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Research Article

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Evaluation of the convolution sums $\sum_{al+bm=n} l\sigma(l)\sigma(m)$ with $ab \leq 9$

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Abstract: The generating functions of divisor functions are quasimodular forms of weight 2 and their products belong to a space of quasimodular forms of higher weight. In this article, we evaluate the convolution sums

$$\sum_{al+bm=n} l\sigma(l)\sigma(m)$$

for all positive integers a, b and n with $ab \leq 9$ and $\gcd(a, b) = 1$.

Keywords: Divisor function, Convolution sum, Quasimodular form

MSC: 11A25, 11F11, 11F20

1 Introduction

Let a and b be positive integers and let ν be a nonnegative integer. Define $W_{a,b}^{(\nu)}(n)$ by the convolution sum

$$W_{a,b}^{(\nu)}(n) := \sum_{\substack{l,m \geq 1 \\ al+bm=n}} l^{\nu}\sigma(l)\sigma(m),$$

where $\sigma(n) := \sigma_1(n)$ and

$$\sigma_s(n) := \sum_{d|n} d^s$$

for any positive integers s and n. Denote $W_{a,b}(n) = W_{a,b}^{(0)}(n)$.

This is a specialized form of the convolution sum which Lahiri introduced in [1]:

$$S[(r_1, \ldots, r_t), (s_1, \ldots, s_t), (a_1, \ldots, a_t)](n) = \sum_{\substack{m_1, \ldots, m_t \in \mathbb{Z}_{>0} \\ a_1 m_1 + \cdots + a_t m_t = n}} m_1^{r_1} \cdots m_t^{r_t} \sigma_{s_1}(m_1) \cdots \sigma_{s_t}(m_t)$$

and clearly,

$$W_{a,b}^{(\nu)}(n) = S[(\nu, 0), (1, 1), (a, b)](n).$$

The convolution sum $W_{a,b}(n)$ has been studied since the mid-nineteenth century. The following table contains references for it:

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Mathematicians (year)	cases
Besge(1862) [2], Glaisher(1885) [3], Ramanujan (1916) [4]	$ab = 1$
Huard-Ou-Spearman-Williams (2002) [5]	$ab = 2, 3, 4$
Lemire-Williams (2005) [6], Cooper-Toh (2009) [7]	$ab = 5, 7$
Alaca-Williams (2007) [8]	$ab = 6$
Williams (2008-9) [9, 10]	$ab = 8, 9$
Royer (2007) [11]	$ab = 11, 13$ and $(a, b) = (1, 10), (1, 14)$
Alaca-Alaca-Williams (2006-8) [12–15]	$ab = 12, 16, 18, 24$
Ramakrishnan-Sahu (2013) [16]	$ab = 15$
Cooper-Ye (2014) [17]	$ab = 20$ and $(a, b) = (2, 5)$
Chan-Cooper (2008) [18]	$ab = 23$
Xia-Tian-Yao (2014) [19]	$ab = 25$
Ntienjem (2015, 2017) [20–22]	$ab = 14, 22, 26, 28, 30, 33, 40, 44, 52, 56$
Alaca-Kesicioglu (2016) [23]	$ab = 27, 32$
Ye (2015) [24]	$ab = 36$

Although it may be possible to evaluate $W_{a,b}(n)$ by means of identities of elementary functions for small a and b, when ab grows, it is easier to use quasimodular forms because $W_{a,b}(n)$ is a coefficient of a certain quasimodular form of level ab and weight 4 and the dimension of the space of quasimodular forms is known.

In this paper we focus on the convolution sum

$$W^{(1)}_{a,b}(n) = \sum_{al+bm=n} l\sigma(l)\sigma(m)$$

for $ab \leq 9$ with $(a, b) = 1$ since $W^{(1)}_{da,db}(n) = W^{(1)}_{a,b}(n/d)$ for $d \mid n$. The convolution sum $W^{(1)}_{1,1}(n)$ is

$$\sum_{l+m=n} l\sigma(l)\sigma(m) = \frac{n}{24}\{5\sigma_3(n) + (1 - 6n)\sigma(n)\}$$

since one can obtain $2W^{(1)}_{1,1}(n) = nW^{(0)}_{1,1}(n)$ easily.

Before stating our main theorem we need the functions $\delta_N(q)$ and $\Delta_N(q)$ defined on the region $|q| < 1$:

Definition 1.1.

$$\Delta_3(q) := qE^6(q)E^6(q^3) = \sum_{n=1}^{\infty} c_3(n)q^n,$$

$$\Delta_4(q) := qE^{12}(q^2) = \sum_{n=1}^{\infty} c_4(n)q^n,$$

$$\delta_5(q) := qE^4(q)E^4(q^5) = \sum_{n=1}^{\infty} b_5(n)q^n,$$

$$\Delta_5(q) := \frac{1}{4}\delta_5(q)\left(5P(q^5) - P(q)\right) = \sum_{n=1}^{\infty} c_5(n)q^n,$$

$$\delta_6(q) := qE^2(q)E^2(q^2)E^2(q^3)E^2(q^6) = \sum_{n=1}^{\infty} b_6(n)q^n,$$

$$\Delta_6(q) := \frac{1}{4}\delta_6(q)\left(6P(q^6) - 3P(q^3) + 2P(q^2) - P(q)\right) = \sum_{n=1}^{\infty} c_6(n)q^n,$$

$$\delta_7(q) := q\left(E^{16}(q)E^8(q^7) + 13qE^{12}(q)E^{12}(q^7) + 49q^2E^8(q)E^{16}(q^7)\right)^{1/3} = \sum_{n=1}^{\infty} b_7(n)q^n,$$

$$\Delta_{7,1}(q) := qE^{10}(q)E^2(q^7) = \sum_{n=1}^{\infty} c_{7,1}(n)q^n,$$

$$\Delta_{7,2}(q) := q^2 E^6(q) E^6(q^7) = \sum_{n=1}^{\infty} c_{7,2}(n) q^n,$$
$$\Delta_{7,3}(q) := q^3 E^2(q) E^{10}(q^7) = \sum_{n=1}^{\infty} c_{7,3}(n) q^n,$$
$$\delta_8(q) := qE^4(q^2) E^4(q^4) = \sum_{n=1}^{\infty} b_8(n) q^n,$$
$$\Delta_8(q) := \delta_8(q)(2P(q^4) - P(q^2)) = \sum_{n=1}^{\infty} c_8(n) q^n,$$
$$\delta_9(q) := qE(q^3)^8 = \sum_{n=1}^{\infty} b_9(n) q^n,$$
$$\Delta_9(q) := \frac{1}{8}\delta_9(q)\left(9P(q^9) - P(q)\right) = \sum_{n=1}^{\infty} c_9(n) q^n,$$

where

$$E(q) := \prod_{n=1}^{\infty}(1-q^n), \quad P(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n,$$
$$Q(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \quad R(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

The following result is our main theorem.

Theorem 1.2. *Let $W_{a,b}^{(1)}(n)$ be the modified convolution sum of divisor functions*

$$W_{a,b}^{(1)}(n) := \sum_{al+bm=n} l\sigma(l)\sigma(m).$$

Assume that a and b are positive integers with $ab \le 9$ and $\gcd(a,b) = 1$. Then $W_{a,b}^{(1)}(n)$ can be explicitly expressed as a linear combination of $\sigma_s(n/d)$, $b_{d'}(n/d)$ and $c_{d'}(n/d)$ for positive integers d, d' with $dd' \mid n$ for all positive integers n and $s = 1, 3$ or 5.

The explicit linear combinations for $W_{a,b}^{(1)}(n)$ are given in Theorems 3.1($a < b$) and 3.2($a > b$) of Section 3.

This paper is organized as follows. We recall in Section 2 the theory of quasimodular forms and find a basis for the space of quasimodular forms which we need in Section 3. In Section 3, we find the identities between the quasimodular forms and the basis (Lemma 3.3) to prove our main theorems (Theorem 3.1 and 3.2) and write $W_{a,b}^{(1)}$ as the linear combination of divisor functions and the coefficients of certain cuspforms. We use the MAPLE program to find the identities between quasimodular forms.

2 Quasimodular forms

For a positive integer N the congruence subgroup $\Gamma_0(N)$ is defined as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We use the theory of quasimodular forms to prove our main theorem. One can refer to [25] for more details. The space $\widetilde{M}_k^{(\le k/2)}(\Gamma_0(N))$ of quasimodular forms of weight k and depth $\le k/2$ on $\Gamma_0(N)$ has the following structure relationship:

$$\widetilde{M}_k^{(\leq k/2)}(\Gamma_0(N)) = \oplus_{j=0}^{k/2-1} D^j M_{k-2j}(\Gamma_0(N)) \oplus < D^{k/2-1}P(q) >,$$

where D is the differential operator defined by $D := q\frac{d}{dq}$ and $D\left(\sum_{n\geq 0} c_n q^n\right) = \sum_{n\geq 0} n c_n q^n$. Thus, the space $\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$ of quasimodular forms of level N of weight 6 and depth ≤ 3 is

$$\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N)) = M_6(\Gamma_0(N)) \oplus DM_4(\Gamma_0(N)) \oplus D^2 M_2(\Gamma_0(N)) \oplus < D^2 P(q) >$$

for any positive integer N. More explicitly, $\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$ is spanned by the functions $D^2P(q^d)$, $DQ(q^d)$, $R(q^d)$, $D\delta_{d'}(q^d)$ and $\Delta_{d'}(q^d)$ for positive divisors d and d' of N with $dd' \mid N$. Hence we get the following basis.

Lemma 2.1. *For $N = 2, \ldots, 9$, let $\mathcal{B}_N$ be the set of quasimodular forms defined by*

$$\mathcal{B}_2 = \left\{D^2P(q), D^2P(q^2), DQ(q), DQ(q^2), R(q), R(q^2)\right\},$$
$$\mathcal{B}_3 = \left\{D^2P(q), D^2P(q^3), DQ(q), DQ(q^3), R(q), R(q^3), \Delta_3(q)\right\},$$
$$\mathcal{B}_4 = \left\{D^2P(q), D^2P(q^2), D^2P(q^4), DQ(q), DQ(q^2), DQ(q^4), R(q), R(q^2), R(q^4), \Delta_4(q)\right\},$$
$$\mathcal{B}_5 = \left\{D^2P(q), D^2P(q^5), DQ(q), DQ(q^5), R(q), R(q^5), D\delta_5(q), \Delta_5(q)\right\},$$
$$\mathcal{B}_6 = \left\{D^2P(q), D^2P(q^2), D^2P(q^3), D^2P(q^6), DQ(q), DQ(q^2), DQ(q^3), DQ(q^6),\right.$$
$$\left. R(q), R(q^2), R(q^3), R(q^6), D\delta_6(q), \Delta_3(q), \Delta_3(q^2), \Delta_6(q)\right\},$$
$$\mathcal{B}_7 = \left\{D^2P(q), D^2P(q^7), DQ(q), DQ(q^7), R(q), R(q^7), D\delta_7(q), \Delta_{7,1}(q), \Delta_{7,2}(q), \Delta_{7,3}(q)\right\},$$
$$\mathcal{B}_8 = \left\{D^2P(q), D^2P(q^2), D^2P(q^4), D^2P(q^8), DQ(q), DQ(q^2), DQ(q^4), DQ(q^8),\right.$$
$$\left. R(q), R(q^2), R(q^4), R(q^8), D\delta_8(q), \Delta_4(q), \Delta_4(q^2), \Delta_8(q)\right\},$$
$$\mathcal{B}_9 = \left\{D^2P(q), D^2P(q^3), D^2P(q^9), DQ(q), DQ(q^3), DQ(q^9), R(q), R(q^3), R(q^9),\right.$$
$$\left. D\delta_9(q), \Delta_3(q), \Delta_3(q^3), \Delta_9(q)\right\}.$$

Then, $\mathcal{B}_N$ is a basis for the space $\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$.

Proof. Note that the dimension of the space $\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$ is

$$\dim \widetilde{M}_6^{(\leq 3)}(\Gamma_0(N)) = 1 + \sum_{j=0}^{2} \dim M_{6-2j}(\Gamma_0(N)).$$

By [26, Proposition 6.1], we have

N	2	3	4	5	6	7	8	9
$\dim \widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$	6	7	10	8	16	10	16	13

It is clear that $P(q^d) \in \widetilde{M}_2^{(\leq 1)}(\Gamma_0(N))$, $Q(q^d) \in M_4(\Gamma_0(N))$, $R(q^d) \in M_6(\Gamma_0(N))$ for a positive divisor d of N. Assume that N is an integer with $2 \leq N \leq 9$. Let $\widetilde{E}(\Gamma_0(N))$ be the subspace of $\widetilde{M}_6^{(\leq 3)}(\Gamma_0(N))$ spanned by the set

$$\mathcal{B}_N(E) := \left\{R(q^d), DQ(q^d), D^2P(q^d) : 0 < d \mid N\right\}$$

of dimension $3\cdot\left(\sum_{d\mid N} 1\right) = 3\sigma_0(N)$. When $0 < dd' \mid N$, the functions $\delta_d(q^{d'})$ and $\Delta_d(q^{d'})$ defined in Definition 1.1 are modular forms of weight 4 and 6 of $\Gamma_0(N)$, respectively. Moreover, the functions $\Delta_{7,j}(q)$ are modular forms of weight 6 of $\Gamma_0(7)$ for $j = 1, 2, 3$. In other words,

$$D\delta_d(q^{d'}) \in \widetilde{M}_6^{(\leq 3)}(\Gamma_0(N)), \Delta_d(q^{d'}) \in M_6(\Gamma_0(N)) \text{ and } \Delta_{7,j}(q) \in M_6(\Gamma_0(7)),$$

where $dd' \mid N$ and $j = 1, 2, 3$. It is easily checked that the set $\mathcal{B}_N$ is linearly independent for each N by the help of the q-expansions of the functions in Appendix.

Since

$$\dim \widetilde{M}_6^{(\le 3)}(\Gamma_0(N)) - \dim \widetilde{E}(\Gamma_0(N)) = 0, 1, 1, 2, 4, 4, 4, 4$$

for $N = 2, 3, 4, 5, 6, 7, 8, 9$, respectively, the set $\mathcal{B}_N$ is a basis of the space $\widetilde{M}_6^{(\le 3)}(\Gamma_0(N))$. □

Remark 2.2. *Let $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. In the theory of modular forms, the functions $\delta_N(q)$ and $\Delta_N(q)$ are newforms with $\delta_N|W_N = \delta_N$ and $\Delta_N|W_N = -\Delta_N$ when $N = 2, 3, 4, 5, 6$ and 8.*

For $N = 7$, $\Delta_{7,1}(q)$, $\Delta_{7,2}(q)$ and $\Delta_{7,3}(q)$ are echelon forms. Instead of them,

$$f_7(q) := \Delta_{7,1}(q) - 49\Delta_{7,3}(q),\ f_{7,\pm}(q) := \Delta_{7,1}(q) + \frac{29 \pm \sqrt{57}}{2}\Delta_{7,2}(q) + 49\Delta_{7,3}(q)$$

are normalized newforms of level 7 with $f_7|W_7 = f_7$ and $(f_{7,\pm})|W_7 = -f_{7,\pm}$. Furthermore, the modular form $-\frac{1}{3}\Delta_3(q) - 9\Delta_3(q^3) + \frac{4}{3}\Delta_9(q)$ is the normalized newform of $\Gamma_0(9)$ with eigenvalue -1 under the action of W_9.

3 Proofs of main results

By using the theory of quasimodular forms we prove Theorem 3.1 and 3.2. These are the explicit linear combinations of Theorem 1.2.

Theorem 3.1. *Let n be a positive integer. Then*

1.
$$\sum_{l+2m=n} l\sigma(l)\sigma(m) = \frac{n}{24}\sigma_3(n) + \frac{n}{6}\sigma_3\left(\frac{n}{2}\right) + \frac{n-2n^2}{24}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{2}\right),$$

2.
$$\sum_{l+3m=n} l\sigma(l)\sigma(m) = \frac{n}{48}\sigma_3(n) + \frac{3n}{16}\sigma_3\left(\frac{n}{3}\right) + \frac{3n-4n^2}{72}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{3}\right) - \frac{1}{144}c_3(n),$$

3.
$$\sum_{l+4m=n} l\sigma(l)\sigma(m) = \frac{n}{96}\sigma_3(n) + \frac{n}{32}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{6}\sigma_3\left(\frac{n}{4}\right) + \frac{n-n^2}{24}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{4}\right) - \frac{1}{96}c_4(n),$$

4.
$$\sum_{l+5m=n} l\sigma(l)\sigma(m)$$
$$= \frac{5n}{624}\sigma_3(n) + \frac{125n}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{5n-4n^2}{120}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{5}\right) - \frac{n}{260}b_5(n) - \frac{1}{80}c_5(n),$$

5.
$$\sum_{l+6m=n} l\sigma(l)\sigma(m) = \frac{n}{240}\sigma_3(n) + \frac{n}{60}\sigma_3\left(\frac{n}{2}\right) + \frac{3n}{80}\sigma_3\left(\frac{n}{3}\right) + \frac{3n}{20}\sigma_3\left(\frac{n}{6}\right)$$
$$+ \frac{3n-2n^2}{72}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{6}\right) - \frac{n}{240}b_6(n) - \frac{1}{144}c_3(n) - \frac{1}{18}c_3\left(\frac{n}{2}\right) - \frac{1}{144}c_6(n),$$

6.
$$\sum_{2l+3m=n} l\sigma(l)\sigma(m) = \frac{n}{480}\sigma_3(n) + \frac{n}{120}\sigma_3\left(\frac{n}{2}\right) + \frac{3n}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{3n}{40}\sigma_3\left(\frac{n}{6}\right)$$
$$+ \frac{3n-4n^2}{144}\sigma\left(\frac{n}{2}\right) - \frac{n^2}{48}\sigma\left(\frac{n}{3}\right) - \frac{n}{480}b_6(n) - \frac{1}{288}c_3(n) - \frac{1}{36}c_3\left(\frac{n}{2}\right) + \frac{1}{288}c_6(n),$$

7.
$$\sum_{l+7m=n} l\sigma(l)\sigma(m) = \frac{n}{240}\sigma_3(n) + \frac{49n}{240}\sigma_3\left(\frac{n}{7}\right) + \frac{7n-4n^2}{168}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{7}\right)$$
$$-\frac{n}{140}b_7(n) - \frac{5}{336}c_{7,1}(n) - \frac{17}{84}c_{7,2}(n) - \frac{35}{48}c_{7,3}(n),$$

8.
$$\sum_{l+8m=n} l\sigma(l)\sigma(m) = \frac{n}{384}\sigma_3(n) + \frac{n}{128}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{32}\sigma_3\left(\frac{n}{4}\right) + \frac{n}{6}\sigma_3\left(\frac{n}{8}\right)$$
$$+\frac{2n-n^2}{48}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{8}\right) - \frac{n}{128}b_8(n) - \frac{1}{128}c_4(n) - \frac{1}{16}c_4\left(\frac{n}{2}\right) - \frac{1}{128}c_8(n),$$

9.
$$\sum_{l+9m=n} l\sigma(l)\sigma(m) = \frac{n}{432}\sigma_3(n) + \frac{n}{54}\sigma_3\left(\frac{n}{3}\right) + \frac{3n}{16}\sigma_3\left(\frac{n}{9}\right)$$
$$+\frac{9n-4n^2}{216}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{9}\right) - \frac{n}{108}b_9(n) + \frac{1}{432}c_3(n) + \frac{1}{16}c_3\left(\frac{n}{3}\right) - \frac{1}{54}c_9(n).$$

For $\nu > 0$, $W_{a,b}^{(\nu)}$ is not symmetric on (a,b) and we obtain results for $W_{a,b}^{(1)}$ $(a > b)$ in the following theorem.

Theorem 3.2. *Let n be a positive integer. Then*

1.
$$\sum_{2l+m=n} l\sigma(l)\sigma(m) = \frac{n}{48}\sigma_3(n) + \frac{n}{12}\sigma_3\left(\frac{n}{2}\right) - \frac{n^2}{48}\sigma(n) + \frac{n-4n^2}{48}\sigma\left(\frac{n}{2}\right),$$

2.
$$\sum_{3l+m=n} l\sigma(l)\sigma(m) = \frac{n}{144}\sigma_3(n) + \frac{n}{16}\sigma_3\left(\frac{n}{3}\right) - \frac{n^2}{108}\sigma(n) + \frac{n-4n^2}{72}\sigma\left(\frac{n}{3}\right) + \frac{1}{432}c_3(n),$$

3.
$$\sum_{4l+m=n} l\sigma(l)\sigma(m) = \frac{n}{384}\sigma_3(n) + \frac{n}{128}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{24}\sigma_3\left(\frac{n}{4}\right) - \frac{n^2}{192}\sigma(n) + \frac{n-4n^2}{96}\sigma\left(\frac{n}{4}\right) + \frac{1}{384}c_4(n),$$

4.
$$\sum_{5l+m=n} l\sigma(l)\sigma(m) = \frac{n}{624}\sigma_3(n) + \frac{25n}{624}\sigma_3\left(\frac{n}{5}\right) - \frac{n^2}{300}\sigma(n) + \frac{n-4n^2}{120}\sigma\left(\frac{n}{5}\right) - \frac{n}{1300}b_5(n) + \frac{1}{400}c_5(n),$$

5.
$$\sum_{6l+m=n} l\sigma(l)\sigma(m) = \frac{n}{1440}\sigma_3(n) + \frac{n}{360}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{n}{40}\sigma_3\left(\frac{n}{6}\right)$$
$$-\frac{n^2}{432}\sigma(n) + \frac{n-4n^2}{144}\sigma\left(\frac{n}{6}\right) - \frac{n}{1440}b_6(n) + \frac{1}{864}c_3(n) + \frac{1}{108}c_3\left(\frac{n}{2}\right) + \frac{1}{864}c_6(n),$$

6.
$$\sum_{3l+2m=n} l\sigma(l)\sigma(m) = \frac{n}{720}\sigma_3(n) + \frac{n}{180}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{80}\sigma_3\left(\frac{n}{3}\right) + \frac{n}{20}\sigma_3\left(\frac{n}{6}\right)$$
$$-\frac{n^2}{108}\sigma\left(\frac{n}{2}\right) + \frac{n-2n^2}{72}\sigma\left(\frac{n}{3}\right) - \frac{n}{720}b_6(n) + \frac{1}{432}c_3(n) + \frac{1}{54}c_3\left(\frac{n}{2}\right) - \frac{1}{432}c_6(n),$$

7.
$$\sum_{7l+m=n} l\sigma(l)\sigma(m) = \frac{n}{1680}\sigma_3(n) + \frac{7n}{240}\sigma_3\left(\frac{n}{7}\right) - \frac{n^2}{588}\sigma(n) + \frac{n-4n^2}{168}\sigma\left(\frac{n}{7}\right)$$
$$-\frac{n}{980}b_7(n) + \frac{5}{2352}c_{7,1}(n) + \frac{17}{588}c_{7,2}(n) + \frac{5}{48}c_{7,3}(n),$$

8.

$$\sum_{8l+m=n} l\sigma(l)\sigma(m) = \frac{n}{3072}\sigma_3(n) + \frac{n}{1024}\sigma_3\left(\frac{n}{2}\right) + \frac{n}{256}\sigma_3\left(\frac{n}{4}\right) + \frac{n}{48}\sigma_3\left(\frac{n}{8}\right)$$

$$-\frac{n^2}{768}\sigma(n) + \frac{n-4n^2}{192}\sigma\left(\frac{n}{8}\right) - \frac{n}{1024}b_8(n) + \frac{1}{1024}c_4(n) + \frac{1}{128}c_4\left(\frac{n}{2}\right) + \frac{1}{1024}c_8(n),$$

9.

$$\sum_{9l+m=n} l\sigma(l)\sigma(m) = \frac{n}{3888}\sigma_3(n) + \frac{n}{486}\sigma_3\left(\frac{n}{3}\right) + \frac{n}{48}\sigma_3\left(\frac{n}{9}\right)$$

$$-\frac{n^2}{972}\sigma(n) + \frac{n-4n^2}{216}\sigma\left(\frac{n}{9}\right) - \frac{n}{972}b_9(n) - \frac{1}{3888}c_3(n) - \frac{1}{144}c_3\left(\frac{n}{3}\right) + \frac{1}{486}c_9(n).$$

Lemma 3.3 gives nine identities involving the functions in Definition 1.1. Using it, Theorems 3.1 and 3.2 are obtained by equating coefficients of q^n.

Lemma 3.3. *Since $DP(q^a)P(q^b)$ is an element of $\widetilde{M}_6^{(\le 3)}(\Gamma_0(ab))$, we have the following identities:*

1.

$$DP(q)P(q^2) = 2D^2P(q) + 8D^2P(q^2) + \frac{1}{10}DQ(q) + \frac{4}{5}DQ(q^2),$$

2.

$$DP(q)P(q^3) = \frac{4}{3}D^2P(q) + 18D^2P(q^3) + \frac{1}{20}DQ(q) + \frac{27}{20}DQ(q^3) - 4\Delta_3(q),$$

3.

$$DP(q)P(q^4) = D^2P(q) + 32D^2P(q^4) + \frac{1}{40}DQ(q) + \frac{3}{20}DQ(q^2) + \frac{8}{5}DQ(q^4) - 6\Delta_4(q),$$

4.

$$DP(q)P(q^5) = \frac{4}{5}D^2P(q) + 50D^2P(q^5) + \frac{1}{52}DQ(q) + \frac{125}{52}DQ(q^5)$$

$$-\frac{144}{65}D\delta_5(q) - \frac{36}{5}\Delta_5(q),$$

5.

$$DP(q)P(q^6) = \frac{2}{3}D^2P(q) + 72D^2P(q^6) + \frac{1}{100}DQ(q) + \frac{2}{25}DQ(q^2)$$

$$+\frac{27}{100}DQ(q^3) + \frac{54}{25}DQ(q^6) - \frac{12}{5}D\delta_6(q) - 4\Delta_3(q) - 32\Delta_3(q^2) - 4\Delta_6(q),$$

6.

$$DP(q^2)P(q^3) = \frac{8}{3}D^2P(q^2) + \frac{9}{2}D^2P(q^3) + \frac{1}{200}DQ(q) + \frac{1}{25}DQ(q^2)$$

$$+\frac{27}{200}DQ(q^3) + \frac{27}{25}DQ(q^6) - \frac{6}{5}D\delta_6(q) - 2\Delta_3(q) - 16\Delta_3(q^2) + 2\Delta_6(q),$$

7.

$$DP(q)P(q^7) = \frac{4}{7}D^2P(q) + 98D^2P(q^7) + \frac{1}{100}DQ(q) + \frac{343}{100}DQ(q^7) - \frac{144}{35}D\delta_7(q)$$

$$-\frac{60}{7}\Delta_{7,1}(q) - \frac{816}{7}\Delta_{7,2}(q) - 420\Delta_{7,3}(q),$$

8.

$$DP(q)P(q^8) = \frac{1}{2}D^2P(q) + 128D^2P(q^8) + \frac{1}{160}DQ(q) + \frac{3}{80}DQ(q^2)$$

$$+\frac{3}{10}DQ(q^4) + \frac{16}{5}DQ(q^8) - \frac{9}{2}\Delta_4(q) - 36\Delta_4(q^2) - \frac{9}{2}\Delta_8(q),$$

9.

$$DP(q)P(q^9) = \frac{4}{9}D^2P(q) + 162D^2P(q^9) + \frac{1}{180}DQ(q) + \frac{2}{15}DQ(q^3) + \frac{81}{20}DQ(q^9)$$

$$-\frac{16}{3}D\delta_9(q) + \frac{4}{3}\Delta_3(q) + 36\Delta_3(q^3) - \frac{32}{3}\Delta_9(q).$$

Proof. By comparing the coefficients of $DP(q^a)P(q^b)$ with ones of the basis of $\widetilde{M}_6^{(\le 3)}(\Gamma_0(ab))$ $((a,b) = (1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(1,7),(1,8),(1,9))$ we gave in Lemma 2.1 we can complete our statement. □

Proof of Theorem 3.1. It is easy to see that

$$DP(q^a)P(q^b) = \sum_{n=1}^{\infty}\left[-\frac{24n}{a}\sigma\left(\frac{n}{a}\right) + 576W_{a,b}^{(1)}(n)\right]q^n.$$

Since the right hand sides of (1)-(9) are written as a linear combination of the functions in Definition 1.1, our theorem is proved. □

Lemma 3.4. *For any relatively prime positive integers a, b, let*

$$S[(\nu,\mu),(1,1),(a,b)](n) = \sum_{\substack{l,m\in\mathbb{Z}_{>0}\\ al+bm=n}} l^\nu m^\mu \sigma(l)\sigma(m).$$

Then

$$S[(0,0),(1,1),(a,b)](n) = n^{-\nu}\sum_{t=0}^{\nu}\binom{\nu}{t}a^{\nu-t}b^t S[(\nu-t,t),(1,1),(a,b)](n).$$

In particular, we have

$$aW_{a,b}^{(1)}(n) + bW_{b,a}^{(1)}(n) = nW_{a,b}^{(0)}(n).$$

Proof. The proof is easy using the binomial theorem:

$$\begin{aligned} n^\nu S[(0,0),(1,1),(a,b)](n) &= \sum_{\substack{l,m\in\mathbb{Z}_{>0}\\ al+bm=n}} (al+bm)^\nu \sigma(l)\sigma(m) \\ &= \sum_{\substack{l,m\in\mathbb{Z}_{>0}\\ al+bm=n}} \sum_{t=0}^{\nu}\binom{\nu}{t}a^{\nu-t}b^t l^{\nu-t}m^t\sigma(l)\sigma(m) \\ &= \sum_{t=0}^{\nu}\binom{\nu}{t}a^{\nu-t}b^t S[(\nu-t,t),(1,1),(a,b)](n). \end{aligned}$$

If $\nu = 1$, then we have that

$$aW_{a,b}^{(1)}(n) + bW_{b,a}^{(1)}(n) = nW_{a,b}^{(0)}(n).$$ □

Hence we need the evaluation of the convolution sums $W_{a,b}(n)$ which already occur in the literature in order to deduce Theorem 3.2.

Lemma 3.5. *1.*

$$\sum_{l+2m=n}\sigma(l)\sigma(m) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \frac{1-3n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{2}\right),$$

2.

$$\sum_{l+3m=n}\sigma(l)\sigma(m) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) + \frac{1-2n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{3}\right),$$

3.

$$\sum_{l+4m=n}\sigma(l)\sigma(m) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) + \frac{2-3n}{48}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{4}\right),$$

4.

$$\sum_{l+5m=n}\sigma(l)\sigma(m) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{5-6n}{120}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{5}\right) - \frac{1}{130}b_5(n),$$

5.
$$\sum_{l+6m=n} \sigma(l)\sigma(m) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) + \frac{1-n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{6}\right) - \frac{1}{120}b_6(n),$$

6.
$$\sum_{2l+3m=n} \sigma(l)\sigma(m) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) + \frac{1-2n}{24}\sigma\left(\frac{n}{2}\right) + \frac{1-3n}{24}\sigma\left(\frac{n}{3}\right) - \frac{1}{120}b_6(n).$$

7.
$$\sum_{l+7m=n} \sigma(l)\sigma(m) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) + \frac{7-6n}{168}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{7}\right) - \frac{1}{70}b_7(n),$$

8.
$$\sum_{l+8m=n} \sigma(l)\sigma(m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) + \frac{4-3n}{96}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{8}\right) - \frac{1}{64}b_8(n),$$

9.
$$\sum_{l+9m=n} \sigma(l)\sigma(m) = \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{3-2n}{72}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{9}\right) - \frac{1}{54}b_9(n).$$

Proof. The evaluations of $W_{a,b}(n) = \sum_{al+bm=n} \sigma(l)\sigma(m)$ are in the following references : (1)-(3) in [5], (4) in (4) is in [6, 7] and (5) and (6) are in [8]. □

Proof of Theorem 3.2. All formulas are linear combinations for $W^{(1)}_{b,a}(n)$ satisfying $a < b$, $(a, b) = 1$ and $ab \le 9$.

$W^{(0)}_{a,b}(n)$ and $W^{(1)}_{a,b}(n)$ are obtained in Lemma 3.5 and Theorem 3.1, respectively. Additionally, by Lemma 3.4, that is,
$$W^{(1)}_{b,a}(n) = \frac{n}{b} W^{(0)}_{a,b}(n) - \frac{a}{b} W^{(1)}_{a,b}(n)$$
we prove our theorem. □

Appendix. q-expansions of the function defined in Definition 1.1

We give the coefficients of the functions defined in Definition 1.1 up to q^{30}:

$$\begin{aligned}
\Delta_3(q) = {} & q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 + 168q^8 + 81q^9 - 36q^{10} - 564q^{11} + 36q^{12} \\
& + 638q^{13} + 240q^{14} + 54q^{15} - 1136q^{16} + 882q^{17} - 486q^{18} - 556q^{19} + 24q^{20} - 360q^{21} \\
& + 3384q^{22} - 840q^{23} + 1512q^{24} - 3089q^{25} - 3828q^{26} + 729q^{27} - 160q^{28} + 4638q^{29} \\
& - 324q^{30} + O\left(q^{31}\right), \\
\Delta_4(q) = {} & q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} - 418q^{13} - 648q^{15} + 594q^{17} + 836q^{19} \\
& + 1056q^{21} - 4104q^{23} - 209q^{25} + 4104q^{27} - 594q^{29} + O\left(q^{31}\right), \\
\delta_5(q) = {} & q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + 20q^{10} + 32q^{11} + 16q^{12} - 38q^{13} \\
& - 24q^{14} - 10q^{15} - 64q^{16} + 26q^{17} + 92q^{18} + 100q^{19} - 40q^{20} + 12q^{21} - 128q^{22} - 78q^{23} \\
& + 25q^{25} + 152q^{26} - 100q^{27} + 48q^{28} - 50q^{29} + 40q^{30} + O\left(q^{31}\right), \\
\Delta_5(q) = {} & q + 2q^2 - 4q^3 - 28q^4 + 25q^5 - 8q^6 + 192q^7 - 120q^8 - 227q^9 + 50q^{10} - 148q^{11}
\end{aligned}$$

$$+112q^{12}+286q^{13}+384q^{14}-100q^{15}+656q^{16}-1678q^{17}-454q^{18}+1060q^{19}-700q^{20}$$
$$-768q^{21}-296q^{22}+2976q^{23}+480q^{24}+625q^{25}+572q^{26}+1880q^{27}-5376q^{28}-3410q^{29}$$
$$-200q^{30}+O\left(q^{31}\right),$$

$$\begin{aligned}\delta_6(q) &= q-2q^2-3q^3+4q^4+6q^5+6q^6-16q^7-8q^8+9q^9-12q^{10}+12q^{11}-12q^{12}+38q^{13}\\ &+32q^{14}-18q^{15}+16q^{16}-126q^{17}-18q^{18}+20q^{19}+24q^{20}+48q^{21}-24q^{22}+168q^{23}\\ &+24q^{24}-89q^{25}-76q^{26}-27q^{27}-64q^{28}+30q^{29}+36q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\Delta_6(q) &= q+4q^2-9q^3+16q^4-66q^5-36q^6+176q^7+64q^8+81q^9-264q^{10}-60q^{11}-144q^{12}\\ &-658q^{13}+704q^{14}+594q^{15}+256q^{16}-414q^{17}+324q^{18}+956q^{19}-1056q^{20}-1584q^{21}\\ &-240q^{22}+600q^{23}-576q^{24}+1231q^{25}-2632q^{26}-729q^{27}+2816q^{28}+5574q^{29}\\ &+2376q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\delta_7(q) &= q-q^2-2q^3-7q^4+16q^5+2q^6-7q^7+15q^8-23q^9-16q^{10}-8q^{11}+14q^{12}+28q^{13}\\ &+7q^{14}-32q^{15}+41q^{16}+54q^{17}+23q^{18}-110q^{19}-112q^{20}+14q^{21}+8q^{22}+48q^{23}\\ &-30q^{24}+131q^{25}-28q^{26}+100q^{27}+49q^{28}-110q^{29}+32q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\Delta_{7,1}(q) &= q-10q^2+35q^3-30q^4-105q^5+238q^6-262q^8-145q^9+70q^{10}+1114q^{11}-560q^{12}\\ &-1071q^{13}-196q^{15}+2502q^{16}+140q^{17}-2078q^{18}-735q^{19}-868q^{20}+2401q^{21}\\ &+1012q^{22}-2684q^{23}+2100q^{24}+501q^{25}-1638q^{26}+2786q^{27}-4802q^{28}+1556q^{29}\\ &-392q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\Delta_{7,2}(q) &= q^2-6q^3+9q^4+10q^5-30q^6+11q^8+36q^9+36q^{10}-124q^{11}-42q^{12}+126q^{13}+49q^{14}\\ &+24q^{15}-243q^{16}-76q^{17}+441q^{18}-18q^{19}-56q^{20}-294q^{21}-360q^{22}+568q^{23}-6q^{24}\\ &-180q^{25}+392q^{26}-324q^{27}+441q^{28}+252q^{29}-720q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\Delta_{7,3}(q) &= q^3-2q^4-q^5+2q^6+q^7+2q^8-2q^9-10q^{10}+18q^{11}+8q^{12}-19q^{13}-10q^{14}-20q^{15}\\ &+22q^{16}+38q^{17}-52q^{18}-13q^{19}+60q^{20}+35q^{21}+68q^{22}-92q^{23}-60q^{24}+10q^{25}\\ &-62q^{26}-26q^{27}-30q^{28}-38q^{29}+152q^{30}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\delta_8(q) &= q-4q^3-2q^5+24q^7-11q^9-44q^{11}+22q^{13}+8q^{15}+50q^{17}+44q^{19}-96q^{21}-56q^{23}\\ &-121q^{25}+152q^{27}+198q^{29}+O\left(q^{31}\right),\end{aligned}$$

$$\begin{aligned}\Delta_8(q) &= q+20q^3-74q^5-24q^7+157q^9+124q^{11}+478q^{13}-1480q^{15}-1198q^{17}+3044q^{19}\\ &-480q^{21}+184q^{23}+2351q^{25}-1720q^{27}-3282q^{29}+O\left(q^{31}\right),\end{aligned}$$

$$\delta_9(q) = q-8q^4+20q^7-70q^{13}+64q^{16}+56q^{19}-125q^{25}-160q^{28}+O\left(q^{31}\right),$$

$$\begin{aligned}\Delta_9(q) &= q+3q^2+9q^3+4q^4-3q^5-54q^6-40q^7-84q^8+81q^9-36q^{10}+282q^{11}+36q^{12}\\ &+638q^{13}-120q^{14}+54q^{15}-1136q^{16}-441q^{17}-486q^{18}-556q^{19}-12q^{20}-360q^{21}\\ &+3384q^{22}+420q^{23}+1512q^{24}-3089q^{25}+1914q^{26}+729q^{27}-160q^{28}-2319q^{29}\\ &-324q^{30}+O\left(q^{31}\right).\end{aligned}$$

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