}$$

takes place for all $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$. f is said to be s -concave if inequality (2) is reversed.

We clearly see that the s -convexity (concavity) defined on $[0, \infty)$ reduces to ordinary convexity (concavity) if $s = 1$.

In [43], the authors established the Hermite-Hadamard type inequality for the s -convex (concave) functions as follows.

Theorem 1.2 ([43]). *Let $s \in (0, 1]$ and $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be an s -convex function on I . Then the double inequality*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1} \tag{3}$$

holds for all $a, b \in I$ with $a < b$. Both inequalities given in (3) hold in the reversed direction if f is s -concave on the interval I .

Both of the upper and lower bounds given in (3) for the s -convex (concave) functions were improved by Jagers in [44].

Hussian et al. [45] provided the Hermite-Hadamard type inequalities for the twice differentiable functions by using the following Lemma 1.3.

Lemma 1.3 ([45]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the identity*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''[ta + (1-t)b]dt$$

is valid if $f'' \in L[a, b]$, where and in what follows I° denotes the interior of the interval I .

Theorem 1.4 ([45]). *Let $s \in (0, 1]$, $q > 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(a-b)^2}{2 \times 6^{(q-1)/q}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{(s+2)(s+3)} \right]^{1/q}$$

holds if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Theorem 1.5 ([45]). *Let $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq 2^{(s-1-a)/q} (b-a)^2 \left| f''\left(\frac{a+b}{2}\right) \right| \left[\frac{\Gamma^2(p+1)}{\Gamma(2p+2)} \right]^p$$

holds if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ [46-50] is the classical gamma function.

In [51], Chu et al. discovered a new identity for the twice differentiable function.

Lemma 1.6 ([51]). *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the identity*

$$\frac{[(x-a)^2 - (b-x)^2]f'(x) + 2(b-x)f(b) + 2(x-a)f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \tag{4}$$

$$= \frac{(x-a)^3}{2(b-a)} \int_0^1 (1-t^2) f''[ta + (1-t)x] dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 (1-t^2) f''[tb + (1-t)x] dt$$

holds for all $x \in [a, b]$ if $f'' \in L[a, b]$.

Next, we recall the definition of the fractional integrals [52].

Let $0 \leq a < b$, $\eta > 0$ and $f \in L[a, b]$. Then the left-sided and right-sided Riemann-Liouville fractional integrals $J_{a+}^\eta f$ and $J_{b-}^\eta f$ of order η are defined by

$$J_{a+}^\eta f(x) = \frac{1}{\Gamma(\eta)} \int_a^x (x-t)^{\eta-1} f(t) dt,$$

$$J_{b-}^\eta f(x) = \frac{1}{\Gamma(\eta)} \int_x^b (t-x)^{\eta-1} f(t) dt,$$

respectively.

We clearly see that $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. In particular, the fractional integral reduces to the classical integral if $\eta = 1$.

In [53], Set dealt with the fractional Ostrowski inequalities involving the Riemann-Liouville fractional integrals. More results and applications for the fractional derivatives and integrals can be found in the literature [54-66]. Sarikaya et al. [67] and Özdemir et al. [68] established the Hermite-Hadamard type inequalities for the Riemann-Liouville fractional integrals as follows.

Theorem 1.7 ([67]). *Let $\eta > 0$, $0 \leq a < b$, $f : [a, b] \rightarrow (0, \infty)$ be a positive real-valued function with $f \in L[a, b]$. Then the double inequality*

$$f\left(\frac{a+b}{2}\right) < \frac{\Gamma(\eta+1)}{2(b-a)^\eta} \left[J_{a+}^\eta f(b) + J_{b-}^\eta f(a) \right] \leq \frac{f(a) + f(b)}{2} \tag{5}$$

holds if f is convex on $[a, b]$.

Theorem 1.8 ([68]). *Let $\eta > 0$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality*

$$\begin{aligned} & \left| \frac{(x-a)^\eta f(a) + (b-x)^\eta f(b)}{b-a} - \frac{\Gamma(\eta+1)}{b-a} \left(J_{x-}^\eta f(a) + J_{x+}^\eta f(b) \right) \right| \\ & \leq \frac{\eta \left((x-a)^{\eta+1} + (b-x)^{\eta+1} \right) |f'(x)|}{(s+1)(\eta+s+1)(b-a)} \\ & + \left(\frac{1}{s+1} - \frac{\Gamma(\eta+1)\Gamma(s+1)}{\Gamma(\eta+s+3)} \right) \frac{(x-a)^{\eta+1} |f'(a)| + (b-x)^{\eta+1} |f'(b)|}{b-a} \end{aligned}$$

is valid for all $x \in [a, b]$ if $f' \in L(a, b)$ and $|f'|$ is s -convex on $[a, b]$.

Remark 1.9. *We clearly see that inequality (5) reduces to inequality (1) if $\eta = 1$.*

The following identity for the twice differentiable function, which was discovered by Chu et al. [51], will be used in the next section.

Lemma 1.10 ([51]). *Let $\eta > 0$, $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the identity*

$$\frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}] f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{b-a} \tag{6}$$

$$\begin{aligned} &= \frac{1}{b-a} \int_a^b f(t)dt + \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) f''[ta+(1-t)x]dt \\ &\quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) f''[tb+(1-t)x]dt \end{aligned}$$

holds for all $x \in [a, b]$ if $f'' \in L(a, b)$.

The main purpose of this paper is to establish several new Hermite-Hadamard type inequalities for s -convex (concave) functions via the classical and Riemann-Liouville fractional integrals, and provide the error estimations for the trapezoidal formula.

2 Hermite-Hadamard type inequalities for s -convex functions via classical integrals

Theorem 2.1. Let $s \in (0, 1]$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\begin{aligned} &\left| \frac{[(x-a)^2 - (b-x)^2]f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{7} \\ &\leq \frac{(x-a)^3}{2(b-a)} \left[\frac{2}{(s+1)(s+3)} |f''(a)| + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)| \right] \\ &\quad + \frac{(b-x)^3}{2(b-a)} \left[\frac{2}{(s+1)(s+3)} |f''(b)| + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)| \right] \end{aligned}$$

holds for all $x \in [a, b]$ if $|f''|$ is s -convex on $[a, b]$ and $f'' \in L[a, b]$.

Proof. It follows from (4) and the triangular inequality together with the s -convexity of $|f''|$ that

$$\begin{aligned} &\left| \frac{[(x-a)^2 - (b-x)^2]f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ &\leq \frac{(x-a)^3}{2(b-a)} \int_0^1 (1-t^2) |f''(ta+(1-t)x)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 (1-t^2) |f''(tb+(1-t)x)| dt \\ &\leq \frac{(x-a)^3}{2(b-a)} \int_0^1 (1-t^2) [t^s |f''(a)| + (1-t)^s |f''(x)|] dt \\ &\quad + \frac{(b-x)^3}{2(b-a)} \int_0^1 (1-t^2) [t^s |f''(b)| + (1-t)^s |f''(x)|] dt \\ &= \frac{(x-a)^3}{2(b-a)} \left[\frac{2}{(s+1)(s+3)} |f''(a)| + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)| \right] \\ &\quad + \frac{(b-x)^3}{2(b-a)} \left[\frac{2}{(s+1)(s+3)} |f''(b)| + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)| \right]. \quad \square \end{aligned}$$

Corollary 2.2. Under the assumptions of Theorem 2.1, one has

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{8} \\ & \leq \frac{(b-a)^2}{8} \left[\frac{1}{(s+1)(s+3)} (|f''(a)| + |f''(b)|) + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} \left| f''\left(\frac{a+b}{2}\right) \right| \right] \\ & \leq \frac{(b-a)^2}{8} \left[\frac{1}{(s+1)(s+3)} + \frac{(s+2)(s+3)-2}{2^s(s+1)(s+2)(s+3)} \right] (|f''(a)| + |f''(b)|). \end{aligned}$$

Proof. Let $x = (a + b)/2$, then the first inequality of (8) follows easily from (7). While the second inequality of (8) can be derived from the s -convexity of $|f''|$. □

Remark 2.3. Let $s = 1$, then the second inequality of (8) becomes

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{24} (|f''(a)| + |f''(b)|).$$

Theorem 2.4. Let $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\begin{aligned} & \left| \frac{[(x-a)^2 - (b-x)^2]f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{9} \\ & \leq \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p} \frac{(x-a)^3 [|f''(a)|^q + |f''(x)|^q]^{1/q} + (b-x)^3 [|f''(b)|^q + |f''(x)|^q]^{1/q}}{2(s+1)^{1/q}(b-a)} \end{aligned}$$

holds for each $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Proof. From (4) together with the triangular and Hölder inequalities we clearly see that

$$\begin{aligned} & \left| \frac{[(x-a)^2 - (b-x)^2]f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{10} \\ & \leq \frac{(x-a)^3}{2(b-a)} \left(\int_0^1 (1-t^2)^p dt \right)^{1/p} \left(\int_0^1 |f''(ta + (1-t)x)|^q dt \right)^{1/q} \\ & \quad + \frac{(b-x)^3}{2(b-a)} \left(\int_0^1 (1-t^2)^p dt \right)^{1/p} \left(\int_0^1 |f''(tb + (1-t)x)|^q dt \right)^{1/q}. \end{aligned}$$

Making use of the s -convexity of $|f''|^q$, we get

$$\int_0^1 |f''(ta + (1-t)x)|^q dt \leq \int_0^1 (t^s |f''(a)|^q + (1-t)^s |f''(x)|^q) dt = \frac{|f''(a)|^q + |f''(x)|^q}{s+1}, \tag{11}$$

$$\int_0^1 |f''(tb + (1-t)x)|^q dt \leq \int_0^1 (t^s |f''(b)|^q + (1-t)^s |f''(x)|^q) dt = \frac{|f''(b)|^q + |f''(x)|^q}{s+1}. \tag{12}$$

Note that

$$\left(\int_0^1 (1-t^2)^p dt \right)^{1/p} = \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p}. \tag{13}$$

Therefore, inequality (9) follows easily from the (10)-(13). □

Corollary 2.5. *Under the assumptions of Theorem 2.4, we have*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{14} \\ & \leq \frac{(b-a)^2}{16(s+1)^{1/q}} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p} \\ & \quad \times \left[\left(|f''(a)|^q + \left| f''\left(\frac{a+b}{2}\right) \right|^q \right)^{1/q} + \left(|f''(b)|^q + \left| f''\left(\frac{a+b}{2}\right) \right|^q \right)^{1/q} \right] \\ & \leq \frac{(b-a)^2}{16(s+1)^{1/q}} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p} \left[\left(\frac{1}{2^s} \right)^{1/q} + \left(1 + \frac{1}{2^s} \right)^{1/q} \right] (|f''(a)| + |f''(b)|). \end{aligned}$$

Proof. Let $x = (a + b)/2$, then inequality (9) leads to the first inequality of (14) immediately. While the second inequality of (14) can be derived easily from the s -convexity of $|f''|^q$ and the elementary inequality

$$\sum_{k=1}^n (\alpha_k + \beta_k)^\sigma \leq \sum_{k=1}^n \alpha_k^\sigma + \sum_{k=1}^n \beta_k^\sigma$$

for $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n \geq 0$ and $0 \leq \sigma \leq 1$. □

Remark 2.6. *If $s = 1$, then the second inequality of (14) becomes*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^2}{32} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} \right)^{1/p} \left[\left(\frac{1}{2} \right)^{1/q} + \left(\frac{3}{2} \right)^{1/q} \right] (|f''(a)| + |f''(b)|). \end{aligned}$$

Theorem 2.7. *Let $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality*

$$\begin{aligned} & \left| \frac{[(x-a)^2 - (b-x)^2]f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{15} \\ & \leq \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p} \frac{(x-a)^3 \left| f''\left(\frac{x+a}{2}\right) \right| + (b-x)^3 \left| f''\left(\frac{x+b}{2}\right) \right|}{2^{1+(1-s)/q}(b-a)} \end{aligned}$$

holds for any $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -concave on $[a, b]$.

Proof. It follows from the s -concavity of $|f''|^q$ and (3) that

$$\int_0^1 |f''(ta + (1-t)x)|^q \leq 2^{s-1} \left| f''\left(\frac{x+a}{2}\right) \right|^q, \tag{16}$$

$$\int_0^1 |f''(tb + (1-t)x)|^q \leq 2^{s-1} \left| f''\left(\frac{x+b}{2}\right) \right|^q. \tag{17}$$

Therefore, inequality (15) follows from (4), (13), (16) and (17) together with the Hölder inequalities

$$\int_0^1 (1-t^2) f''(ta + (1-t)x) dt \leq \left(\int_0^1 (1-t^2)^p dt \right)^{1/p} \left(\int_0^1 |f''(ta + (1-t)x)|^q dt \right)^{1/q},$$

$$\int_0^1 (1-t^2) f''(tb + (1-t)x) dt \leq \left(\int_0^1 (1-t^2)^p dt \right)^{1/p} \left(\int_0^1 |f''(tb + (1-t)x)|^q dt \right)^{1/q}. \quad \square$$

Corollary 2.8. *Under the assumptions of Theorem 2.7, one has*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{18} \\ & \leq \frac{2^{s/q} (b-a)^2}{32} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} \right)^{1/p} \left[\left| f''\left(\frac{3a+b}{4}\right) \right| + \left| f''\left(\frac{a+3b}{4}\right) \right| \right] \\ & \leq \frac{2^{(1+1/q)s} (b-a)^2}{32} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} \right)^{1/p} \left| f''\left(\frac{a+b}{2}\right) \right|. \end{aligned}$$

Proof. Let $x = (a + b)/2$, then inequality (15) leads to the first inequality of (18) immediately. While the second inequality of (18) can be obtained by the s -concavity of $|f''|$ due to the fact that $|f''|^q$ is s -concave, indeed, the s -concavity of $|f''|^q$ leads to the conclusion that

$$(t^s |f''(a)| + (1-t)^s |f''(b)|)^q \leq t^s |f''(a)|^q + (1-t)^s |f''(b)|^q \leq |f''(ta + (1-t)b)|^q. \quad \square$$

Remark 2.9. *Let $s = 1$, then from the second inequality of (18), we get*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{2^{1/q} (b-a)^2}{16} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} \right)^{1/p} \left| f''\left(\frac{a+b}{2}\right) \right|.$$

Theorem 2.10. *Let $s \in (0, 1]$, $q > 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality*

$$\begin{aligned} & \left| \frac{[(x-a)^2 - (b-x)^2] f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{19} \\ & \leq \frac{\left(\frac{3}{2}\right)^{1/q} (x-a)^3 \left[\frac{2}{(s+1)(s+3)} |f''(a)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)|^q \right]^{1/q}}{3(b-a)} \\ & \quad + \frac{\left(\frac{3}{2}\right)^{1/q} (b-x)^3 \left[\frac{2}{(s+1)(s+3)} |f''(b)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)|^q \right]^{1/q}}{3(b-a)} \end{aligned}$$

holds for any $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Proof. It follows from (4) and the power-mean inequality that

$$\begin{aligned} & \left| \frac{[(x-a)^2 - (b-x)^2] f'(x) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{20} \\ & \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 (1-t^2) |f''(ta + (1-t)x)| dt \\ & \quad + \frac{(b-x)^3}{2(b-a)} \int_0^1 (1-t^2) |f''(tb + (1-t)x)| dt \end{aligned}$$

$$\begin{aligned} &\leq \frac{(x-a)^3}{2(b-a)} \left(\int_0^1 (1-t^2) dt \right)^{1-1/q} \left(\int_0^1 (1-t^2) |f''(ta + (1-t)x)|^q dt \right)^{1/q} \\ &+ \frac{(b-x)^3}{2(b-a)} \left(\int_0^1 (1-t^2) dt \right)^{1-1/q} \left(\int_0^1 (1-t^2) |f''(tb + (1-t)x)|^q dt \right)^{1/q}. \end{aligned}$$

From the s -convexity of $|f''|^q$ on $[a, b]$ we get

$$\begin{aligned} &\int_0^1 (1-t^2) |f''(ta + (1-t)x)|^q dt \tag{21} \\ &\leq \int_0^1 (1-t^2) [t^s |f''(a)|^q + (1-t)^s |f''(x)|^q] dt \\ &= \frac{2}{(s+1)(s+2)} |f''(a)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)|^q \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 (1-t^2) |f''(tb + (1-t)x)|^q dt \tag{22} \\ &\leq \int_0^1 (1-t^2) [t^s |f''(b)|^q + (1-t)^s |f''(x)|^q] dt \\ &= \frac{2}{(s+1)(s+2)} |f''(b)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} |f''(x)|^q. \end{aligned}$$

Note that

$$\int_0^1 (1-t^2) dt = \frac{2}{3}. \tag{23}$$

Therefore, inequality (19) follows from (20)-(23). □

Corollary 2.11. *Under the assumptions of Theorem 2.10, one has*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{24} \\ &\leq \frac{\left(\frac{3}{2}\right)^{1/q} (b-a)^2}{24} \left[\frac{2}{(s+1)(s+3)} |f''(a)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \\ &+ \frac{\left(\frac{3}{2}\right)^{1/q} (b-a)^2}{24} \left[\frac{2}{(s+1)(s+3)} |f''(b)|^q + \frac{(s+2)(s+3)-2}{(s+1)(s+2)(s+3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \\ &\leq \frac{\left(\frac{3}{2}\right)^{1/q} (b-a)^2}{24} \left[\frac{2}{(s+1)(s+3)} + \frac{(s+2)(s+3)-2}{2^s (s+1)(s+2)(s+3)} \right]^{1/q} (|f''(a)| + |f''(b)|) \\ &+ \frac{\left(\frac{3}{2}\right)^{1/q} (b-a)^2}{24} \left[\frac{(s+2)(s+3)-2}{2^s (s+1)(s+2)(s+3)} \right]^{1/q} (|f''(a)| + |f''(b)|). \end{aligned}$$

Proof. Let $x = (a + b)/2$, then the first inequality of (24) can be obtained from inequality (19) immediately. While the second inequality of (24) follows from the s -convexity of $|f''|^q$ and the inequality

$$\sum_{k=1}^n (\alpha_k + \beta_k)^\sigma \leq \sum_{k=1}^n \alpha_k^\sigma + \sum_{k=1}^n \beta_k^\sigma$$

for $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n \geq 0$ and $0 \leq \sigma \leq 1$. □

Remark 2.12. If $s = 1$, the inequality (24) leads to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{\left(\frac{3}{2}\right)^{1/q} (b-a)^2}{24} \left[\left(\frac{11}{24}\right)^{1/q} + \left(\frac{5}{24}\right)^{1/q} \right] (|f''(a)| + |f''(b)|).$$

3 Hermite-Hadamard type inequalities for fractional integrals

Theorem 3.1. Let $s \in (0, 1]$, $\eta > 0$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\begin{aligned} & \left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \quad (25) \\ & \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \left[\frac{\eta+1}{(s+1)(s+\eta+2)} |f''(a)| + \left(\frac{1}{s+1} - \frac{\Gamma(\eta+2)\Gamma(s+1)}{\Gamma(\eta+s+3)} \right) |f''(x)| \right] \\ & \quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \left[\frac{\eta+1}{(s+1)(s+\eta+2)} |f''(b)| + \left(\frac{1}{s+1} - \frac{\Gamma(\eta+2)\Gamma(s+1)}{\Gamma(\eta+s+3)} \right) |f''(x)| \right]. \end{aligned}$$

holds for all $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|$ is s -convex on $[a, b]$.

Proof. It follows from (6) and the triangle inequality together with the s -convexity of $|f''|$ that

$$\begin{aligned} & \left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ & \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) |f''(ta + (1-t)x)| dt \\ & \quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) |f''(tb + (1-t)x)| dt \\ & \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) [t^s |f''(a)| + (1-t)^s |f''(x)|] dt \\ & \quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) [t^s |f''(b)| + (1-t)^s |f''(x)|] dt \\ & = \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \left[\frac{\eta+1}{(s+1)(s+\eta+2)} |f''(a)| + \left(\frac{1}{s+1} - \frac{\Gamma(\eta+2)\Gamma(s+1)}{\Gamma(\eta+s+3)} \right) |f''(x)| \right] \\ & \quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \left[\frac{\eta+1}{(s+1)(s+\eta+2)} |f''(b)| + \left(\frac{1}{s+1} - \frac{\Gamma(\eta+2)\Gamma(s+1)}{\Gamma(\eta+s+3)} \right) |f''(x)| \right]. \quad \square \end{aligned}$$

Remark 3.2. Let $\eta = 1$ in Theorem 3.1, then we get inequality (7) given in Theorem 2.1.

Corollary 3.3. Under the assumptions of Theorem 3.1, we have

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a)+f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^{\eta} f \left(\frac{a+b}{2} \right) + J_{b^-}^{\eta} f \left(\frac{a+b}{2} \right) \right] \right| \tag{26} \\ & \leq \frac{(b-a)^{\eta+1}}{2^{\eta+2}(\eta+1)} \left[\frac{(\eta+1)(|f''(a)|+|f''(b)|)}{(s+1)(s+\eta+1)} + 2 \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \left| f'' \left(\frac{a+b}{2} \right) \right| \right) \right] \\ & \leq \frac{(b-a)^{\eta+1}(|f''(a)|+|f''(b)|)}{2^{\eta+2}(\eta+1)} \left[\frac{\eta+1}{(s+1)(s+\eta+1)} + 2^{1-s} \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \right) \right]. \end{aligned}$$

Proof. Let $x = (a + b)/2$, then inequality (25) leads to the first inequality of (26). While the second inequality of (26) can be derived from the s -convexity of $|f''|$. □

Remark 3.4. Let $s = 1$, then the second inequality of (26) leads to

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a)+f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^{\eta} f \left(\frac{a+b}{2} \right) + J_{b^-}^{\eta} f \left(\frac{a+b}{2} \right) \right] \right| \\ & \leq \frac{(b-a)^{\eta+1}(|f''(a)|+|f''(b)|)}{2^{\eta+3}(\eta+1)} \left[\frac{\eta+1}{\eta+2} - \frac{2\Gamma(\eta+2)}{\Gamma(\eta+4)} + 1 \right]. \end{aligned}$$

Theorem 3.5. Let $\eta > 0, s \in (0, 1], p, q > 1$ with $1/p + 1/q = 1, M = \Gamma(1+p)\Gamma(1/(\eta+1))/[(\eta+1)\Gamma(1+p+1/(\eta+1))], f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\begin{aligned} & \left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{27} \\ & \leq \frac{(x-a)^{\eta+2} [|f''(a)|^q + |f''(x)|^q]^{1/q} + (b-x)^{\eta+2} [|f''(b)|^q + |f''(x)|^q]^{1/q}}{(\eta+1)(s+1)^{1/q}(b-a)} M^{1/p} \end{aligned}$$

holds for all $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Proof. It follows from (6) and the Hölder inequality together with the s -convexity of $|f''|^q$ that

$$\begin{aligned} & \left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{28} \\ & \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \left(\int_0^1 (1-t^{\eta+1})^p dt \right)^{1/p} \left(\int_0^1 |f''(ta + (1-t)x)|^q dt \right)^{1/q} \\ & \quad + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \left(\int_0^1 (1-t^{\eta+1})^p dt \right)^{1/p} \left(\int_0^1 |f''(tb + (1-t)x)|^q dt \right)^{1/q}, \end{aligned}$$

and inequalities (11) and (12) hold.

Note that

$$\int_0^1 (1-t^{\eta+1})^p dt = \frac{1}{\eta+1} \int_0^1 u^{1/(\eta+1)-1} (1-u)^p du = M. \tag{29}$$

Therefore, inequality (27) follows from (11), (12), (28) and (29). □

Remark 3.6. Let $\eta = 1$, then Theorem 3.5 leads to Theorem 2.4.

Let $x = (a + b)/2$, then the following Corollary 3.7 can be obtained from (27) and the s -convexity of $|f''|^q$ together with the inequality

$$\sum_{k=1}^n (\alpha_k + \beta_k)^\sigma \leq \sum_{k=1}^n \alpha_k^\sigma + \sum_{k=1}^n \beta_k^\sigma$$

for $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n \geq 0$ and $0 \leq \sigma \leq 1$.

Corollary 3.7. Under the assumptions of Theorem 3.5, we have the inequality as follows:

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^\eta f \left(\frac{a+b}{2} \right) + J_{b^-}^\eta f \left(\frac{a+b}{2} \right) \right] \right| \tag{30} \\ & \leq \frac{(b-a)^{\eta+1} M^{1/p}}{2^{\eta+2}(\eta+1)(s+1)^{1/q}} \left[\left(|f''(a)|^q + \left| f'' \left(\frac{a+b}{2} \right) \right|^q \right)^{1/q} + \left(|f''(b)|^q + \left| f'' \left(\frac{a+b}{2} \right) \right|^q \right)^{1/q} \right] \\ & \leq \frac{(b-a)^{\eta+1} (|f''(a)| + |f''(b)|) M^{1/p}}{2^{\eta+2}(\eta+1)(s+1)^{1/q}} \left[\left(\frac{1}{2^s} \right)^{1/q} + \left(1 + \frac{1}{2^s} \right)^{1/q} \right]. \end{aligned}$$

Remark 3.8. Let $s = 1$, then the second inequality of (30) leads to

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^\eta f \left(\frac{a+b}{2} \right) + J_{b^-}^\eta f \left(\frac{a+b}{2} \right) \right] \right| \\ & \leq \frac{(b-a)^{\eta+1} (|f''(a)| + |f''(b)|) M^{1/p}}{2^{\eta+2+1/q}(\eta+1)} \left[\left(\frac{1}{2} \right)^{1/q} + \left(\frac{3}{2} \right)^{1/q} \right]. \end{aligned}$$

Theorem 3.9. Let $\eta > 0$, $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $M = \Gamma(1+p)\Gamma(1/(\eta+1))/[(\eta+1)\Gamma(1+p+1/(\eta+1))]$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\begin{aligned} & \left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{31} \\ & \leq \frac{(x-a)^{\eta+2} \left| f'' \left(\frac{x+a}{2} \right) \right| + (b-x)^{\eta+2} \left| f'' \left(\frac{x+b}{2} \right) \right|}{(\eta+1)2^{(1-s)/q}(b-a)} M^{1/p} \end{aligned}$$

holds for all $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -concave on $[a, b]$.

Proof. Theorem 3.9 follows easily from (16), (17), (28) and (29). □

Remark 3.10. Let $\eta = 1$, then Theorem 3.9 becomes Theorem 2.7.

Letting $x = (a + b)/2$ and making use of the s -concavity of $|f''|$, then inequality (31) leads to Corollary 3.11 immediately.

Corollary 3.11. Under the assumptions of Theorem 3.9, one has

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^\eta f \left(\frac{a+b}{2} \right) + J_{b^-}^\eta f \left(\frac{a+b}{2} \right) \right] \right| \tag{32} \\ & \leq \frac{2^{(s-1)/q}(b-a)^{\eta+1} M^{1/p}}{2^{\eta+2}(\eta+1)} \left[\left| f'' \left(\frac{3a+b}{4} \right) \right| + \left| f'' \left(\frac{a+3b}{4} \right) \right| \right] \\ & \leq \frac{2^{[s(q+1)-1]/q}(b-a)^{\eta+1} M^{1/p}}{2^{\eta+2}(\eta+1)} \left| f'' \left(\frac{a+b}{2} \right) \right|. \end{aligned}$$

Remark 3.12. Let $s = 1$ in the second inequality of (32), then we get

$$\left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^{\eta} f \left(\frac{a+b}{2} \right) + J_{b^-}^{\eta} f \left(\frac{a+b}{2} \right) \right] \right| \leq \frac{(b-a)^{\eta+1} M^{1/p}}{2^{\eta+1}(\eta+1)} \left| f'' \left(\frac{a+b}{2} \right) \right|.$$

Theorem 3.13. Let $\eta > 0, s \in (0, 1], q > 1, f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. Then the inequality

$$\left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{33} \leq \frac{(x-a)^{\eta+2} \left[\frac{(\eta+1)|f''(a)|^q}{(s+1)(s+\eta+2)} + \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \right) |f''(x)|^q \right]^{1/q}}{(\eta+2)(b-a)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q} + \frac{(b-x)^{\eta+2} \left[\frac{(\eta+1)|f''(b)|^q}{(s+1)(s+\eta+2)} + \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \right) |f''(x)|^q \right]^{1/q}}{(\eta+2)(b-a)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q}$$

holds for all $x \in [a, b]$ if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Proof. By use of (6) and the power-mean inequality, we have

$$\left| \frac{[(x-a)^{\eta+1} - (b-x)^{\eta+1}]f'(x) + (\eta+1)f(b)(b-x) + (\eta+1)f(a)(x-a)}{(\eta+1)(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{34} \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) |f''(ta + (1-t)x)| dt + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \int_0^1 (1-t^{\eta+1}) |f''(tb + (1-t)x)| dt \leq \frac{(x-a)^{\eta+2}}{(\eta+1)(b-a)} \left(\int_0^1 (1-t^{\eta+1}) dt \right)^{1-1/q} \left(\int_0^1 (1-t^{\eta+1}) |f''(ta + (1-t)x)|^q dt \right)^{1/q} + \frac{(b-x)^{\eta+2}}{(\eta+1)(b-a)} \left(\int_0^1 (1-t^{\eta+1}) dt \right)^{1-1/q} \left(\int_0^1 (1-t^{\eta+1}) |f''(tb + (1-t)x)|^q dt \right)^{1/q}.$$

It follows from the s -convexity of $|f''|^q$ on $[a, b]$ that

$$\int_0^1 (1-t^{\eta+1}) |f''(ta + (1-t)x)|^q dt \leq \frac{(\eta+1)|f''(a)|^q}{(s+1)(s+\eta+2)} + \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \right) |f''(x)|^q, \tag{35}$$

$$\int_0^1 (1-t^{\eta+1}) |f''(tb + (1-t)x)|^q dt \leq \frac{(\eta+1)|f''(b)|^q}{(s+1)(s+\eta+2)} + \left(\frac{1}{s+1} - \frac{\Gamma(s+1)\Gamma(\eta+2)}{\Gamma(s+\eta+3)} \right) |f''(x)|^q. \tag{36}$$

Note that

$$\int_0^1 (1-t^{\eta+1}) dt = \frac{\eta+1}{\eta+2}. \tag{37}$$

Therefore, Theorem 3.13 follows from (34)-(37). □

Remark 3.14. Let $\eta = 1$, then Theorem 3.13 becomes Theorem 2.10.

Let $x = (a + b)/2$, then from (33) and the s -convexity of $|f''|^q$ we get Corollary 3.15 immediately.

Corollary 3.15. Under the assumptions of Theorem 3.13, one has

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^{\eta} f \left(\frac{a+b}{2} \right) + J_{b^-}^{\eta} f \left(\frac{a+b}{2} \right) \right] \right| \tag{38} \\ & \leq \frac{(b-a)^{\eta+1}}{2^{\eta+2}(\eta+2)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q} \left[M_1 |f''(a)|^q + M_2 \left| f'' \left(\frac{a+b}{2} \right) \right|^q \right]^{1/q} \\ & \quad + \frac{(b-a)^{\eta+1}}{2^{\eta+2}(\eta+2)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q} \left[M_1 |f''(b)|^q + M_2 \left| f'' \left(\frac{a+b}{2} \right) \right|^q \right]^{1/q} \\ & \leq \frac{(b-a)^{\eta+1}}{2^{\eta+2}(\eta+2)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q} \left[M_1^{1/q} + 2 \left(\frac{M_2}{2^s} \right)^{1/q} \right] (|f''(a)| + |f''(b)|), \end{aligned}$$

where $M_1 = (\eta + 1)/[(s + 1)(s + \eta + 2)]$ and $M_2 = 1/(s + 1) - \Gamma(s + 1)\Gamma(\eta + 2)/\Gamma(s + \eta + 3)$.

Remark 3.16. Let $s = 1$, then inequality (38) leads to

$$\begin{aligned} & \left| \left(\frac{b-a}{2} \right)^{\eta-1} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\eta+1)}{b-a} \left[J_{a^+}^{\eta} f \left(\frac{a+b}{2} \right) + J_{b^-}^{\eta} f \left(\frac{a+b}{2} \right) \right] \right| \\ & \leq \frac{(b-a)^{\eta+1}}{2^{\eta+2}(\eta+2)} \left(\frac{\eta+2}{\eta+1} \right)^{1/q} \left[\left(\frac{\eta+1}{2\eta+6} \right)^{1/q} + 2 \left(\frac{1}{4} - \frac{\Gamma(\eta+2)}{2\Gamma(\eta+4)} \right)^{1/q} \right] (|f''(a)| + |f''(b)|). \end{aligned}$$

4 Applications to trapezoidal formula

Let d be a division $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T(f, d) + E(f, d),$$

where

$$T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

is the trapezoidal version and $E(f, d)$ denotes the associated approximation error.

Theorem 4.1. Let $s \in (0, 1]$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and d be a division $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$. Then the inequality

$$|E(f, d)| \leq \left[\frac{1}{(s+1)(s+3)} + \frac{(s+2)(s+3)-2}{2^s(s+1)(s+2)(s+3)} \right] \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3 [|f''(x_i)| + |f''(x_{i+1})|]}{8}$$

holds if $f'' \in L[a, b]$ and $|f''|$ is s -convex on $[a, b]$.

Proof. Let $i \in \{0, 1, 2, \dots, n - 1\}$, then applying Corollary 2.2 on the interval $[x_i, x_{i+1}]$ we get

$$\left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right|$$

$$\leq \frac{(x_{i+1} - x_i)^2}{8} \left[\frac{1}{(s+1)(s+3)} + \frac{(s+2)(s+3) - 2}{2^s(s+1)(s+2)(s+3)} \right] [|f''(x_i)| + |f''(x_{i+1})|].$$

Therefore,

$$\begin{aligned} |E(f, d)| &= \left| \int_a^b f(x) dx - T(f, d) \right| \\ &= \left| \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right] \right| \\ &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right| \\ &\leq \left[\frac{1}{(s+1)(s+3)} + \frac{(s+2)(s+3) - 2}{2^s(s+1)(s+2)(s+3)} \right] \\ &\quad \times \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3 [|f''(x_i)| + |f''(x_{i+1})|]}{8}. \end{aligned} \quad \square$$

Making use of the similar arguments as in Theorem 4.1, we can get Theorems 4.2-4.4 from the Corollaries 2.5, 2.8 and 2.11 immediately.

Theorem 4.2. Let $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and d be a division $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$. Then the inequality

$$\begin{aligned} |E(f, d)| &\leq \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)} \right)^{1/p} \left[\left(\frac{1}{2^s} \right)^{1/q} + \left(1 + \frac{1}{2^s} \right)^{1/q} \right] \\ &\quad \times \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3 [|f''(x_i)| + |f''(x_{i+1})|]}{16(s+1)^{1/q}} \end{aligned}$$

holds if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$.

Theorem 4.3. Let $s \in (0, 1]$, $p, q > 1$ with $1/p + 1/q = 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and d be a division $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$. Then the inequality

$$|E(f, d)| \leq \frac{2^{(q+1)s/q}}{32} \left(\frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} \right)^{1/p} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left| f'' \left(\frac{x_{i+1} + x_i}{2} \right) \right|$$

holds if $f'' \in L[a, b]$ and $|f''|^q$ is s -concave on $[a, b]$.

Theorem 4.4. Let $s \in (0, 1]$, $q > 1$, $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and d be a division $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$. Then the inequality

$$\begin{aligned} |E(f, d)| &\leq \left(\frac{3}{2} \right)^{1/q} \left[\left(\frac{2}{(s+1)(s+3)} + \Delta(s) \right)^{1/q} + \Delta^{1/q}(s) \right] \\ &\quad \times \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3 [|f''(x_i)| + |f''(x_{i+1})|]}{24} \end{aligned}$$

holds if $f'' \in L[a, b]$ and $|f''|^q$ is s -convex on $[a, b]$, where

$$\Delta(s) = \frac{(s+2)(s+3) - 2}{2^s(s+1)(s+2)(s+3)}.$$

5 Conclusion

In the article, we present several new Hermite-Hadamard type inequalities and error estimations for the trapezoidal formula involving the s -convex and s -concave functions for the classical and Riemann-Liouville fractional integrals.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript, and they read and approved the final manuscript.

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