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## Research Article

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# A study on soft rough semigroups and corresponding decision making applications

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**Abstract:** In this paper, we study a kind of soft rough semigroups according to Shabir's idea. We define the upper and lower approximations of a subset of a semigroup. According to Zhan's idea over hemirings, we also define a kind of new  $C$ -soft sets and  $CC$ -soft sets over semigroups. In view of this theory, we investigate the soft rough ideals (prime ideals, bi-ideals, interior ideals, quasi-ideals, regular semigroups). Finally, we give two decision making methods: one is for looking a best a parameter which is to the nearest semigroup, the other is to choose a parameter which keeps the maximum regularity of regular semigroups.

**Keywords:** Soft set, Soft rough semigroup, Soft rough ideal, Regular semigroup, Decision making method

**MSC:** 20N25, 20M99, 20M17

## 1 Introduction

Since rough sets were proposed by Pawlak in [1], some famous academics turned their attention to rough sets. Rough sets, a new-style mathematical tools, are widely applied to handle incertitude and incomplete data in many fields, such as cognitive science, patten recognition, machine learning, and so on; for example in [2–4] the applications of rough sets were given. In addition, an equivalence relation (briefly, ER), as an indispensable part in Pawlak rough sets, is also investigated highly by many researchers. With the development of rough sets, some models of generalized rough sets were investigated, just as in [5, 6]. Later, rough sets were also established over algebraic structures, such as in [7–10].

Soft set theory, as another new-style mathematical tool for handling uncertainties, was firstly put forward by Molodsov in [11]. There is no doubt that soft sets play an important part in real life. Therefore, some operations were proposed over soft sets to make the best option, such as in [12–14]. Alcantud [15] discussed some relations among soft sets and other theories. Similarly, setting up soft sets over algebraic structures and studying the relevant properties were introduced by some researchers. Especially in [16], Ali et al. gave a detailed account about soft ideals, soft bi-ideals, soft quai-ideals, regular semigroups, and the relations between them. In recent years, some kinds of hybrid soft set models have been investigated by some researchers. For examples, Alcantud [17] raised a new algorithm for fuzzy soft sets with respect to decision making (briefly, DM) from multiobserver IPD-sets. In [18, 19], soft rough fuzzy sets (briefly, SRF-sets) and soft fuzzy rough sets (briefly, SFR-sets) were investigated, and their applications in decision making were given.

As we all know, both rough sets and soft sets are tools for dealing with incompleteness problems, and an ER can be replaced by other relations, such as binary relations in [5], therefore Feng et al. [20–22] built rough sets based on soft

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sets rather than an ER, which were called SR-sets. Following that, some studies were made on SR-sets. However, there is a limit on SR-sets, that is, the soft set must be full. Therefore, in [23], Shabir et al. presented another approach to SR-sets, which avoided the drawbacks occurring in [20, 22]. Shabir gave the modified SR-sets, which still have the properties of Pawlak-rough sets and similarly we also can solve uncertain problems and make decision problems. More recently, Zhang in [24] gave a way of multi-attribute decision making based on SR-sets. According to this kind of SR-set, Zhan et al. [25] discussed SR-hemirings under  $C$ -soft sets and  $CC$ -soft sets and obtained some different conclusions. At the same time, Zhan et al. made corresponding multicriteria for group decision making. Recently, Zhan et al. [26] put forth a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods.

As a special algebraic structure with wide applications, semigroups (for more details, see [27, 28]) have been studied extensively, such as in [29–31]. In this paper, according to the idea of Shabir about SR-sets, we study the rough sets over soft semigroups. In Section 2, we firstly give some relevant concepts about semigroups, SR-sets and soft sets. In Section 3, we study SR-semigroups and get some conclusions about the upper soft rough approximations and lower soft rough approximations under  $C$ -soft sets and  $CC$ -soft sets. In section 4, SR-semigroups (ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals, regular semigroups) are studied. In Section 5, we give two decision making methods. The method I looks for a best parameter to meet the nearest semigroup, and method II looks for keeping the maximum regularity of regular semigroups.

## 2 Preliminaries

Recall that a nonempty set  $S$  with a binary operation “ $\cdot$ ” is called a semigroup if it satisfies: (i)  $s \cdot t \in S$ , for all  $s, t \in S$ ; (ii)  $(m \cdot n) \cdot t = m \cdot (n \cdot t)$ , for all  $m, n, t \in S$ . For more details, see [28].

We know that  $A$  is called a subsemigroup of  $S$  if  $\emptyset \neq A \subseteq S$ , and  $A \cdot A \subseteq S$ ; a subsemigroup  $A$  of  $S$  is called a left (resp. right) ideal if  $SA \subseteq A$  (resp.  $AS \subseteq A$ ); a subsemigroup  $I$  is called an ideal, if  $SI \subseteq I$  and  $IS \subseteq I$ ; an ideal  $I$  is called a prime ideal, if  $m \cdot n \in I$  implies  $m \in I$  or  $n \in I$ , for all  $m, n \in S$ ; a subsemigroup  $T$  of  $S$  is called a bi-ideal, if  $TST \subseteq T$ ; a non-empty subset  $Q$  of  $S$  is called a quasi-ideal, if  $QS \cap SQ \subseteq Q$ ; a subsemigroup  $T$  of  $S$  is called an interior ideal, if it satisfies  $STS \subseteq T$ .

Let  $S$  be a semigroup, for  $m \in S$ , if there exists an  $n \in S$  s.t.  $m = mn$ , then  $m$  is a regular element. If for all  $x \in S$ ,  $x$  is regular, then  $S$  is called a regular semigroup. If for all  $a \in S$ , there exists  $b \in S$  such that  $aba = a$  and  $bab = b$ , we say that  $b$  is an inverse of  $a$ . It is well known that in a regular semigroup, every element has an inverse. For more details, see [27, 28].

Throughout this paper,  $S$  denotes a semigroup.

**Definition 2.1** ([11]). A pair  $\mathfrak{T} = (G, B)$  is called a soft set over  $S$ , where  $B \subseteq E$  and  $G : B \rightarrow \mathcal{P}(S)$  is a set-valued mapping, where the symbol  $\mathcal{P}(S)$  denotes the power set of  $S$ .

**Definition 2.2** ([20]). A soft set  $\mathfrak{T} = (G, B)$  over  $S$  is called full if  $\bigcup_{b \in B} G(b) = S$ .

**Definition 2.3** ([9]). Let  $(G, B)$  be a soft set over  $S$ . Then:

- (1)  $(G, B)$  is called a soft semigroup (regular semigroup) over  $S$  if  $G(x)$  is a subsemigroup (regular subsemigroup) of  $S$ , for all  $x \in \text{Supp}(G, B)$ ,
- (2)  $(G, B)$  is called a soft ideal (prime ideal, bi-ideal, quasi-ideal, interior ideal) if  $G(x)$  is an ideal (prime ideal, bi-ideal, quasi-ideal, interior ideal) of  $S$ , for all  $x \in \text{Supp}(G, B)$ , where  $\text{Supp}(G, B) = \{x \in B \mid G(x) \neq \emptyset\}$  is called a soft support of the soft set  $(G, B)$ .

**Definition 2.4** ([23]). Consider  $(F, A)$  be a soft set over  $S$  and  $\eta : S \rightarrow \mathcal{P}(A)$  be a map defined as  $\eta(x) = \{a \in A \mid x \in F(a)\}$ . Then the pair  $(S, \eta)$  is called an  $MS$ -approximation space, for any  $X \subseteq S$ , the lower  $MSR$ -approximation and upper  $MSR$ -approximation of  $X$  are denoted by  $\underline{X}_\eta$  and  $\overline{X}_\eta$ .

$$\underline{X}_\eta = \{x \in X \mid \eta(x) \neq \eta(y), \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\eta = \{x \in S \mid \eta(x) = \eta(y), \text{ for some } y \in X\}.$$

If  $\underline{X}_\eta = \overline{X}_\eta$ , then the  $X$  is said to be *MS-definable*, otherwise  $X$  is said to be *MSR-set*.

### 3 Soft rough approximations

In this section, we study some operations and fundamental properties of modified SR-sets over semigroups, in order to illustrate the roughness in a semigroup  $S$  with respect to an SR-approximation space over semigroups.

**Definition 3.1.** Let  $\mathfrak{S} = (F, A)$  be a soft set over  $S$  and  $\eta : S \rightarrow \mathcal{P}(A)$  be a set-valued mapping defined as  $\eta(x) = \{a \in A \mid x \in F(a)\}$ . Then  $\mathfrak{S}$  is called a *C-soft set* over  $S$  if  $\eta(a) = \eta(b)$  and  $\eta(m) = \eta(n)$  imply  $\eta(am) = \eta(bn)$ , for all  $a, b, c, d \in S$ .

**Example 3.2.** Let  $S = \{a, b, c, d\}$  be a semigroup as in Table 1:

**Table 1.** Table for semigroup  $S$

$\cdot$	$a$	$b$	$c$	$d$
$a$	$a$	$a$	$a$	$a$
$b$	$a$	$c$	$c$	$c$
$c$	$a$	$c$	$c$	$c$
$d$	$a$	$c$	$c$	$c$

Define a soft set  $\mathfrak{S} = (F, A)$  over  $S$  as in Table 2:

**Table 2.** Table for soft set  $\mathfrak{S}$

	$a$	$b$	$c$	$d$
$e_1$	$0$	$0$	$0$	$1$
$e_2$	$1$	$1$	$1$	$0$
$e_3$	$1$	$1$	$1$	$1$

Then the mapping  $\eta : S \rightarrow \mathcal{P}(A)$  of MS-approximation space  $(S, \eta)$  is given by  $\eta(a) = \eta(b) = \eta(c) = \{e_2, e_3\}$ ,  $\eta(d) = \{e_1, e_3\}$ . Obviously,  $\mathfrak{S} = (F, A)$  is a C-soft set over  $S$ .

Let  $M, N \subseteq S$ ,  $M, N \neq \emptyset$ . Denote  $M \cdot N = \{x \cdot y \mid \forall x \in M, y \in N\}$ .

**Proposition 3.3.** Let  $\mathfrak{S} = (F, A)$  be a C-soft set over  $S$  and  $(S, \eta)$  an MS-approximation space. Then for any two nonempty subsets  $M, N$  in  $S$ ,

$$\overline{M}_\eta \cdot \overline{N}_\eta \subseteq \overline{M \cdot N}_\eta.$$

*Proof.* Let  $a \in \overline{M}_\eta \cdot \overline{N}_\eta$ . Then  $a = mn$ , where  $m \in \overline{M}_\eta$  and  $n \in \overline{N}_\eta$ , and so there exist  $x \in M$  and  $y \in N$  such that  $\eta(m) = \eta(x)$  and  $\eta(n) = \eta(y)$ . Since  $\mathfrak{S}$  is a C-soft set,  $\eta(mn) = \eta(xy)$  for  $xy \in M \cdot N$ . Hence  $a = mn \in \overline{M \cdot N}_\eta$ . That is  $\overline{M}_\eta \cdot \overline{N}_\eta \subseteq \overline{M \cdot N}_\eta$ .  $\square$

We claim that the containment in Proposition 3.3 is proper by the following example.

**Example 3.4.** Assume that  $S$  and  $\mathfrak{S} = (F, A)$  are in Example 3.2. Define two subsets  $M$  and  $N$  over  $S$ , here  $M = \{a, b\}$  and  $N = \{a, c, d\}$ . Then  $M \cdot N = \{a, c\}$ ,  $\overline{M}_\eta = \{a, b, c\}$ ,  $\overline{N}_\eta = \{a, b, c, d\}$ ,  $\overline{M \cdot N}_\eta = \{a, b, c\}$  and  $\overline{M}_\eta \cdot \overline{N}_\eta = \{a, c\}$ . Thus  $\overline{M}_\eta \cdot \overline{N}_\eta \subseteq \overline{M \cdot N}_\eta$ .

**Definition 3.5.** Assume that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$  and  $\eta : S \rightarrow \mathcal{P}(A)$  is a set-valued map defined as  $\eta(x) = \{e \in A \mid x \in F(e)\}$ . Then  $\mathfrak{S}$  is said to be a  $CC$ -soft set over  $S$  if for all  $x \in S$ ,  $\eta(x) = \eta(ab)$  for  $a, b \in S$ , there exist  $m, n \in S$  such that  $\eta(a) = \eta(m)$  and  $\eta(b) = \eta(n)$  satisfying  $x = mn$ .

**Remark 3.6.** We point out that  $\mathfrak{S}$  is a  $C$ -soft set over  $S$  in Example 3.2, but it is not a  $CC$ -soft set. Since  $\eta(a) = \eta(dd)$ , there only exist  $d \in S$  such that  $\eta(d) = \eta(d)$  but  $a \neq dd$ .

In the following, we give an example of  $CC$ -soft sets over  $S$ .

**Example 3.7.** Let  $S = \{a, b, c, d\}$  be a semigroup as in Table 3:

**Table 3.** Table for semigroup  $S$

$\cdot$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$b$	$b$	$b$
$c$	$c$	$c$	$c$	$c$
$d$	$d$	$c$	$b$	$a$

Define a soft set  $\mathfrak{S} = (F, A)$  over  $S$  as in Table 4:

**Table 4.** Table for soft set  $\mathfrak{S}$

	$a$	$b$	$c$	$d$
$e_1$	1	1	1	1
$e_2$	0	1	1	0
$e_3$	1	0	0	1

Then the set-valued mapping  $\eta : S \rightarrow \mathcal{P}(A)$  of  $MS$ -approximation space  $(S, \eta)$  is given by  $\eta(a) = \eta(d) = \{e_1, e_3\}$  and  $\eta(b) = \eta(c) = \{e_1, e_2\}$ . Obviously,  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ .

The obtained conclusions based on  $CC$ -soft sets over  $S$  are different from Proposition 3.3 which are obtained from  $C$ -soft sets over  $S$ .

**Theorem 3.8.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $(S, \eta)$  is an  $MS$ -approximation space. Then for any two nonempty subsets  $X$  and  $Y$  in  $S$ ,

$$\overline{X}_\eta \cdot \overline{Y}_\eta = \overline{X \cdot Y}_\eta.$$

*Proof.* From Proposition 3.3,  $\overline{X}_\eta \cdot \overline{Y}_\eta \subseteq \overline{X \cdot Y}_\eta$  holds. Now let  $a \in \overline{X \cdot Y}_\eta$ , so  $\eta(a) = \eta(mn)$  where  $m \in X$  and  $n \in Y$ . Since  $\mathfrak{S}$  is a  $CC$ -soft set, there exist  $x, y \in S$  such that  $\eta(m) = \eta(x)$  and  $\eta(n) = \eta(y)$  satisfying  $a = xy$ . Thus  $x \in \overline{X}_\eta$  and  $y \in \overline{Y}_\eta$ . Hence  $a = xy \in \overline{X}_\eta \cdot \overline{Y}_\eta$ , that is,  $\overline{X \cdot Y}_\eta \subseteq \overline{X}_\eta \cdot \overline{Y}_\eta$ . Therefore,  $\overline{X}_\eta \cdot \overline{Y}_\eta = \overline{X \cdot Y}_\eta$  holds.  $\square$

**Theorem 3.9.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $(S, \eta)$  is an  $MS$ -approximation space. Then for any two nonempty subsets  $X$  and  $Y$  in  $S$ ,

$$\underline{X}_\eta \cdot \underline{Y}_\eta \subseteq \underline{X \cdot Y}_\eta.$$

*Proof.* Suppose that  $\underline{X}_\eta \cdot \underline{Y}_\eta \subseteq \underline{X \cdot Y}_\eta$  does not hold. Then there exists  $a \in \underline{X}_\eta \cdot \underline{Y}_\eta$ , such that  $a \notin \underline{X \cdot Y}_\eta$ . Hence  $a = xy$ , where  $x \in \underline{X}_\eta$  and  $y \in \underline{Y}_\eta$ . This means that  $\eta(x) \neq \eta(m)$  and  $\eta(y) \neq \eta(n)$  for all  $m \in X^c$  and  $n \in Y^c$ . ( $\Delta$ ) On the other hand, since  $a \notin \underline{X \cdot Y}_\eta$ , we may have the following two conditions:

(i)  $a \notin X \cdot Y$ , which is in contradicts with  $a \in \underline{X}_\eta \cdot \underline{Y}_\eta \subseteq X \cdot Y$ ;  
(ii)  $a \in X \cdot Y$ , but  $\eta(a) = \eta(x' \cdot y')$  for some  $x' \cdot y' \in (X \cdot Y)^c$ . Thus  $x' \in X^c$  or  $y' \in Y^c$ . In fact, if  $x' \notin X^c$  and  $y' \notin Y^c$ , we have  $x' \cdot y' \in X \cdot Y$ , a contradiction. Since  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , there exist  $a', b' \in S$  such that  $\eta(a') = \eta(x')$  and  $\eta(b') = \eta(y')$  satisfying  $a' \cdot b' = a$ , for some  $x' \in X^c$  and  $y' \in Y^c$ . This is in contradiction with ( $\Delta$ ). Hence  $\underline{X}_\eta \cdot \underline{Y}_\eta \subseteq \underline{X \cdot Y}_\eta$ .  $\square$

**Example 3.10.** Assume that  $S$  and  $\mathfrak{S} = (F, A)$  are in Example 3.7. Define two subsets  $X$  and  $Y$  over  $S$ , here  $X = \{a, b, d\}$  and  $Y = \{b, c, d\}$ . Then  $X \cdot Y = \{a, b, c, d\}$ ,  $\underline{X}_\eta = \{a, d\}$ ,  $\underline{Y}_\eta = \{b, c\}$ ,  $\underline{X \cdot Y}_\eta = \{a, b, c, d\}$ , and  $\underline{X}_\eta \cdot \underline{Y}_\eta = \{b, c\}$ . Thus  $\underline{X}_\eta \cdot \underline{Y}_\eta \subsetneq \underline{X \cdot Y}_\eta$ .

We claim that the containment in Theorem 3.8 is proper when  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  by the above example. Now we consider the case when  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ , if we can get the similar conclusion as Theorem 3.8.

**Example 3.11.** Assume that the semigroup  $S$  and the soft set  $\mathfrak{S} = (F, A)$  are in Example 3.7. Define two subsets  $X$  and  $Y$  over  $S$ , here  $X = \{a, b\}$  and  $Y = \{a, b\}$ , then  $X \cdot Y = \{a\}$ . And so,  $\underline{X}_\eta = \{a, b\}$ ,  $\underline{Y}_\eta = \{a, b\}$ ,  $\underline{X \cdot Y}_\eta = \emptyset$ ,  $\underline{X}_\eta \cdot \underline{Y}_\eta = \{a, b\}$ . Thus  $\underline{X}_\eta \cdot \underline{Y}_\eta \not\subseteq \underline{X \cdot Y}_\eta$ . Obviously, when  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ , Theorem 3.8 is not proper.

## 4 SR-semigroups

In this Section, we discuss the operations of lower and upper  $MSR$ -approximations of  $SR$ -semigroups.

**Definition 4.1.** Suppose that  $\mathfrak{S} = (F, A)$  is a soft set over  $S$  and  $(S, \eta)$  is an  $MS$ -approximation space. For any  $X \subseteq S$ , the lower  $MSR$ -approximation and upper  $MSR$ -approximation of  $X$  are denoted by  $\underline{X}_\eta$  and  $\overline{X}_\eta$ , respectively, which two operations are given as:

$$\underline{X}_\eta = \{x \in X \mid \eta(x) \neq \eta(y), \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\eta = \{x \in S \mid \eta(x) = \eta(y), \text{ for some } y \in X\}.$$

If  $\underline{X}_\eta \neq \overline{X}_\eta$ , then

- (i)  $X$  is called a lower (upper)  $SR$ -semigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal, regular semigroup) over  $S$ , if  $\underline{X}_\eta$  ( $\overline{X}_\eta$ ) is a subsemigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal, regular semigroup) of  $S$ .
- (ii)  $X$  is called a  $SR$ -semigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal, regular semigroup) over  $S$ , if  $\underline{X}_\eta$  and  $\overline{X}_\eta$  are subsemigroups (resp., ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals, regular semigroup) of  $S$ .

**Example 4.2.** Let  $S = \{a, b, c, d, e\}$  be a semigroup as in Table 5:

**Table 5.** Table for semigroup  $S$ 

$\cdot$	$a$	$b$	$c$	$d$	$e$
$a$	$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$	$b$	$c$
$c$	$a$	$b$	$c$	$a$	$a$
$d$	$a$	$a$	$a$	$d$	$e$
$e$	$a$	$d$	$e$	$a$	$a$

Define a soft set  $\mathfrak{S} = (F, A)$  over  $S$  as in Table 6:

**Table 6.** Table for soft set  $\mathfrak{S}$ 

	$a$	$b$	$c$	$d$	$e$
$e_1$	0	1	1	1	1
$e_2$	0	1	1	0	0
$e_3$	1	0	0	1	1

Then the set-valued mapping  $\eta : S \rightarrow \mathcal{P}(A)$  of soft rough approximation space  $(S, \eta)$  is given by  $\eta(a) = \{e_3\}$ ,  $\eta(b) = \eta(c) = \{e_1, e_2\}$  and  $\eta(d) = \eta(e) = \{e_1, e_3\}$ . Let  $X = \{a, b, d\}$  and  $Y = \{a, b, d, e\}$ , then we have  $\underline{X}_\eta = \{a\}$  and  $\overline{X}_\eta = \{a, b, c, d, e\}$ . Obviously,  $\underline{X}_\eta$  and  $\overline{X}_\eta$  are ideals of  $S$  and so  $X$  is a SR-ideal over  $S$ . Moreover,  $\underline{Y}_\eta = \{a, d, e\}$  and  $\overline{Y}_\eta = \{a, b, c, d, e\}$ . Obviously,  $\underline{Y}_\eta$  and  $\overline{Y}_\eta$  are subsemigroups of  $S$  and so  $Y$  is a SR-semigroup over  $S$ .

Similarly, we can construct SR-prime ideals, bi-ideals, quasi-ideals and interior ideals over  $S$ .

**Proposition 4.3.** Let  $(S, \eta)$  be an MS-approximation space. Suppose that  $X$  and  $Y$  are lower SR-semigroups (resp., ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals) over  $S$ , then so is  $X \cap Y$ .

*Proof.* Suppose that  $X$  and  $Y$  are lower SR-semigroups (resp., ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals) over  $S$ , then  $\underline{X}_\eta$  and  $\underline{Y}_\eta$  are subsemigroups (resp., ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals) of  $S$ , so  $\underline{X}_\eta \cap \underline{Y}_\eta$  is a subsemigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal) of  $S$ .

By Theorem 3 in [23], we have  $\underline{X} \cap \underline{Y}_\eta = \underline{X}_\eta \cap \underline{Y}_\eta$  is also a subsemigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal) of  $S$ . Hence  $X \cap Y$  is a lower SR-subsemigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal) over  $S$ .  $\square$

In reality, although  $X$  and  $Y$  are upper SR-semigroups (resp., ideals, prime ideals, bi-ideals, quasi-ideals, interior ideals) over  $S$ ,  $X \cap Y$  may not be upper SR-subsemigroup (resp., ideal, prime ideal, bi-ideal, quasi-ideal, interior ideal). The example in the following can explain this case.

**Example 4.4.** Suppose that the semigroup  $S$  is in Example 3.7 and we can define a new soft set as in Table 7:

**Table 7.** Table for soft set  $\mathfrak{S}$ 

	$a$	$b$	$c$	$d$
$e_1$	1	1	1	1
$e_2$	1	1	0	0
$e_3$	0	0	1	1

Then the mapping  $\eta : S \rightarrow \mathcal{P}(A)$  of MS-approximation space  $(S, \eta)$  is given by  $\eta(a) = \eta(b) = \{e_1, e_2\}$ ,  $\eta(c) = \eta(d) = \{e_1, e_3\}$ . Now we give two subsets of  $S$ ,  $X = \{a, d\}$  and  $Y = \{b, d\}$ , then  $X \cap Y = \{d\}$ . Thus,  $\overline{X}_\eta =$

$\overline{Y}_\eta = \{a, b, c, d\}$ ,  $\overline{X \cap Y}_\eta = \{c, d\}$ ,  $X$  and  $Y$  are upper SR-semigroups of  $S$ . But  $\overline{X \cap Y}_\eta = \{c, d\}$  is not a subsemigroup of  $S$ .

**Theorem 4.5.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ . Let  $X$  be a subsemigroup of  $S$ , then  $X$  is an upper SR-subsemigroup over  $S$ .

*Proof.* Clearly,  $\overline{X}_\eta \neq \emptyset$  since  $X \subseteq \overline{X}_\eta$ . For any  $a, b \in \overline{X}_\eta$ , then  $\eta(a) = \eta(m)$  and  $\eta(b) = \eta(n)$  for some  $m, n \in X$ . Since  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ ,  $\eta(ab) = \eta(mn)$ , here  $mn \in X$  since  $X$  is a subsemigroup of  $S$ , so  $ab \in \overline{X}_\eta$ . Hence,  $\overline{X}_\eta$  is a subsemigroup of  $S$ , that is,  $X$  is an upper SR-subsemigroup over  $S$ .  $\square$

**Theorem 4.6.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $X$  is a subsemigroup of  $S$ . Then  $X$  is a lower SR-semigroup over  $S$ , where  $\underline{X}_\eta \neq \emptyset$ .

*Proof.* Let  $\underline{X}_\eta \neq \emptyset$ , and for  $a, b \in \underline{X}_\eta$ , we suppose  $ab \notin \underline{X}_\eta$ . Then we have  $\eta(a) \neq \eta(x)$  for all  $x \in X^c$  and  $\eta(b) \neq \eta(y)$  for all  $y \in X^c$ . Since  $ab \notin \underline{X}_\eta$ , we may have the following two conditions:

- (1)  $ab \notin X$ , which contradicts with  $ab \in \underline{X}_\eta \cdot \underline{X}_\eta \subseteq X \cdot X \subseteq X$ ;
- (2)  $ab \in X$  and  $\eta(x') = \eta(ab)$  for some  $x' \in X^c$ . Since  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , then there exist  $m, n \in S$  such that  $\eta(a) = \eta(m)$  and  $\eta(b) = \eta(n)$  satisfying  $mn = x' \in X^c$ . Thus  $m \in X^c$  or  $n \in X^c$ . In fact, if  $m \notin X^c$  and  $n \notin X^c$ , we have  $mn \in X \cdot X \subseteq X$ , a contradiction. That is there exist  $m \in X^c$  such that  $\eta(a) = \eta(m)$  or  $n \in X^c$  such that  $\eta(b) = \eta(n)$ , so  $a \notin \underline{X}_\eta$  or  $b \notin \underline{X}_\eta$ , this is contradictory to  $a, b \in \underline{X}_\eta$ . Therefore, the assumption is incorrect, that is,  $ab \in \underline{X}_\eta$ . Hence  $\underline{X}_\eta$  is a subsemigroup of  $S$ , this means that  $X$  is a lower SR-semigroup over  $S$ .  $\square$

**Theorem 4.7.** (1) Suppose that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ . If  $X$  is an ideal of  $S$ , then  $\overline{X}_\eta$  is an ideal of  $S$ .  
 (2) Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ . If  $X$  is an ideal of  $S$ , then  $X$  is a lower SR-ideal over  $S$ ,  $\underline{X}_\eta \neq \emptyset$ .

*Proof.* (1) According to Theorem 4.5 and  $X$  is an ideal of  $S$ , we have  $\overline{X}_\eta$  is subsemigroup of  $S$ . For  $s \in S$  and  $x \in \overline{X}_\eta$ , then  $\eta(x) = \eta(x')$  for some  $x' \in X$ . Since  $\eta(s) = \eta(s)$  and  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ ,  $\eta(x's) = \eta(xs)$ . Moreover,  $X$  is an ideal of  $S$ , then  $x's \in X$ , and so,  $xs \in \overline{X}_\eta$ . Hence  $\overline{X}_\eta$  is a right ideal. We can obtain that  $\overline{X}_\eta$  is a left ideal in the same way. Therefore,  $\overline{X}_\eta$  is an ideal of  $S$ .

(2) According to Theorem 4.6 and the fact that  $X$  is an ideal of  $S$ , we have  $\underline{X}_\eta$  is a subsemigroup of  $S$ . And by Theorem 3.9, we have  $S \cdot \underline{X}_\eta = \underline{S}_\eta \cdot \underline{X}_\eta \subseteq \underline{S} \cdot \underline{X}_\eta \subseteq \underline{X}_\eta$ . Similarly, we can obtain  $\underline{X}_\eta \cdot S \subseteq \underline{X}_\eta$ . Thus,  $\underline{X}_\eta$  is an ideal of  $S$ , that is,  $X$  is a lower SR-ideal over  $S$ .  $\square$

We claim that the converse of Theorem 4.7 is incorrect by in the following example.

**Example 4.8.** Suppose that the semigroup  $S$  and the soft set  $\mathfrak{S} = (F, A)$  are in Example 3.7. Define a subset  $X$  over  $S$ , here  $X = \{b, c, d\}$ . Then  $\underline{X}_\eta = \{b, c\}$  and  $\overline{X}_\eta = \{a, b, c, d\}$ . Obviously,  $\underline{X}_\eta$  and  $\overline{X}_\eta$  are ideals of  $S$ , but  $X$  is not an ideal of  $S$ .

**Theorem 4.9.** (1) Suppose that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$  and  $X$  is a bi-ideal of  $S$ . Then  $\overline{X}_\eta$  is a bi-ideal of  $S$ .

(2) Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $X$  is a bi-ideal of  $S$ . Then  $X$  is a lower SR-bi-ideal over  $S$ , where  $\underline{X}_\eta \neq \emptyset$ .

*Proof.* (1) According to Theorem 4.5 and the fact that  $X$  is a bi-ideal of  $S$ , we have that  $\overline{X}_\eta$  is a subsemigroup of  $S$ . By Proposition 3.3 and by that  $X$  is a bi-ideal of  $S$ , we get  $\overline{T}_\eta S \overline{T}_\eta \subseteq \overline{T}_\eta \overline{S}_\eta \overline{T}_\eta \subseteq \overline{T} S \overline{T} \subseteq \overline{T}_\eta$ . Therefore,  $\overline{X}_\eta$  is a bi-ideal of  $S$ .

(2) According to Theorem 4.6 and to the fact that  $X$  is a bi-ideal of  $S$ , we have that  $\underline{X}_\eta$  is a subsemigroup of  $S$ . And by Proposition 3.9, we have  $\underline{X}_\eta \cdot S \cdot \underline{X}_\eta \subseteq \underline{X}_\eta \cdot \underline{S}_\eta \cdot \underline{X}_\eta \subseteq \underline{X} \cdot \underline{S} \cdot \underline{X} \subseteq \underline{X}_\eta$ . Hence  $\underline{X}_\eta$  is a bi-ideal of  $S$ , that is  $X$  is a lower SR-semigroup over  $S$ .  $\square$

**Theorem 4.10.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $X$  is a prime ideal of  $S$ . Then  $\overline{X}_\eta$  is a prime ideal of  $S$ .

*Proof.* According to Theorem 4.7 and to the fact that  $X$  is a prime ideal of  $S$ , we have that  $\overline{X}_\eta$  is an ideal of  $S$ . For  $ab \in \overline{X}_\eta$ , we have  $\eta(x) = \eta(ab)$  for some  $x \in X$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , there exist  $m, n \in S$  such that  $\eta(a) = \eta(m)$  and  $\eta(b) = \eta(n)$ , here  $x = mn$ . Moreover,  $X$  is a prime ideal of  $S$ , then  $m \in X$  or  $n \in X$ , and so,  $a \in \overline{X}_\eta$  or  $b \in \overline{X}_\eta$ . Therefore,  $\overline{X}_\eta$  is a prime ideal of  $S$ .  $\square$

**Theorem 4.11.** Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $X$  is a prime ideal of  $S$ . Then  $X$  is a lower SR-prime ideal over  $S$ ,  $\underline{X}_\eta \neq \emptyset$ .

*Proof.* According to Theorem 4.7 and the fact that  $X$  is a prime ideal of  $S$ , we have that  $\underline{X}_\eta$  is an ideal of  $S$ . For  $ab \in \underline{X}_\eta$ , suppose that  $\underline{X}_\eta$  is not a prime ideal of  $S$ , that is,  $a \notin \underline{X}_\eta$  and  $b \notin \underline{X}_\eta$ , then we may have the following conditions:

(1)  $a, b \notin X$ : since  $ab \in \underline{X}_\eta \subseteq X$  and  $X$  is a prime ideal of  $S$ , we have  $a \in X$  or  $b \in X$ , which contradicts  $a, b \notin X$ .

(2)  $a, b \in X$ : but  $\eta(a) = \eta(m)$  and  $\eta(b) = \eta(n)$  for some  $m, n \in X^c$ , by  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ , so  $\eta(ab) = \eta(mn)$ . By using  $m, n \in X^c$ , then  $mn \in X^c$ . If not,  $mn \in X$ , since  $X$  is a prime ideal of  $S$ , we have  $m \in X$  or  $n \in X$ , this is contradictory to  $m, n \in X^c$ . Therefore  $mn \in X^c$  ( $\blacktriangle$ ) and so  $ab \notin \underline{X}_\eta$ , which is contradictory to  $ab \in \underline{X}_\eta$ .

(3)  $a \in X$  or  $b \in X$ . We consider only  $a \in X$ ,  $b \notin X$ , that is,  $b \in X^c$ , the proof of  $a \notin X$ ,  $b \in X$  is similar. Since  $a \in X$ ,  $a \notin \underline{X}_\eta$ , there exists  $m \in X^c$  such that  $\varphi(a) = \varphi(m)$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ , then  $\varphi(ab) = \varphi(mb)$ . By using  $b, m \in X^c$ , we can prove  $mb \in X^c$  as ( $\blacktriangle$ ). Hence  $ab \notin \underline{X}_\eta$ , this contradicts with our assumption that  $ab \in \underline{X}_\eta$ . Therefore  $a \in X$ ,  $b \notin X$  is incorrect.

To summarize, the assumption is incorrect. Hence  $a \in \underline{X}_\eta$  or  $b \in \underline{X}_\eta$ , that is,  $\underline{X}_\eta$  is a prime ideal of  $S$  and  $X$  is a lower SR-prime ideal over  $S$ .  $\square$

**Theorem 4.12.** (1) Suppose that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$  and  $Q$  is a quasi-ideal of  $S$ . Then  $\overline{Q}_\eta$  is a quasi-ideal of  $S$ .

(2) Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $Q$  is a quasi-ideal of  $S$ . Then  $\underline{Q}_\eta$  is a quasi-ideal of  $S$ ,  $\underline{Q}_\eta \neq \emptyset$ .

*Proof.* (1) According to Theorem 4.7 and that  $Q$  is a prime ideal of  $S$ , we have that  $\overline{Q}_\eta$  is an ideal of  $S$ . By Theorem 3 in [23] and Proposition 3.3, we get  $\overline{Q}_\eta S \cap S \overline{Q}_\eta \subseteq \overline{Q}_\eta \cdot \overline{S}_\eta \cap \overline{S}_\eta \cdot \overline{Q}_\eta \subseteq \overline{Q} \overline{S}_\eta \cap \overline{S} \overline{Q}_\eta \subseteq \overline{Q} S \cap S \overline{Q}_\eta \subseteq \overline{Q}_\eta$ . Therefore  $\overline{Q}_\eta$  is a quasi-ideal of  $S$ .

(2) According to Theorem 4.7 and that  $Q$  is a prime ideal of  $S$ , we have that  $\underline{Q}_\eta$  is ideal of  $S$ . By Theorem 3 in [23], Theorem 3.9 and by that  $Q$  is a quasi-ideal of  $S$ , we get  $\underline{Q}_\eta S \cap S \underline{Q}_\eta \subseteq \underline{Q}_\eta \cdot \underline{S}_\eta \cap \underline{S}_\eta \cdot \underline{Q}_\eta \subseteq \underline{Q} \underline{S}_\eta \cap \underline{S} \underline{Q}_\eta \subseteq \underline{Q} S \cap S \underline{Q}_\eta \subseteq \underline{Q}_\eta$ . Therefore  $\underline{Q}_\eta$  is a quasi-ideal of  $S$ .  $\square$

**Theorem 4.13.** (1) Suppose that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $T$  is an interior ideal of  $S$ . Then  $\underline{T}_\eta$  is an interior ideal of  $S$ , when  $\underline{T}_\eta \neq \emptyset$ .

(2) Suppose that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ . Let  $T$  be an interior ideal of  $S$ , then  $T$  is an upper SR-interior ideal over  $S$ .

*Proof.* (1) Since  $T$  is an interior ideal of  $S$  and Theorem 4.6, we obtain  $\underline{T}_\eta$  is a subsemigroup of  $S$ . According to Theorem 3.9,  $S \cdot \underline{T}_\eta \cdot S \subseteq \underline{S}_\eta \cdot \underline{T}_\eta \cdot \underline{S}_\eta \subseteq \underline{S} \underline{T} \underline{S}_\eta \subseteq \underline{T}_\eta$ , so  $\underline{T}_\eta$  is an interior ideal of  $S$ .

(2) According to Theorem 4.5 and by that  $T$  is an interior ideal of  $S$ , we get that  $\overline{T}_\eta$  is a subsemigroup of  $S$ . By Proposition 3.3, we have  $S \cdot \overline{T}_\eta \cdot S \subseteq \overline{S}_\eta \cdot \overline{T}_\eta \cdot \overline{S}_\eta \subseteq \overline{S} \overline{T} \overline{S}_\eta \subseteq \overline{T}_\eta$ , thus  $\overline{T}_\eta$  is an interior ideal of  $S$ , that is,  $T$  is an upper SR-interior ideal over  $S$ .  $\square$

In the following, we investigate the upper and lower SR-regular semigroups over ordinary semigroups and give the conditions of the upper and lower SR-regular semigroup.

**Theorem 4.14.** Assume that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$  and  $N$  is a regular subsemigroup of  $S$ . Then  $N$  is an upper  $SR$ -regular semigroup over  $S$ .

*Proof.* According to Theorem 4.5 and to the fact that  $N$  is a regular subsemigroup of  $S$ , we have that  $\overline{N}_\eta$  is a subsemigroup of  $S$ . For  $a \in \overline{N}_\eta$ , there exists  $k \in N$  such that  $\eta(a) = \eta(k)$ . For this  $k \in N$ , there exists  $n \in N$  such that  $k = knk$  since  $N$  is regular, and so,  $\eta(a) = \eta(knk)$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , there exists  $x, y \in S$  such that  $\eta(k) = \eta(x)$  and  $\eta(n) = \eta(y)$  and  $a = xyx$ , which implies,  $x, y \in \overline{N}_\eta$ . This means that  $N$  is an upper  $SR$ -regular subsemigroup over  $S$ .  $\square$

**Theorem 4.15.** Assume that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ ,  $N$  is a regular subsemigroup of  $S$  and  $N^c$  is an interior ideal of  $S$ . Then  $N$  is a lower  $SR$ -regular semigroup over  $S$ , where  $\underline{N}_\eta \neq \emptyset$ .

*Proof.* According to Theorem 4.6 and to the fact that  $N$  is a regular subsemigroup of  $S$ , we have that  $\underline{N}_\eta$  is subsemigroup of  $S$ . Since  $\underline{N}_\eta \neq \emptyset$ , for any  $a \in \underline{N}_\eta$  and for all  $m \in N^c$ , we have  $\eta(a) \neq \eta(m)$ . And since  $a \in \underline{N}_\eta \subseteq N$ , there exists  $n \in N$  such that  $a = ana$ . We suppose that  $n \notin \underline{N}_\eta$ , then there exists  $k \in N^c$  such that  $\eta(n) = \eta(k)$ . Moreover,  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set, then  $\eta(a) = \eta(ana) = \eta(aka)$ . By hypothesis,  $N^c$  is an interior ideal of  $S$  and  $k \in N^c$ , then  $aka \in N^c$ , and so  $a \notin \underline{N}_\eta$ , which contradicts with  $a \in \underline{N}_\eta$ . This means that  $n \notin \underline{N}_\eta$  is incorrect, that is,  $n \in \underline{N}_\eta$ . Hence,  $\underline{N}_\eta$  is a regular subsemigroup of  $S$ , that is,  $N$  is a lower  $SR$ -regular subsemigroup over  $S$ .  $\square$

Finally, we investigate the properties over a regular semigroup.

**Theorem 4.16.** Assume that  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over a regular semigroup  $S$  and  $X$  is an interior ideal of  $S$ . Then  $X$  is an upper  $SR$ -regular semigroup over  $S$ .

*Proof.* Since  $X$  is an interior ideal of  $S$ , then  $X$  is a subsemigroup of  $S$ . By Theorem 4.5,  $X$  is an upper  $SR$ -subsemigroup over  $S$ . For any  $x \in \overline{X}_\eta$ , since  $S$  is a regular semigroup, there exists  $m$  such that  $x = xmx$ , also we have  $m = mxm$ . Next, we prove  $m \in \overline{X}_\eta$ . Since  $x \in \overline{X}_\eta$ , there exists  $y \in X$  such that  $\eta(x) = \eta(y)$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $C$ -soft set over  $S$ , then  $\eta(m) = \eta(xmx) = \eta(mym)$ . Since  $X$  is an interior ideal of  $S$ ,  $mym \in X$ , and so,  $m \in \overline{X}_\eta$ . Hence  $X$  is an upper  $SR$ -regular semigroup over  $S$ .  $\square$

**Theorem 4.17.** Assume that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over a regular semigroup of  $S$  and  $X$  is an interior ideal of  $S$ . Then  $X$  is a lower  $SR$ -regular semigroup over  $S$ , when  $\underline{X}_\eta \neq \emptyset$ .

*Proof.* Firstly, from Theorem 4.13 (1) and by the fact that  $X$  is an interior ideal of  $S$ , we have that  $X$  is a lower  $SR$ -interior ideal over  $S$ . Since  $S$  is a regular semigroup, for any  $x \in \underline{X}_\eta$ , there exists  $m \in S$  such that  $x = xmx$  and  $m = mxm$ . And as  $X$  is an interior ideal of  $S$ ,  $m = mxm \in \underline{X}_\eta$ . Hence,  $X$  is a lower  $SR$ -regular semigroup over  $S$ .  $\square$

In the following, we suppose that  $S$  is a semigroup with an identity 1.

**Lemma 4.18** ([27]). A semigroup  $S$  is regular if and only if for every right ideal  $A$  and every left ideal  $B$ ,  $AB = A \cap B$ .

**Theorem 4.19.** Assume that  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over a regular semigroup  $S$ . Then:

- (1)  $\overline{X} \cap \overline{Y}_\eta \subseteq \overline{X \cdot Y}_\eta$ , for every upper  $SR$ -right ideal  $X$  and every upper  $SR$ -left ideal  $Y$ .
- (2)  $\underline{X} \cap \underline{Y}_\eta \subseteq \underline{X \cdot Y}_\eta$ , for every lower  $SR$ -right ideal  $X$  and every lower  $SR$ -left ideal  $Y$ .

*Proof.* (1) By hypothesis,  $X$  and  $Y$  are an upper  $SR$ -right ideal and an upper  $SR$ -left ideal over  $S$ , respectively, then  $\overline{X}_\eta$  and  $\overline{Y}_\eta$  are a right ideal and a left ideal of  $S$ , respectively. By Lemma 4.18,  $\overline{X}_\eta \cdot \overline{Y}_\eta = \overline{X}_\eta \cap \overline{Y}_\eta$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , then by Theorem 3.8,  $\overline{X \cdot Y}_\eta = \overline{X}_\eta \cdot \overline{Y}_\eta = \overline{X}_\eta \cap \overline{Y}_\eta$ . By Theorem 3(5) in [23], we have  $\overline{X \cap Y}_\eta \subseteq \overline{X}_\eta \cap \overline{Y}_\eta$ . Therefore  $\overline{X \cap Y}_\eta \subseteq \overline{X}_\eta \cap \overline{Y}_\eta = \overline{X \cdot Y}_\eta$ .

(2) By hypothesis,  $X$  and  $Y$  are a lower SR-right ideal and a lower SR-left ideal over  $S$ , respectively, then  $\underline{X}_\eta$  and  $\underline{Y}_\eta$  are a right ideal and a left ideal of  $S$ , respectively. By Lemma 4.18,  $\underline{X}_\eta \cdot \underline{Y}_\eta = \underline{X}_\eta \cap \underline{Y}_\eta$ . By Theorem 3(4) in [23], we have  $\underline{X} \cap \underline{Y}_\eta = \underline{X}_\eta \cap \underline{Y}_\eta$ . Hence,  $\underline{X} \cap \underline{Y}_\eta = \underline{X}_\eta \cdot \underline{Y}_\eta$ . By hypothesis,  $\mathfrak{S} = (F, A)$  is a  $CC$ -soft set over  $S$ , then by Theorem 3.9,  $\underline{X}_\eta \cdot \underline{Y}_\eta \subseteq \underline{X} \cdot \underline{Y}_\eta$ . Therefore  $\underline{X} \cap \underline{Y}_\eta \subseteq \underline{X} \cdot \underline{Y}_\eta$ .  $\square$

**Example 4.20.** Assume that the semigroup  $S$  and the soft set  $\mathfrak{S} = (F, A)$  are in Example 3.7, we can find  $S$  is a regular semigroup. Define subsets  $X = \{a\}$  and  $Y = \{a, b\}$  over  $S$ . Then  $X \cap Y = \{a\}$ ,  $X \cdot Y = \{a, d\}$ ,  $\overline{X} \cap \overline{Y}_\eta = \{a, d\}$ ,  $\overline{X} \cdot \overline{Y}_\eta = \{a, b, c, d\}$ ,  $\underline{X} \cap \underline{Y}_\eta = \emptyset$  and  $\underline{X} \cdot \underline{Y}_\eta = \{a, d\}$ . Obviously,  $\overline{X} \cap \overline{Y}_\eta \subsetneq \overline{X} \cdot \overline{Y}_\eta$  and  $\underline{X} \cap \underline{Y}_\eta \subsetneq \underline{X} \cdot \underline{Y}_\eta$ .

Similarly, we can obtain the same conclusions which are similar to Theorem 4.19, when  $\overline{X}_\varphi$  and  $\overline{Y}_\varphi$  are upper (lower) SR-left (bi-ideals) ideals and upper (lower) SR-quasi-ideals (left ideals).

## 5 SR-semigroups (SR-regular semigroups) in DM-methods

In this section, we introduce the definitions of SR-semigroups (SR-regular semigroups) based on another soft set. We put forth two DM-methods for Shabir's SR-sets to semigroups and regular semigroups, respectively.

**Definition 5.1.** Assume that  $\mathfrak{S} = (F, A)$  is an original soft set over  $S$  and  $(S, \eta)$  is an MS-approximation space. Let  $\mathfrak{T} = (G, B)$  be another soft set defined over  $S$  with  $B = \{e'_1, e'_2, \dots, e'_n\}$ . The lower MSR-approximation and upper MSR-approximation of  $\mathfrak{T}$  with respect to  $\mathfrak{T}$  are denoted by  $\underline{(G, B)}_\eta = (\underline{G}_\eta, B)$  and  $\overline{(G, B)}_\eta = (\overline{G}_\eta, B)$ , respectively, which two operations are defined as

$$\underline{G(e)}_\eta = \{x \in G(e) \mid \eta(x) \neq \eta(y), \text{ for all } y \in S - G(e)\}$$

and

$$\overline{G(e)}_\eta = \{x \in S \mid \eta(x) = \eta(y), \text{ for some } y \in G(e)\}.$$

for all  $e \in B$ .

- (1) If  $\underline{(G, B)}_\eta = \overline{(G, B)}_\eta$ , then  $\mathfrak{T}$  is called definable;
- (2) If  $\underline{(G, B)}_\eta \neq \overline{(G, B)}_\eta$ , then  $\mathfrak{T}$  is called a lower (upper) SR-semigroup (resp., regular semigroup, ideal) over  $S$ , if  $\underline{G(e)}_\eta$  ( $\overline{G(e)}_\eta$ ) is a semigroup (resp., regular semigroup, ideal) of  $S$ , for all  $e \in \text{Supp}(G, B)$ ; Moreover,  $\mathfrak{T}$  is called a SR-semigroup (resp., regular semigroup, ideal) over  $S$ , if  $\underline{G(e)}_\eta$  and  $\overline{G(e)}_\eta$  are semigroups (resp., regular semigroups, ideals) of  $S$ , for all  $e \in \text{Supp}(G, B)$ .

Let  $S$  be a semigroup and  $E$  a set of related parameters. Let  $A = \{e_1, e_2, \dots, e_m\} \subseteq E$ ,  $\mathfrak{S} = (F, A)$  be a soft set over  $S$  which is the original properties of  $S$  and  $(S, \eta)$  be an MS-approximation space. Let  $\mathfrak{T} = (G, B)$  be another soft set defined over  $S$  with  $B = \{e'_1, e'_2, \dots, e'_n\}$ . Then we present the decision algorithm for SR-semigroups as follows:

### DM-method I:

**Input.** Soft rough set systems  $(S, \mathfrak{S}, \mathfrak{T})$ ;

**Output.** The optimal decision goal;

**Step 1** Compute the lower and upper RS-approximation operators  $\underline{(G, B)}_\eta$  and  $\overline{(G, B)}_\eta$  with respect to  $\mathfrak{T}$ , respectively.

**Step 2** Compute the values of  $\|G(e_i)\|$ , where  $\|G(e_i)\| = \frac{|\overline{G(e_i)}_\eta| - |\underline{G(e_i)}_\eta|}{|\underline{G(e_i)}_\eta|}$ , when  $|G(e_i)|$  denote the cardinality of  $G(e_i)$ .

**Step 3** Find the minimum value  $\|G(e_k)\|$  of  $\|G(e_i)\|$ , where  $\|G(e_k)\| = \min_i \|G(e_i)\|$ .

**Step 4** The decision goal is  $G(e_k)$ .

**Example 5.2.** Assume that the semigroup  $S = \{a, b, c, d, e\}$  is in Example 4.2 as in Table 5 and we want to find  $G(e_k)$  to be the nearest semigroup.

Define a soft set  $\mathfrak{S} = (F, A)$  over  $S$  as in Table 8:

**Table 8.** Table for semigroup  $S$

$\cdot$	$a$	$b$	$c$	$d$	$e$
$e_1$	1	0	1	1	0
$e_2$	1	1	1	1	1
$e_3$	1	0	0	1	0
$e_4$	0	1	1	0	1

Then we can easily get  $\eta(a) = \eta(d) = \{e_1, e_2, e_3\}$ ,  $\eta(b) = \eta(e) = \{e_2, e_4\}$  and  $\eta(c) = \{e_1, e_2, e_4\}$ . Now, define another soft set  $\mathfrak{T} = (G, B)$  over  $S$  as in Table 9:

**Table 9.** Table for semigroup  $S$

$\cdot$	$a$	$b$	$c$	$d$	$e$
$e_1$	1	1	0	0	0
$e_2$	1	0	1	0	0
$e_3$	0	0	1	1	1
$e_4$	0	1	1	0	0
$e_5$	1	1	1	1	0

Then we obtain  $G(e_1) = \{a, b\}$ ,  $G(e_2) = \{a, c\}$ ,  $G(e_3) = \{c, d, e\}$ ,  $G(e_4) = \{b, c\}$  and  $G(e_5) = \{a, b, c, d\}$ . By calculation,  $\underline{G(e_1)}_\eta = \emptyset$ ,  $\overline{G(e_1)}_\eta = \{a, b, d, e\}$ ,  $\underline{G(e_2)}_\eta = \{c\}$ ,  $\overline{G(e_2)}_\eta = \{a, c, d\}$ ,  $\underline{G(e_3)}_\eta = \{c\}$ ,  $\overline{G(e_3)}_\eta = \{a, b, c, d, e\}$ ,  $\underline{G(e_4)}_\eta = \{c\}$ ,  $\overline{G(e_4)}_\eta = \{b, c, e\}$ ,  $\underline{G(e_5)}_\eta = \{a, c, d\}$  and  $\overline{G(e_5)}_\eta = \{a, b, c, d, e\}$ .

Then,  $\|G(e_1)\| = 2$ ,  $\|G(e_2)\| = 1$ ,  $\|G(e_3)\| = 1.33$ ,  $\|G(e_4)\| = 1$ ,  $\|G(e_5)\| = 0.5$ . This means  $\|G(e_5)\|$  has the minimum value of  $\|G(e_i)\|$ . That is,  $G(e_5) = \{a, b, c, d\}$  is the closest to  $S$ .

**Remark 5.3.** In DM-method I, we can find the best parameter  $e$  of  $\mathfrak{T} = (G, B)$  such that  $G(e)$  is the nearest accurate semigroup.

First of all, let  $X$  be a subalgebraic system of  $S$ , that is, the operations in regular semigroup  $S$  also suit to  $G(e_i)$ . In DM-method I, we choose a best parameter  $e$  such that  $G(e)$  is the nearest semigroup based on Shabir's SR-sets.

Now, we consider a special case when  $S$  is a regular semigroup. In this condition, we want to know if the  $G(e_i)$  keeps the nearest regularity of  $S$ . In DM-method II, we put up a new way to choose the best  $e$  such that  $G(e_i)$  is the nearest regular semigroup  $S$ . For a semigroup  $S$ , if for any  $s \in S$  is a regular element, then we call  $S$  is a regular semigroup. Therefore, the more the number of regular elements of  $G(e_i)$  has, the nearer it is to be the regular semigroup  $S$ .

**Definition 5.4.** Let  $S$  be a regular semigroup. Assume that  $\mathfrak{S} = (F, A)$  is an original soft set over  $S$  with  $A = \{e_1, e_2, \dots, e_m\}$  and  $(S, \eta)$  be an MS-approximation space. Let  $\mathfrak{T} = (G, B)$  be another soft set defined over  $S$  with  $B = \{e'_1, e'_2, \dots, e'_n\}$ .

(1) For any  $a \in \underline{G(e)}_\eta$  for some  $e \in B$ , there exists  $b \in \underline{G(e)}_\eta$  such that  $a = aba$ . We call  $a$  the regular element of  $\underline{G(e)}_\eta$  and  $N(\underline{G(e)}_\eta)$  represents all regular elements of  $\underline{G(e)}_\eta$ . We denote  $|N(\underline{G(e)}_\eta)|$  to represent the number of all regular elements of  $\underline{G(e)}_\eta$ .

(2) For any  $a \in \overline{G(e)}_\eta$  for some  $e \in B$ , there exists  $b \in \overline{G(e)}_\eta$  such that  $a = aba$ . We call  $a$  the regular element of  $\overline{G(e)}_\eta$  and  $N(\overline{G(e)}_\eta)$  represents all regular elements of  $\overline{G(e)}_\eta$ . We denote  $|N(\overline{G(e)}_\eta)|$  to represent the number of all regular elements of  $\overline{G(e)}_\eta$ .

**Example 5.5.** Assume that the semigroup  $S = \{a, b, c, d, e\}$  is in Example 4.2 as in Table 5, the soft set  $\mathfrak{S} = (F, A)$  as in Table 8 and  $\mathfrak{T} = (G, B)$  as in Table 9. We can easily prove that  $S$  is a regular semigroup.

Then we obtain

(1)  $\overline{G(e_1)}_\eta = \{a, b, d, e\}$ ,  $\overline{G(e_2)}_\eta = \{a, c, d\}$ ,  $\overline{G(e_3)}_\eta = \{a, b, c, d, e\}$ ,  $\overline{G(e_4)}_\eta = \{b, c, e\}$  and  $\overline{G(e_5)}_\eta = \{a, b, c, d, e\}$ . Then  $N(\overline{G(e_1)}_\eta) = \{a, b, d, e\}$ ,  $N(\overline{G(e_2)}_\eta) = \{a, c, d\}$ ,  $N(\overline{G(e_3)}_\eta) = \{a, b, c, d, e\}$ ,  $N(\overline{G(e_4)}_\eta) = \{b, c, e\}$  and  $N(\overline{G(e_5)}_\eta) = \{a, b, c, d, e\}$ . Thus,  $|N(\overline{G(e_1)}_\eta)| = 4$ ,  $|N(\overline{G(e_2)}_\eta)| = 3$ ,  $|N(\overline{G(e_3)}_\eta)| = 5$ ,  $|N(\overline{G(e_4)}_\eta)| = 3$  and  $|N(\overline{G(e_5)}_\eta)| = 5$ .

(2)  $\underline{G(e_1)}_\eta = \emptyset$ ,  $\underline{G(e_2)}_\eta = \{c\}$ ,  $\underline{G(e_3)}_\eta = \{c\}$ ,  $\underline{G(e_4)}_\eta = \{c\}$  and  $\underline{G(e_5)}_\eta = \{a, c, d\}$ . Then  $N(\underline{G(e_1)}_\eta) = \emptyset$ ,  $N(\underline{G(e_2)}_\eta) = \{c\}$ ,  $N(\underline{G(e_3)}_\eta) = \{c\}$ ,  $N(\underline{G(e_4)}_\eta) = \{c\}$  and  $N(\underline{G(e_5)}_\eta) = \{a, c, d\}$ . Thus,  $|N(\underline{G(e_1)}_\eta)| = 0$ ,  $|N(\underline{G(e_2)}_\eta)| = 1$ ,  $|N(\underline{G(e_3)}_\eta)| = 1$ ,  $|N(\underline{G(e_4)}_\eta)| = 1$  and  $|N(\underline{G(e_5)}_\eta)| = 3$ .

Next, we present the decision algorithm for SR-regular semigroups as follows:

#### DM-method II:

**Input.** Soft rough set systems  $(S, \mathfrak{S}, \mathfrak{T})$ ;

**Output.** The optimal decision goal;

**Step 1** Compute the lower and upper RS-approximation operators  $\underline{(G, B)}_\eta$  and  $\overline{(G, B)}_\eta$  with respect to  $\mathfrak{T}$ , respectively.

**Step 2** Compute the values of  $\|G(e_i)\|_N$ , where  $\|G(e_i)\|_N = \frac{|N(\overline{G(e_i)}_\eta)| + |N(\underline{G(e_i)}_\eta)|}{|G(e_i)|}$ , when  $|G(e_i)|$  denote the cardinality of  $G(e_i)$ .

**Step 3** Find the maximum value  $\|G(e_k)\|_N$  of  $\|G(e_i)\|_N$ .

**Step 4**  $\|G(e_k)\|_N$  is the one which keeps the maximum regularity of  $S$ .

**Example 5.6.** Consider the semigroup  $S = \{a, b, c, d, e\}$  in Example 4.2 as in Table 5, soft set  $\mathfrak{S} = (F, A)$  as in Table 8,  $\mathfrak{T} = (G, B)$  as in Table 9. We can easily prove that  $S$  is a regular semigroup. Then,  $\|G(e_1)\|_N = 2$ ,  $\|G(e_2)\|_N = 0.5$ ,  $\|G(e_3)\|_N = 1.68$ ,  $\|G(e_4)\|_N = 0.5$  and  $\|G(e_5)\|_N = 0.75$ . This means  $\|G(e_1)\|_N$  has the maximum value. That is,  $G(e_1)$  keeps the maximum regularity of  $S$ .

**Remark 5.7.** (1) Obviously, in DM-method I, we only treat semigroup  $S$  as a set. And although we find  $G(e_i)$  to be the nearest accurate semigroup  $S$ , the properties of  $G(e_i)$  may not meet the demand of a semigroup when  $G(e_i)$  is a subalgebraic system of  $S$ .

(2) When  $S$  is a regular semigroup, we should consider the regularity of  $G(e_i)$ . That is, we get the upper soft approximations and the lower soft approximations of  $G(e_i)$  based on another soft set  $\mathfrak{T} = (G, B)$ , when the greater is the sum of the numbers of regular elements of the upper soft approximation and the lower soft approximation divided by the cardinality of  $G(e_i)$ , we say it keeps the better regularity of  $G(e_i)$ .

(3) From DM-method I, we obtain that  $G(e_5)$  in Example 5.2 is the nearest to be the semigroup  $S$ . In DM-method II, although the number of elements of  $G(e_1)$  in Example 5.6 is very low, the number of regular elements is higher than others under soft set  $\mathfrak{T} = (G, B)$ . This means that it has the most value in this case. By analogy, if we employ five people to finish one thing, but another two people can also finish the thing at the same time, there is no doubt that all of us will choose the latter.

## 6 Conclusion

Since 1999, when Molodsov in [11] put forward soft sets, the study about soft sets has started. The applications of soft sets are rather important. Then some researchers built soft sets over algebraic structure, such as in [16] in 2013. In 2011, Feng et al. [20] made soft sets as an ER to build rough set which was called an SR-set. However, this kind

of soft rough set must be full. Therefore, Shabir *et al.* [23] put forward another SR-set which avoids the limits of Feng's SR-set, that is, Shabir's SR-set does not demand that the soft set is full. After Shabir's SR-set, some better different conclusions were obtained than the Feng's SR-set. It is pointed out that Zhan in [25] made use of Shabir's SR-sets to study SR-hemirings and discussed the properties of SR-hemirings. At the same time, Zhan *et al.* also put up a corresponding multicriteria group decision making.

In this paper, according to Zhan's  $C$ -soft sets and  $CC$ -soft sets, we establish  $C$ -soft sets and  $CC$ -soft sets over semigroups. At the same time, we also establish the relations between the upper soft approximations and the lower soft approximations about  $\cdot$  and  $\cap$ . Some conclusions about SR-semigroups (prime ideals, bi-ideals, quasi-ideals, interior ideals, regular semigroups) are obtained. Finally, we give two decision making methods: one is to look for a best a parameter which is the nearest semigroups, the other is to choose a parameter which keeps the nearest regularity of regular semigroups.

Our extension of these topics are considered as follows:

- (1) To establish a new hybrid soft set models by means of Zhan's idea and make use of some new classes of ideals to describe regular semigroups.
- (2) To propose a new DM-method which not only makes  $G(e_i)$  to be the nearest regular semigroup, but also keeps the regularity as far as possible.
- (3) To investigate soft rough fuzzy semigroups.
- (4) To discuss soft fuzzy rough semigroups.

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