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Asymptotic approximation for the solution to a semi-linear elliptic problem in a thin aneurysm-type domain

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Abstract: A semi-linear boundary-value problem with nonlinear Robin boundary conditions is considered in a thin $3D$ aneurysm-type domain that consists of thin curvilinear cylinders that are joined through an aneurysm of diameter $\mathcal{O}(\varepsilon)$. Using the multi-scale analysis, the asymptotic approximation for the solution is constructed and justified as the parameter $\varepsilon \to 0$. Namely, we derive the limit problem ($\varepsilon = 0$) in the corresponding graph, define other terms of the asymptotic approximation and prove energetic and uniform pointwise estimates. These estimates allow us to observe the impact of the aneurysm on some properties of the solution.

Keywords: Multiscale analysis, Thin aneurysm-type domains, Asymptotic approximation, Semi-linear elliptic problem

MSC: 35B25, 35J65, 35B40, 74K30

1 Introduction

Investigations of various physical and biological processes in channels, junctions and networks are urgent for numerous fields of natural sciences (see, e.g., [1–14] and the references therein). Especial interest is the investigation of the influence of a local geometrical heterogeneity in vessels on the blood flow. This is both an aneurysm (a pathological extension of an artery like bulge) and a stenosis (a pathological restriction of an artery). The understanding of the impact of a local geometric irregularity on properties of solutions to boundary-value problems in such domains can have useful applications in medicine and many areas of applied science. In [15] the authors classified 12 different aneurysms and proposed computational approach for this study. The aneurysm models have been meshed with 800,000 – 1,200,000 tetrahedral cells containing three boundary layers. It was showed that the geometric aneurysm form essentially impacts on the hemodynamics of the blood flow. However, as was noted by the authors, the question *how to model blood flow with sufficient accuracy is still open*.

This question was the main motivation for us to develop a new approach (asymptotic one) for the study of boundary-value problems in domains of such type, since numerical methods do not give good approximations through the presence of a local geometric irregularity. It is clear that such domains are prototypes of many other biological and engineering structures, but we prefer to call them *thin aneurysm-type domains* as more comprehensive and concise.

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There are several asymptotic approaches to study such problems (see [1, 3, 10–13, 16–18]) with special assumptions, namely: the uniform boundary conditions on the lateral surfaces of the thin cylinders, the right-hand sides depend only on the longitudinal variable in the direction of the corresponding cylinder and they are constant in neighbourhoods of the nodes and vertices, the right-hand sides satisfy especial orthogonality conditions (for more detail see [19, 20]). These assumptions significantly narrow the class of problems that can be studied by such methods.

In the present paper, we continue to develop the asymptotic method proposed in [19], where the complete asymptotic expansion was constructed for the solution to a linear boundary-value problem for the Poisson equation with a nonuniform Neumann boundary conditions in a thin $2D$ aneurysm-type domain, and in [20], where similar results were obtained for the Poisson equation in a thin $3D$ aneurysm-type domain, which does not need the above mentioned assumptions. Here, we have adapted this method to semi-linear elliptic problems with nonlinear perturbed Robin boundary conditions in thin aneurysm-type domains. These results were presented on the conference [21].

1.1 Statement of the problem

The model thin aneurysm-type domain Ω_ε consists of three thin curvilinear cylinders

$$\Omega_\varepsilon^{(i)} = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ \varepsilon\ell < x_i < 1, \quad \sum_{j=1}^{3} (1-\delta_{ij}) x_j^2 < \varepsilon^2 h_i^2(x_i) \right\}, \quad i = 1, 2, 3,$$

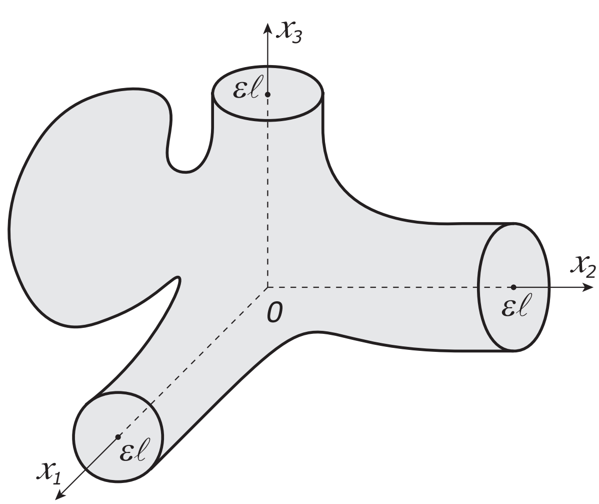
that are joined through a domain $\Omega_\varepsilon^{(0)}$ (referred in the sequel "aneurysm"). Here ε is a small parameter; $\ell \in (0, \frac{1}{3})$; the positive functions $\{h_i\}_{i=1}^3$ belong to the space $C^1([0,1])$ and they are equal to some constants in neighborhoods at the points $x = 0$ and $x_i = 1, \ i = 1,2,3$; the symbol δ_{ij} is the Kroneker delta, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

The aneurysm $\Omega_\varepsilon^{(0)}$ (see Fig. 1) is formed by the homothetic transformation with coefficient ε from a bounded domain $\Xi^{(0)} \in \mathbb{R}^3$, i.e., $\Omega_\varepsilon^{(0)} = \varepsilon\, \Xi^{(0)}$. In addition, we assume that its boundary contains the disks

$$\Upsilon_\varepsilon^{(i)}(\varepsilon\ell) = \left\{ x \in \mathbb{R}^3 \ : \ x_i = \varepsilon\ell, \quad \sum_{j=1}^{3} (1-\delta_{ij}) x_j^2 < \varepsilon^2 h_i^2(\varepsilon\ell) \right\}, \quad i = 1, 2, 3,$$

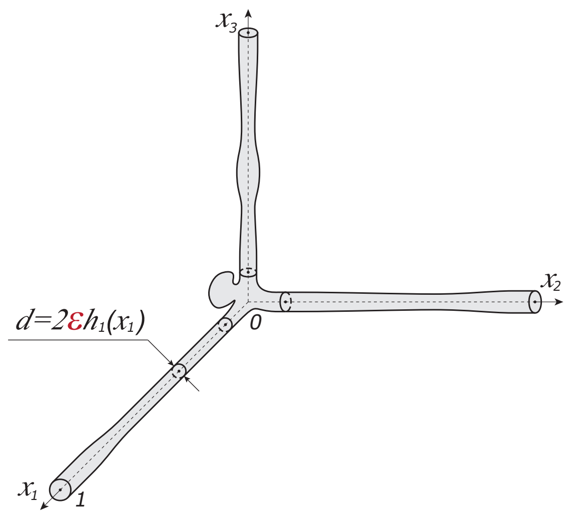
and denote $\Gamma_\varepsilon^{(0)} := \partial\Omega_\varepsilon^{(0)} \setminus \left\{ \overline{\Upsilon_\varepsilon^{(1)}(\varepsilon\ell)} \cup \overline{\Upsilon_\varepsilon^{(2)}(\varepsilon\ell)} \cup \overline{\Upsilon_\varepsilon^{(3)}(\varepsilon\ell)} \right\}$.

Fig. 1. The aneurysm $\Omega_\varepsilon^{(0)}$



Thus the model thin aneurysm-type domain Ω_ε (see Fig. 2) is the interior of the union $\bigcup_{k=0}^{3} \overline{\Omega_\varepsilon^{(k)}}$ and we assume that it has the Lipschitz boundary.

Fig. 2. The model thin aneurysm-type domain Ω_ε



Remark 1.1. *We can consider more general thin aneurysm-type domains with arbitrary orientation of thin cylinders (their number can be also arbitrary). But to avoid technical and huge calculations and to demonstrate the main steps of the proposed asymptotic approach we consider a such kind of the thin aneurysm-type domain, when the cylinders are placed on the coordinate axes.*

In Ω_ε, we consider the following semi-linear elliptic problem:

$$\begin{cases} -\Delta u_\varepsilon(x) + \kappa_0\big(u_\varepsilon(x)\big) = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\boldsymbol{\nu}} u_\varepsilon(x) = 0, & x \in \Gamma_\varepsilon^{(0)}, \\ -\partial_{\boldsymbol{\nu}} u_\varepsilon(x) - \varepsilon\kappa_i\big(u_\varepsilon(x)\big) = \varphi_\varepsilon(x), & x \in \Gamma_\varepsilon^{(i)}, \quad i = 1,2,3, \\ u_\varepsilon(x) = 0, & x \in \Upsilon_\varepsilon^{(i)}(1),\ i = 1,2,3, \end{cases} \tag{1}$$

where $\Gamma_\varepsilon^{(i)} = \partial\Omega_\varepsilon^{(i)} \cap \{x \in \mathbb{R}^3 \ : \ \varepsilon\ell < x_i < 1\}$, $\partial_{\boldsymbol{\nu}}$ is the outward normal derivative. The the given functions satisfy the following assumptions:

1. the functions $\{\kappa_j\}_{j=0}^3$ belong to the space $C^1(\mathbb{R})$ and there exist positive constants $\kappa_- > 0$ and $\kappa_+ > 0$ such that
$$\kappa_- \le \kappa_j'(s) \le \kappa_+ \quad \text{for} \quad s \in \mathbb{R}, \quad j = 0,1,2,3; \tag{2}$$
2. $\varphi_\varepsilon(x) := \varepsilon\,\varphi^{(i)}\left(x_i, \dfrac{\overline{x}_i}{\varepsilon}\right)$, $x \in \Gamma_\varepsilon^{(i)}$, $i = 1,2,3$, where
$$\overline{x}_i = \begin{cases} (x_2, x_3), \ i = 1, \\ (x_1, x_3), \ i = 2, \\ (x_1, x_2), \ i = 3, \end{cases}$$
and $\varphi^{(i)} \in C\left(\Omega_{\hat\varepsilon_0}^{(i)}\right)$, $i = 1,2,3$;
3. the function $f \in C\left(\Omega_{\hat\varepsilon_0}\right)$ and its restrictions on the curvilinear cylinders $\Omega_{\hat\varepsilon_0}^{(i)}$ belong to the spaces $C^1_{\overline{x}_i}\left(\Omega_{\hat\varepsilon_0}^{(i)}\right)$, $i = 1,2,3$, respectively.

Here $\hat\varepsilon_0$ is a fixed positive number and in what follows all values of the small parameter ε belong to the interval $(0, \hat\varepsilon_0)$.

Recall that a function u_ε from the Sobolev space $\mathcal{H}_\varepsilon = \{u \in H^1(\Omega_\varepsilon) : u|_{\Upsilon_\varepsilon^{(i)}(1)} = 0,\ i = 1,2,3\}$ is called a weak solution to the problem (1) if it satisfies the integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\, dx + \int\limits_{\Omega_\varepsilon} \kappa_0(u_\varepsilon) v\, dx + \varepsilon \sum_{i=1}^{3} \int\limits_{\Gamma_\varepsilon^{(i)}} \kappa_i(u_\varepsilon) v\, d\sigma_x = \int\limits_{\Omega_\varepsilon} f\, v\, dx - \sum_{i=1}^{3} \int\limits_{\Gamma_\varepsilon^{(i)}} \varphi_\varepsilon\, v\, d\sigma_x \tag{3}$$

for any function $v \in \mathcal{H}_\varepsilon$.

The aim of the present paper is to

- construct the asymptotic approximation for the solution to the problem (1) as the parameter $\varepsilon \to 0$;
- derive the corresponding limit problem ($\varepsilon = 0$);
- prove the corresponding asymptotic estimates from which the influence of the aneurysm will be observed.

1.2 Existence and uniqueness of the weak solution

In order to obtain operator statement for the problem (1) we introduce the new norm $\|\cdot\|_\varepsilon$ in $\mathcal{H}_\varepsilon$, which is generated by the scalar product

$$(u, v)_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u \cdot \nabla v\, dx, \quad u, v \in \mathcal{H}_\varepsilon.$$

Due to the uniform Dirichlet condition on $\Upsilon_\varepsilon^{(i)}(1),\ i = 1,2,3$, the norm $\|\cdot\|_\varepsilon$ and the ordinary norm $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ are uniformly equivalent, *i.e.*, there exist constants $C_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $u \in \mathcal{H}_\varepsilon$ the following estimates hold:

$$\|u\|_\varepsilon \le \|u\|_{H^1(\Omega_\varepsilon)} \le C_1 \|u\|_\varepsilon. \tag{4}$$

Remark 1.2. *Here and in what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.*

In the following we will often use the identities (see [22])

$$\varepsilon \int\limits_{\Gamma_\varepsilon^{(i)}} v^2\, d\sigma_x \le C_1 \Bigg(\varepsilon^2 \int\limits_{\Omega_\varepsilon^{(i)}} |\nabla_{\overline{x}_i} v|^2\, dx + \int\limits_{\Omega_\varepsilon^{(i)}} v^2\, dx \Bigg), \qquad \forall v \in H^1(\Omega_\varepsilon^{(i)}), \quad i = 1,2,3; \tag{5}$$

and inequalities

$$\kappa_-\, s^2 + \kappa_j(0)s \le \kappa_j(s)s \le \kappa_+\, s^2 + \kappa_j(0)s \quad \forall\, s \in \mathbb{R}, \quad j = 0,1,2,3, \tag{6}$$

that can be deduced from the conditions (2) (see [22]).

Denote by $\mathcal{H}_\varepsilon^*$ the dual space to $\mathcal{H}_\varepsilon$ and define a nonlinear operator $\mathcal{A}_\varepsilon : \mathcal{H}_\varepsilon \longrightarrow \mathcal{H}_\varepsilon^*$ through the relation

$$\big\langle \mathcal{A}_\varepsilon(u), v \big\rangle_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u \cdot \nabla v\, dx + \int\limits_{\Omega_\varepsilon} \kappa_0(u) v\, dx + \varepsilon \sum_{i=1}^{3} \int\limits_{\Gamma_\varepsilon^{(i)}} \kappa_i(u) v\, d\sigma_x \qquad \forall u, v \in \mathcal{H}_\varepsilon, \tag{7}$$

where $\langle \cdot, \cdot \rangle_\varepsilon$ is the duality pairing of $\mathcal{H}_\varepsilon^*$ and $\mathcal{H}_\varepsilon$.

In this case the integral identity (3) can be rewritten as follows

$$\big\langle \mathcal{A}_\varepsilon(u_\varepsilon), v \big\rangle_\varepsilon = \big\langle F_\varepsilon, v \big\rangle_\varepsilon \qquad \forall v \in \mathcal{H}_\varepsilon,$$

where $F_\varepsilon \in \mathcal{H}_\varepsilon^*$ is defined by

$$\big\langle F_\varepsilon, v \big\rangle_\varepsilon = \int\limits_{\Omega_\varepsilon} f\, v\, dx - \sum_{i=1}^{3} \int\limits_{\Gamma_\varepsilon^{(i)}} \varphi_\varepsilon\, v\, d\sigma_x \qquad \forall v \in \mathcal{H}_\varepsilon.$$

To prove the well-posedness result, we verify some properties of the operator $\mathcal{A}_\varepsilon$.

1. With the help of (6) and Cauchy's inequality with δ ($ab \le \delta a^2 + \frac{b^2}{4\delta}$, $a, b > 0$), we obtain

$$\begin{aligned}
&\langle \mathcal{A}_\varepsilon(v), v\rangle_\varepsilon \\
&\ge \int_{\Omega_\varepsilon} |\nabla v|^2\,dx + \int_{\Omega_\varepsilon} \kappa_-\,|v|^2\,dx + \int_{\Omega_\varepsilon} \kappa_0(0)\,|v|\,dx + \varepsilon\sum_{i=1}^{3}\left(\int_{\Gamma_\varepsilon^{(i)}} \kappa_-\,|v|^2\,d\sigma_x + \int_{\Gamma_\varepsilon^{(i)}} \kappa_i(0)\,|v|\,d\sigma_x\right) \\
&\ge \|v\|_\varepsilon^2 - \delta\left(|\kappa_0(0)|\int_{\Omega_\varepsilon} v^2\,dx + \varepsilon\sum_{i=1}^{3}|\kappa_i(0)|\int_{\Gamma_\varepsilon^{(i)}} v^2\,d\sigma_x\right) - \frac{1}{4\delta}\left(|\kappa_0(0)||\Omega_\varepsilon| + \varepsilon\sum_{i=1}^{3}|\kappa_i(0)||\Gamma_\varepsilon^{(i)}|\right).
\end{aligned}$$

Then using (5), we can select appropriate δ such that

$$\langle \mathcal{A}_\varepsilon(v), v\rangle_\varepsilon \ge C_2\|v\|_\varepsilon^2 - C_3 \quad \forall v \in \mathcal{H}_\varepsilon.$$

This inequality means that the operator $\mathcal{A}_\varepsilon$ is coercive.

2. Let us show that it is monotone. Taking into account (2), we get

$$\begin{aligned}
&\langle \mathcal{A}_\varepsilon(u_1) - \mathcal{A}_\varepsilon(u_2), u_1 - u_1\rangle_\varepsilon \\
&\ge \int_{\Omega_\varepsilon} |\nabla u_1 - \nabla u_2|^2\,dx + \kappa_- \int_{\Omega_\varepsilon} |u_1 - u_2|^2\,dx + \kappa_-\,\varepsilon\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}} |u_1 - u_2|^2\,d\sigma_x \ge \|u_1 - u_2\|_\varepsilon^2.
\end{aligned}$$

3. The operator $\mathcal{A}_\varepsilon$ is hemicontinuous. Ineed, the real valued function

$$[0,1] \ni \tau \to \langle \mathcal{A}_\varepsilon(u_1 + \tau v), u_2\rangle_\varepsilon$$

is continuous on $[0, 1]$ for all fixed $u_1, u_2, v \in \mathcal{H}_\varepsilon$ due to the continuity of the functions $\kappa_j,\ j = 0, 1, 2, 3$ and Lebesque's dominated convergence theorem.

4. Let us prove that operator $\mathcal{A}_\varepsilon$ is bounded. Using Cauchy-Bunyakovsky integral inequality, (4) and (6), we deduce the following inequality:

$$\begin{aligned}
&\langle \mathcal{A}_\varepsilon(u), v\rangle_\varepsilon \\
&\le \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v\,dx + \int_{\Omega_\varepsilon} \big(\kappa_+\,|u| + |\kappa_0(0)|\big)|v|\,dx + \varepsilon\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}} \big(\kappa_+\,|u| + |\kappa_i(0)|\big)|v|\,d\sigma_x \\
&\le \|u\|_\varepsilon\|v\|_\varepsilon + \kappa_+\,\|u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} + |\kappa_0(0)|\sqrt{|\Omega_\varepsilon|}\,\|v\|_{L^2(\Omega_\varepsilon)} \\
&\quad + \varepsilon\sum_{i=1}^{3}\left(\kappa_+\,\|u\|_{L^2(\Gamma_\varepsilon^{(i)})}\|v\|_{L^2(\Gamma_\varepsilon^{(i)})} + |\kappa_i(0)|\sqrt{|\Gamma_\varepsilon^{(i)}|}\,\|v\|_{L^2(\Gamma_\varepsilon^{(i)})}\right)
\end{aligned}$$

Now with the help of (5), we obtain

$$\langle \mathcal{A}_\varepsilon(u), v\rangle_\varepsilon \le C_5\big(1 + \|u\|_\varepsilon\big)\|v\|_\varepsilon \quad \forall u, v \in \mathcal{H}_\varepsilon.$$

Thus, the existence and uniqueness of the weak solution for every fixed value ε follow directly from Theorem 2.1 (see [23, Section 2]).

2 Formal asymptotic approximation

In this section we assume that the functions f and φ_ε are smooth enough. Following the approach of [19, 20], we propose ansatzes of the asymptotic approximation for the solution to the problem (1) in the following form:

1. the regular part of the approximation
$$\omega_0^{(i)}(x_i)+\varepsilon\omega_1^{(i)}(x_i)+\varepsilon^2 u_2^{(i)}\left(x_i,\frac{\overline{x}_i}{\varepsilon}\right)+\varepsilon^3 u_3^{(i)}\left(x_i,\frac{\overline{x}_i}{\varepsilon}\right) \tag{8}$$
is located inside each thin cylinder $\Omega_\varepsilon^{(i)}$ and their terms depend both on the corresponding longitudinal variable x_i and so-called "fast variables" $\dfrac{\overline{x}_i}{\varepsilon}$ $(i=1,2,3)$;
2. and the inner part of the approximation
$$N_0\left(\frac{x}{\varepsilon}\right)+\varepsilon N_1\left(\frac{x}{\varepsilon}\right)+\varepsilon^2 N_2\left(\frac{x}{\varepsilon}\right) \tag{9}$$
is located in a neighborhood of the aneurysm $\Omega_\varepsilon^{(0)}$.

2.1 Regular part

Substituting the representation (8) for each fixed index $i\in\{1,2,3\}$ into the differential equation of the problem (1), using Taylor's formula for the function κ_0 at $s=\omega_0^{(i)}(x_i)$ and the function f at the point $\overline{x}_i=(0,0)$, and collecting coefficients at ε^0, we obtain
$$-\Delta_{\overline{\xi}_i}u_2^{(i)}(x_i,\overline{\xi}_i)=\frac{d^2\omega_0^{(i)}}{dx_i{}^2}(x_i)-\kappa_0\Big(\omega_0^{(i)}(x_i)\Big)+f_0^{(i)}(x_i), \tag{10}$$
where $\overline{\xi}_i=\dfrac{\overline{x}_i}{\varepsilon}$ and $f_0^{(i)}(x_i):=f(x)|_{\overline{x}_i=(0,0)}$.

It is easy to calculate the outer unit normal to $\Gamma_\varepsilon^{(i)}$:
$$\boldsymbol{\nu}^{(i)}(x_i,\,\overline{\xi}_i)=\frac{1}{\sqrt{1+\varepsilon^2|h_i'(x_i)|^2}}\big(-\varepsilon h_i'(x_i),\ \overline{\nu}_i(\overline{\xi}_i)\big)=\begin{cases}\dfrac{\big(-\varepsilon h_1'(x_1),\ \nu_2^{(1)}(\overline{\xi}_1),\ \nu_3^{(1)}(\overline{\xi}_1)\big)}{\sqrt{1+\varepsilon^2|h_1'(x_1)|^2}}, & i=1,\\ \dfrac{\big(\nu_1^{(2)}(\overline{\xi}_2),\ -\varepsilon h_2'(x_2),\ \nu_3^{(2)}(\overline{\xi}_2)\big)}{\sqrt{1+\varepsilon^2|h_2'(x_2)|^2}}, & i=2,\\ \dfrac{\big(\nu_1^{(3)}(\overline{\xi}_3),\ \nu_2^{(3)}(\overline{\xi}_3),\ -\varepsilon h_3'(x_3)\big)}{\sqrt{1+\varepsilon^2|h_3'(x_3)|^2}}, & i=3,\end{cases}$$
where $\overline{\nu}_i(\frac{\overline{x}_i}{\varepsilon})$ is the outward normal for the disk $\Upsilon_\varepsilon^{(i)}(x_i):=\{\overline{\xi}_i\in\mathbb{R}^2\ :\ |\overline{\xi}_i|<h_i(x_i)\}$.

Taking the view of the outer unit normal into account and putting the sum (8) into the third relation of the problem (1), we get with the help of Taylor's formula for the function κ_i at $s=\omega_0^{(i)}(x_i)$ the following relation:
$$-\varepsilon\partial_{\overline{\nu}_i(\overline{\xi}_i)}u_2^{(i)}(x_i,\overline{\xi}_i)=-h_i'(x_i)\varepsilon\frac{d\omega_0^{(i)}}{dx_i}(x_i)+\varepsilon\left(\kappa_i\Big(\omega_0^{(i)}(x_i)\Big)+\varphi^{(i)}(x_i,\overline{\xi}_i)\right). \tag{11}$$

Relations (10) and (11) form the linear inhomogeneous Neumann boundary-value problem
$$\begin{cases}-\Delta_{\overline{\xi}_i}u_2^{(i)}(x_i,\overline{\xi}_i)=\dfrac{d^2\omega_0^{(i)}}{dx_i{}^2}(x_i)-\kappa_0\Big(\omega_0^{(i)}(x_i)\Big)+f_0^{(i)}(x_i), & \overline{\xi}_i\in\Upsilon_i(x_i),\\ -\partial_{\boldsymbol{\nu}_{\overline{\xi}_i}}u_2^{(i)}(x_i,\overline{\xi}_i)=-\,h_i'(x_i)\dfrac{d\omega_0^{(i)}}{dx_i}(x_i)+\kappa_i\Big(\omega_0^{(i)}(x_i)\Big)+\varphi^{(i)}(x_i,\overline{\xi}_i), & \overline{\xi}_i\in\partial\Upsilon_i(x_i),\\ \langle u_2^{(i)}(x_i,\cdot)\rangle_{\Upsilon_i(x_i)}=0, & \end{cases} \tag{12}$$

to define $u_2^{(i)}$. Here $\langle u(x_i,\cdot)\rangle_{\Upsilon_i(x_i)} := \int_{\Upsilon_i(x_i)} u(x_i,\overline{\xi}_i)d\overline{\xi}_i$, the variable x_i is regarded as a parameter from the interval $I_\varepsilon^{(i)} := \{x : \ x_i \in (\varepsilon\ell,1), \ \overline{x_i} = (0,0)\}$. We add the third relation in (12) for the uniqueness of a solution.

Writing down the necessary and sufficient conditions for the solvability of the problem (12), we derive the differential equation

$$-\pi \frac{d}{dx_i}\left(h_i^2(x_i)\frac{d\omega_0^{(i)}}{dx_i}(x_i)\right) + \pi h_i^2(x_i)\kappa_0\Big(\omega_0^{(i)}(x_i)\Big) + 2\pi h_i(x_i)\kappa_i\Big(\omega_0^{(i)}(x_i)\Big) = \pi h_i^2(x_i) f_0^{(i)}(x_i) - \int\limits_{\partial\Upsilon_i(x_i)} \varphi^{(i)}(x_i,\overline{\xi}_i)\, dl_{\overline{\xi}_i}, \quad x_i \in I_\varepsilon^{(i)}, \tag{13}$$

to define $\omega_0^{(i)}$ $(i \in \{1,2,3\})$.

Let $\omega_0^{(i)}$ be a solution of the differential equation (13) (the existence will be proved in the subsection 2.2.1). Thus, there exists a unique solution to the problem (12) for each $i \in \{1,2,3\}$.

For determination of the coefficients $u_3^{(i)}$, $i = 1,2,3$, we similarly obtain the following problems:

$$\begin{cases} -\Delta_{\overline{\xi}_i} u_3^{(i)}(x_i,\overline{\xi}_i) = \dfrac{d^2\omega_1^{(i)}}{dx_i^{\,2}}(x_i) - \kappa_0'\Big(\omega_0^{(i)}(x_i)\Big)\,\omega_1^{(i)}(x_i) + f_1^{(i)}(x_i,\overline{\xi}_i), & \overline{\xi}_i \in \Upsilon_i(x_i), \\ -\partial_{\boldsymbol{\nu}_{\overline{\xi}_i}} u_3^{(i)}(x_i,\overline{\xi}_i) = -\,h_i'(x_i)\dfrac{d\omega_1^{(i)}}{dx_i}(x_i) + \kappa_i'\Big(\omega_0^{(i)}(x_i)\Big)\,\omega_1^{(i)}(x_i), & \overline{\xi}_i \in \partial\Upsilon_i(x_i), \\ \langle u_3^{(i)}(x_i,\cdot)\rangle_{\Upsilon_i(x_i)} = 0, \end{cases} \tag{14}$$

for each $i \in \{1,2,3\}$. Here

$$f_1^{(i)}(x_i,\overline{\xi}_i) = \sum_{j=1}^{3}(1-\delta_{ij})\,\xi_j\,\frac{\partial}{\partial x_j} f(x)|_{\overline{x}_i=(0,0)}. \tag{15}$$

Repeating the previous reasoning, we find that the coefficients $\{\omega_1^{(i)}\}_{i=1}^3$ have to be solutions to the respective linear ordinary differential equation

$$-\pi \frac{d}{dx_i}\left(h_i^2(x_i)\frac{d\omega_1^{(i)}}{dx_i}(x_i)\right) + \pi h_i^2(x_i)\kappa_0'\Big(\omega_0^{(i)}(x_i)\Big)\,\omega_1^{(i)}(x_i) + 2\pi h_i(x_i)\kappa_i'\Big(\omega_0^{(i)}(x_i)\Big)\,\omega_1^{(i)}(x_i) = \int\limits_{\Upsilon_i(x_i)} f_1^{(i)}(x_i,\overline{\xi}_i)\, d\overline{\xi}_i, \quad x_i \in I_\varepsilon^{(i)} \quad (i \in \{1,2,3\}). \tag{16}$$

2.2 Inner part

To obtain conditions for the functions $\{\omega_k^{(i)}\}$, $i = 1,2,3$, $k \in \{0,1\}$ at the point $(0,0,0)$, we introduce the inner part of the asymptotic approximation (9) in a neighborhood of the aneurysm $\Omega_\varepsilon^{(0)}$. If we pass to the "fast variables" $\xi = \frac{x}{\varepsilon}$ and tend ε to 0, the domain Ω_ε is transformed into the unbounded domain Ξ that is the union of the domain $\Xi^{(0)}$ and three semibounded cylinders

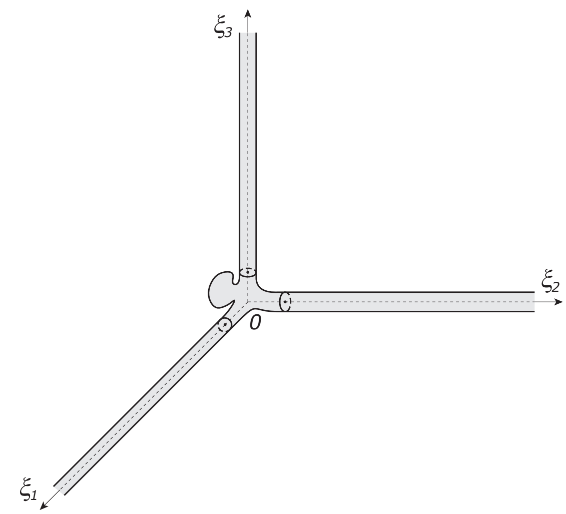
$$\Xi^{(i)} = \{\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \quad \ell < \xi_i < +\infty, \quad |\overline{\xi}_i| < h_i(0)\}, \qquad i = 1,2,3,$$

i.e., Ξ is the interior of $\bigcup_{i=0}^{3} \overline{\Xi^{(i)}}$ (see Fig. 3).

Let us introduce the following notation for parts of the boundary of the domain Ξ:

- $\Gamma_i = \{\xi \in \mathbb{R}^3 : \quad \ell < \xi_i < +\infty, \quad |\overline{\xi}_i| = h_i(0)\}, \quad i = 1,2,3,$
- $\Gamma_0 = \partial\Xi \setminus \left(\bigcup_{i=1}^{3} \Gamma_i\right).$

Fig. 3. Domain Ξ



Substituting (9) into the problem (1) and equating coefficients at the same powers of ε, we derive the following relations for N_k, $(k \in \{0,1,2\})$:

$$\begin{cases} -\Delta_\xi N_k(\xi) = F_k(\xi), & \xi \in \Xi, \\ \partial_{\boldsymbol{\nu}_\xi} N_k(\xi) = 0, & \xi \in \Gamma_0, \\ -\partial_{\overline{\nu}_{\overline{\xi}_i}} N_k(\xi) = B_k^{(i)}(\xi), & \xi \in \Gamma_i, \quad i = 1,2,3, \\ N_k(\xi) \sim \omega_k^{(i)}(0) + \Psi_k^{(i)}(\xi), & \xi_i \to +\infty, \ \overline{\xi}_i \in \Upsilon_i(0), \quad i = 1,2,3. \end{cases} \tag{17}$$

Here

$$F_0 \equiv F_1 \equiv 0, \quad F_2(\xi) = -\kappa_0(N_0) + f(0), \quad \xi \in \Xi,$$
$$B_0^{(i)} \equiv B_1^{(i)} \equiv 0, \qquad B_2^{(i)}(\xi) = \kappa_i(N_0) + \varphi^{(i)}(0, \overline{\xi}_i), \quad \xi \in \Gamma_i, \ i = 1,2,3.$$

The right hand sides in the differential equation and boundary conditions on $\{\Gamma_i\}$ of the problem (17) are obtained with the help of the Taylor decomposition of the functions f and $\varphi^{(i)}$ at the points $x = 0$ and $x_i = 0, \ i = 1,2,3,$ respectively.

The fourth condition in (17) appears by matching the regular and inner asymptotics in a neighborhood of the aneurysm, namely the asymptotics of the terms $\{N_k\}$ as $\xi_i \to +\infty$ have to coincide with the corresponding asymptotics of the terms $\{\omega_k^{(i)}\}$ as $x_i = \varepsilon \xi_i \to +0, \ i = 1,2,3,$ respectively. Expanding formally each term of the regular asymptotics in the Taylor series at the points $x_i = 0$ and collecting the coefficients of the same powers of ε, we get

$$\Psi_0^{(i)} \equiv 0, \qquad \Psi_1^{(i)}(\xi) = \xi_i \frac{d\omega_0^{(i)}}{dx_i}(0), \quad i = 1,2,3,$$
$$\Psi_2^{(i)}(\xi) = \frac{\xi_i^2}{2} \frac{d^2\omega_0^{(i)}}{dx_i^2}(0) + \xi_i \frac{d\omega_1^{(i)}}{dx_i}(0) + u_2^{(i)}(0, \overline{\xi}_i), \quad i = 1,2,3. \tag{18}$$

A solution of the problem (17) at $k = 1,2$ is sought in the form

$$N_k(\xi) = \sum_{i=1}^{3} \Psi_k^{(i)}(\xi)\chi_i(\xi_i) + \widetilde{N}_k(\xi), \tag{19}$$

where $\chi_i \in C^\infty(\mathbb{R}_+), \ 0 \le \chi_i \le 1$ and

$$\chi_i(\xi_i) = \begin{cases} 0, \text{ if } & \xi_i \le 1 + \ell, \\ 1, \text{ if } & \xi_i \ge 2 + \ell, \end{cases} \qquad i = 1,2,3.$$

Then $\widetilde{N}_k$ has to be a solution of the problem

$$\begin{cases} -\Delta_\xi \widetilde{N}_k(\xi) = \widetilde{F}_k(\xi), & \xi \in \Xi, \\ \partial_{\boldsymbol{\nu}_\xi} \widetilde{N}_k(\xi) = 0, & \xi \in \Gamma_0, \\ -\partial_{\boldsymbol{\nu}_{\overline{\xi}_i}} \widetilde{N}_k(\xi) = \widetilde{B}_k^{(i)}(\xi), & \xi \in \Gamma_i, \quad i = 1,2,3, \end{cases} \tag{20}$$

where

$$\widetilde{F}_1(\xi) = \sum_{i=1}^{3} \Big(\xi_i \frac{d\omega_0^{(i)}}{dx_i}(0)\chi_i''(\xi_i) + 2\frac{d\omega_0^{(i)}}{dx_i}(0)\chi_i'(\xi_i) \Big),$$

$$\widetilde{F}_2(\xi) = \sum_{i=1}^{3} \Bigg[\bigg(\frac{\xi_i^2}{2} \frac{d^2\omega_0^{(i)}}{dx_i^2}(0) + \xi_i \frac{d\omega_1^{(i)}}{dx_i}(0) + u_2^{(i)}(0,\overline{\xi}_i) \bigg) \chi_i''(\xi_i) + 2\bigg(\xi_i \frac{d^2\omega_0^{(i)}}{dx_i^2}(0) + \frac{d\omega_1^{(i)}}{dx_i}(0) \bigg) \chi_i'(\xi_i) \Bigg]$$
$$+ \Big(1 - \sum_{i=1}^{3} \chi_i(\xi_i)\Big) f(0) - \kappa_0\big(N_0\big) + \sum_{i=1}^{3} \kappa_0\big(\omega_0^{(i)}(0)\big)\chi_i(\xi_i)$$

and

$$\widetilde{B}_1^{(i)} \equiv 0, \qquad \widetilde{B}_2^{(i)}(\xi) = \big(1 - \chi_i(\xi_i)\big)\varphi^{(i)}(0,\overline{\xi}_i) + \kappa_i\big(N_0\big) - \sum_{i=1}^{3} \kappa_i\big(\omega_0^{(i)}(0)\big)\chi_i(\xi_i), \quad i = 1,2,3.$$

In addition, we demand that $\widetilde{N}_k$ satisfies the following stabilization conditions:

$$\widetilde{N}_k(\xi) \to \omega_k^{(i)}(0) \quad \text{as} \quad \xi_i \to +\infty, \ \overline{\xi}_i \in \Upsilon_i(0), \quad i = 1,2,3. \tag{21}$$

The existence of a solution to the problem (20) in the corresponding energetic space can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity (see e.g. [8, 24]). We will use approach proposed in [6, 8].

Let $C^\infty_{0,\xi}(\overline{\Xi})$ be a space of functions infinitely differentiable in $\overline{\Xi}$ and finite with respect to ξ, i.e.,

$$\forall\, v \in C^\infty_{0,\xi}(\overline{\Xi}) \quad \exists\, R > 0 \quad \forall\, \xi \in \overline{\Xi} \quad \xi_i \geq R, \ i = 1,2,3: \quad v(\xi) = 0.$$

We now define a space $\mathcal{H} := \overline{\Big(C^\infty_{0,\xi}(\overline{\Xi}), \ \|\cdot\|_{\mathcal{H}}\Big)}$, where

$$\|v\|_{\mathcal{H}} = \sqrt{\int_\Xi |\nabla v(\xi)|^2\, d\xi + \int_\Xi |v(\xi)|^2 |\rho(\xi)|^2\, d\xi}\,,$$

and the weight function $\rho \in C^\infty(\mathbb{R}^3), \ 0 \leq \rho \leq 1$ and

$$\rho(\xi) = \begin{cases} 1, & \text{if} \quad \xi \in \Xi^{(0)}, \\ |\xi_i|^{-1}, & \text{if} \quad \xi_i \geq \ell + 1, \ \xi \in \Xi^{(i)}, \quad i = 1,2,3. \end{cases}$$

Definition 2.1. *A function $\widetilde{N}_k$ from the space $\mathcal{H}$ is called a weak solution of the problem (20) if the identity*

$$\int_\Xi \nabla \widetilde{N}_k \cdot \nabla v\, d\xi = \int_\Xi \widetilde{F}_k\, v\, d\xi - \sum_{i=1}^{3} \int_{\Gamma_i} \widetilde{B}_k^{(i)}\, v\, d\sigma_\xi. \tag{22}$$

holds for all $v \in \mathcal{H}$.

Similarly as in [6], we prove the following proposition.

Proposition 2.2. *Let* $\rho^{-1}\widetilde{F}_k \in L^2(\Xi)$, $\rho^{-1}\widetilde{B}_k^{(i)} \in L^2(\Gamma_i)$, $\quad i = 1,2,3$. *Then there exists a weak solution of problem (20) if and only if*

$$\int\limits_{\Xi} \widetilde{F}_k \, d\xi = \sum_{i=1}^{3} \int\limits_{\Gamma_i} \widetilde{B}_k^{(i)} \, d\sigma_\xi. \tag{23}$$

This solution is defined up to an additive constant. The additive constant can be chosen to guarantee the existence and uniqueness of a weak solution of problem (20) with the following differentiable asymptotics:

$$\widehat{N}_k(\xi) = \begin{cases} \mathcal{O}(\exp(-\gamma_1\xi_1)) \text{ as } \xi_1 \to +\infty, \\ \delta_k^{(2)} + \mathcal{O}(\exp(-\gamma_2\xi_2)) \text{ as } \xi_2 \to +\infty, \\ \delta_k^{(3)} + \mathcal{O}(\exp(-\gamma_3\xi_3)) \text{ as } \xi_3 \to +\infty, \end{cases} \tag{24}$$

where γ_i, $i = 1,2,3$ *are positive constants.*

The constants $\delta_k^{(2)}$ and $\delta_k^{(3)}$ in (24) are defined as follows:

$$\delta_k^{(i)} = \int\limits_{\Xi} \mathfrak{N}_i \, \widetilde{F}_k(\xi) \, d\xi - \sum_{j=1}^{3} \int\limits_{\Gamma_j} \mathfrak{N}_i \, \widetilde{B}_k^{(j)}(\xi) \, d\sigma_\xi, \quad i = 2,3, \ k \in \{0,1,2\}, \tag{25}$$

where $\mathfrak{N}_2$ and $\mathfrak{N}_3$ are special solutions to the corresponding homogeneous problem

$$-\Delta_\xi \mathfrak{N} = 0 \ \text{ in } \ \Xi, \qquad \partial_\nu \mathfrak{N} = 0 \ \text{ on } \ \partial\Xi, \tag{26}$$

for the problem (20).

Proposition 2.3. *The problem (26) has two linearly independent solutions* $\mathfrak{N}_2$ *and* $\mathfrak{N}_3$ *that do not belong to the space* $\mathcal{H}$ *and they have the following differentiable asymptotics:*

$$\mathfrak{N}_2(\xi) = \begin{cases} -\dfrac{\xi_1}{\pi h_1^2(0)} + \mathcal{O}(\exp(-\gamma_1\xi_1)) \text{ as } \xi_1 \to +\infty, \\ C_2^{(2)} + \dfrac{\xi_2}{\pi h_2^2(0)} + \mathcal{O}(\exp(-\gamma_2\xi_2)) \text{ as } \xi_2 \to +\infty, \\ C_2^{(3)} + \mathcal{O}(\exp(-\gamma_3\xi_3)) \text{ as } \xi_3 \to +\infty, \end{cases} \tag{27}$$

$$\mathfrak{N}_3(\xi) = \begin{cases} -\dfrac{\xi_1}{\pi h_1^2(0)} + \mathcal{O}(\exp(-\gamma_1\xi_1)) \text{ as } \xi_1 \to +\infty, \\ C_3^{(2)} + \mathcal{O}(\exp(-\gamma_2\xi_2)) \text{ as } \xi_2 \to +\infty, \\ C_3^{(3)} + \dfrac{\xi_3}{\pi h_3^2(0)} + \mathcal{O}(\exp(-\gamma_3\xi_3)) \text{ as } \xi_3 \to +\infty, \end{cases} \tag{28}$$

Any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as a linear combination $\alpha_1 + \alpha_2\mathfrak{N}_2 + \alpha_3\mathfrak{N}_3$.

Proof. The solution $\mathfrak{N}_2$ is sought in the form of a sum

$$\mathfrak{N}_2(\xi) = -\frac{\xi_1}{\pi h_1^2(0)} \chi_1(\xi_1) + \frac{\xi_2}{\pi h_2^2(0)} \chi_2(\xi_2) + \widetilde{\mathfrak{N}}_2(\xi),$$

where $\widetilde{\mathfrak{N}}_2 \in \mathcal{H}$ and $\widetilde{\mathfrak{N}}_2$ is the solution to the problem (20) with right-hand sides

$$\widetilde{F}_2^*(\xi) = \begin{cases} \dfrac{1}{\pi h_1^2(0)} \Big(\big(\xi_1 \, \chi_1'(\xi_1)\big)' + \chi_1'(\xi_1) \Big), & \xi \in \Xi^{(1)}, \\ -\dfrac{1}{\pi h_2^2(0)} \Big(\big(\xi_2 \, \chi_2'(\xi_2)\big)' + \chi_2'(\xi_2) \Big), & \xi \in \Xi^{(2)}, \\ 0, & \xi \in \Xi^{(0)} \cup \Xi^{(3)}. \end{cases}$$

It is easy to verify that the solvability condition (23) is satisfied. Thus, by virtue of Proposition 2.1 there exist a unique solution $\widetilde{\mathfrak{N}}_2 \in \mathcal{H}$ that has the asymptotics

$$\widetilde{\mathfrak{N}}_2(\xi) = (1-\delta_{1j})C_2^{(j)} + \mathcal{O}(\exp(-\gamma_j \xi_j)) \quad \text{as} \ \ \xi_j \to +\infty, \qquad j = 1,2,3.$$

Similarly we can prove the existence of the solution $\mathfrak{N}_3$ with the asymptotics (28).

Obviously, $\mathfrak{N}_2$ and $\mathfrak{N}_3$ are linearly independent and any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as $\alpha_1 + \alpha_2\mathfrak{N}_2 + \alpha_3\mathfrak{N}_3$. □

Remark 2.4. *To obtain formulas (25) for the constants $\delta_k^{(2)}$ and $\delta_k^{(3)}$, it is necessary to substitute the functions $\widehat{N}_k, \mathfrak{N}_2$ and $\widehat{N}_k, \mathfrak{N}_3$ in the second Green-Ostrogradsky formula*

$$\int\limits_{\Xi_R} \left(\widehat{N}\, \Delta_\xi \mathfrak{N} - \mathfrak{N}\, \Delta_\xi \widehat{N}\right) d\xi = \int\limits_{\partial\Xi_R} \left(\widehat{N}\, \partial_{\nu_\xi} \mathfrak{N} - \mathfrak{N}\, \partial_{\nu_\xi} \widehat{N}\right) d\sigma_\xi$$

respectively, and then pass to the limit as $R \to +\infty$. Here $\Xi_R = \Xi \cap \{\xi : \ |\xi_i| < R, \ i = 1,2,3\}$.

2.2.1 Limit problem

The problem (17) at $k = 0$ is as follows:

$$\begin{cases} -\Delta_\xi N_0(\xi) \ = \ 0, & \xi \in \Xi, \\ \partial_{\boldsymbol{\nu}_\xi} N_0(\xi) \ = \ 0, & \xi \in \Gamma_0, \\ -\partial_{\boldsymbol{\nu}_{\overline{\xi}_i}} N_0(\xi) \ = \ 0, & \xi \in \Gamma_i, \quad i = 1,2,3, \\ N_0(\xi) \longrightarrow \omega_0^{(i)}(0), & \xi_i \to +\infty, \ \ \overline{\xi}_i \in \Upsilon_i(0), \quad i = 1,2,3, \end{cases} \tag{29}$$

It is easy to verify that $\delta_0^{(2)} = \delta_0^{(3)} = 0$ and $\widehat{N}_0 \equiv 0$. Thus, this problem has a solution in $\mathcal{H}$ if and only if

$$\omega_0^{(1)}(0) = \omega_0^{(2)}(0) = \omega_0^{(3)}(0); \tag{30}$$

in this case $N_0 \equiv \widetilde{N}_0 \equiv \omega_0^{(1)}(0)$.

In the problem (20) at $k = 1$ the solvability condition (23) reads as follows:

$$\pi h_1^2(0) \frac{d\omega_0^{(1)}}{dx_1}(0) + \pi h_2^2(0) \frac{d\omega_0^{(2)}}{dx_2}(0) + \pi h_3^2(0) \frac{d\omega_0^{(3)}}{dx_3}(0) = 0. \tag{31}$$

Substituting (8) into the fourth condition in (1) and neglecting terms of order of $\mathcal{O}(\varepsilon)$, we arrive at the following boundary conditions:

$$\omega_0^{(i)}(1) = 0, \quad i = 1,2,3. \tag{32}$$

Thus, taking into account (13), (30), (31) and (32), we obtain for $\{\omega_0^{(i)}\}_{i=1}^3$ the following semi-linear problem:

$$\begin{cases} -\pi \dfrac{d}{dx_i}\left(h_i^2(x_i) \dfrac{d\omega_0^{(i)}}{dx_i}(x_i)\right) & \\ +\pi h_i^2(x_i)\kappa_0\Big(\omega_0^{(i)}(x_i)\Big) + 2\pi h_i(x_i)\kappa_i\Big(\omega_0^{(i)}(x_i)\Big) = \widehat{F}_0^{(i)}(x_i), \ x_i \in I_i, & i = 1,2,3, \\ \omega_0^{(i)}(1) = 0, & i = 1,2,3, \\ \omega_0^{(1)}(0) \ = \ \omega_0^{(2)}(0) = \omega_0^{(3)}(0), & \\ \sum\limits_{i=1}^{3} \pi h_i^2(0) \dfrac{d\omega_0^{(i)}}{dx_i}(0) = 0, & \end{cases} \tag{33}$$

where $I_i := \{x : \ x_i \in (0,1), \ \overline{x_i} = (0,0)\}$ and

$$\widehat{F}_0^{(i)}(x_i) := \pi h_i^2(x_i)\, f(x)\,|_{\overline{x}_i=(0,0)} - \int\limits_{\partial\Upsilon_i(x_i)} \varphi^{(i)}(x_i,\overline{\xi}_i)\, dl_{\overline{\xi}_i}, \quad x \in I_i. \tag{34}$$

The problem (33) is called *limit problem* for problem (1).

For functions

$$\widetilde{\phi}(x) = \begin{cases} \phi^{(1)}(x_1), \text{ if } x_1 \in I_1, \\ \phi^{(2)}(x_2), \text{ if } x_2 \in I_2, \\ \phi^{(3)}(x_3), \text{ if } x_3 \in I_3, \end{cases}$$

defined on the graph $I_1 \cup I_2 \cup I_3$, we introduce the Sobolev space

$$\mathcal{H}_0 := \Big\{\widetilde{\phi} : \ \phi^{(i)} \in H^1(I_i), \ \phi^{(i)}(1) = 0, \ i = 1,2,3, \ \text{and } \phi^{(1)}(0) = \phi^{(2)}(0) = \phi^{(3)}(0)\Big\}$$

with the scalar product

$$(\widetilde{\phi}, \widetilde{\psi})_0 := \sum_{i=1}^{3} \pi \int\limits_0^1 h_i^2(x_i)\, \frac{d\phi^{(i)}}{dx_i}\, \frac{d\psi^{(i)}}{dx_i}\, dx_i, \qquad \widetilde{\phi}, \widetilde{\psi} \in \mathcal{H}_0.$$

Definition 2.5. *A function $\widetilde{\omega} \in \mathcal{H}_0$ is called a weak solution to the problem (33) if it satisfies the integral identity*

$$(\widetilde{\omega}, \widetilde{\psi})_0 + \sum_{i=1}^{3} \left(\pi \int\limits_0^1 h_i^2(x_i)\kappa_0(\omega^{(i)}(x_i))\psi^{(i)}(x_i)\, dx_i + 2\pi \int\limits_0^1 h_i(x_i)\kappa_i(\omega^{(i)}(x_i))\psi^{(i)}(x_i)\, dx_i \right)$$
$$= \sum_{i=1}^{3} \int\limits_0^1 \widehat{F}_0^{(i)}(x_i)\, \psi^{(i)}(x_i)\, dx_i \qquad \forall\, \widetilde{\psi} \in \mathcal{H}_0. \tag{35}$$

Similarly as was done in Section 1.2, the integral identity (35) can be rewritten as follows

$$\langle \mathcal{A}_0(\widetilde{\omega}), \widetilde{\psi}\rangle_0 = \langle F_0, \widetilde{\psi}\rangle_0 \qquad \forall \widetilde{\psi} \in \mathcal{H}_0. \tag{36}$$

where the nonlinear operator $\mathcal{A}_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_0^*$ is defined through the relation

$$\langle \mathcal{A}_0(\phi^{(i)}), \psi^{(i)}\rangle_0 = (\widetilde{\phi}, \widetilde{\psi})_0 + \sum_{i=1}^{3} \left(\pi \int\limits_0^1 h_i^2\kappa_0(\phi^{(i)})\psi^{(i)}\, dx_i + 2\pi \int\limits_0^1 h_i\kappa_i(\phi^{(i)})\psi^{(i)}\, dx_i \right) \qquad \forall\, \widetilde{\phi}, \widetilde{\psi} \in \mathcal{H}_0,$$

and $F_0 \in \mathcal{H}_0^*$ is defined by

$$\langle F_0, \widetilde{\psi}\rangle_0 = \sum_{i=1}^{3} \int\limits_0^1 \widehat{F}_0^{(i)}\, \psi^{(i)}\, dx_i \qquad \forall\, \widetilde{\psi} \in \mathcal{H}_0,$$

where $\langle\cdot,\cdot\rangle_0$ is the duality pairing of the dual space $\mathcal{H}_0^*$ and $\mathcal{H}_0$.

Using (2) and (6), we can prove that the operator $\mathcal{A}_0$ is bounded, strongly monotone, hemicontinuous and coercive. As a result, the existence and uniqueness of the weak solution to the problem (33) follow directly from Theorem 2.1 (see [23, Section 2]).

2.2.2 Problem for $\{\omega_1\}$

Let us verify the solvability condition (23) for the problem (20) at $k = 2$. Knowing that $N_0 \equiv \omega_0^{(1)}(0)$ and taking into account the third relation in problem (12), the equality (23) can be re-written as follows:

$$\sum_{i=1}^{3} \left[\pi h_i^2(0) \int\limits_{\ell+1}^{\ell+2} \left(\xi_i \frac{d^2\omega_0^{(i)}}{dx_i^2}(0) + \frac{d\omega_1^{(i)}}{dx_i}(0) \right) \chi_i'(\xi_i)\, d\xi_i \right.$$

$$+\int_{\ell}^{\ell+2}(1-\chi_i(\xi_i))\int_{\Upsilon_i(0)}\Big(f(0)-\kappa_0\big(\omega_0^{(i)}(0)\big)\Big)\,d\overline{\xi}_i\,d\xi_i$$

$$-\int_{\ell}^{\ell+2}(1-\chi_i(\xi_i))\int_{\partial\Upsilon_i(0)}\Big(\varphi^{(i)}(0,\overline{\xi}_i)+\kappa_i\big(\omega_0^{(i)}(0)\big)\Big)\,dl_{\overline{\xi}_i}\,d\xi_i\Bigg]$$

$$+\int_{\Xi^{(0)}}\Big(f(0)-\kappa_0\big(\omega_0^{(i)}(0)\big)\Big)\,d\xi \;=\; 0.$$

Whence, integrating by parts in the first integrals with regard to (13), we obtain the following relations for $\{\omega_1^{(i)}\}$:

$$\sum_{i=1}^{3}\pi h_i^2(0)\frac{d\omega_1^{(i)}}{dx_i}(0)=d_1^*, \tag{37}$$

where

$$d_1^*=\ell\sum_{i=1}^{3}\left(\pi h_i^2(0)\Big(f(0)-\kappa_0\big(\omega_0^{(i)}(0)\big)\Big)-2\pi h_i(0)\,\kappa_i\big(\omega_0^{(i)}(0)\big)-\int_{\partial\Upsilon_i(0)}\varphi^{(i)}(0,\overline{\xi}_i)\,dl_{\overline{\xi}_i}\right)$$
$$-\big|\Xi^{(0)}\big|\Big(f(0)-\kappa_0\big(\omega_0^{(i)}(0)\big)\Big). \tag{38}$$

Hence, if the functions $\{\omega_1^{(i)}\}_{i=1}^3$ satisfy (37), then there exists a weak solution $\widetilde{N}_2$ of the problem (20). According to Proposition 2.2, it can be chosen in a unique way to guarantee the asymptotics (24).

It remains to satisfy the stabilization conditions (21) at $k=1$. For this, we represent a weak solution of the problem (20) in the following form:

$$\widetilde{N}_1=\omega_1^{(1)}(0)+\widehat{N}_1.$$

Taking into account the asymptotics (24), we have to put

$$\omega_1^{(1)}(0)=\omega_1^{(2)}(0)-\delta_1^{(2)}=\omega_1^{(3)}(0)-\delta_1^{(3)}. \tag{39}$$

As a result, we get the solution of the problem (17) with the following asymptotics:

$$N_1(\xi)=\omega_1^{(i)}(0)+\Psi_1^{(i)}(\xi)+\mathcal{O}(\exp(-\gamma_i\xi_i))\quad\text{as}\ \ \xi_i\to+\infty,\quad i=1,2,3. \tag{40}$$

Let us denote by

$$G_1(\xi):=\omega_1^{(i)}(0)+\Psi_1^{(i)}(\xi),\quad \xi\in\Xi^{(i)},\quad i=1,2,3.$$

Remark 2.6. *Due to (40), the function N_1-G_1 is exponentially decrease as $\xi_i\to+\infty$, $i=1,2,3$.*

Relations (39) and (37) are the first and second transmission conditions for $\{\omega_1^{(i)}\}_{i=1}^3$ at $x=0$. Thus, the coefficients $\{\omega_1^{(1)},\ \omega_1^{(2)},\ \omega_1^{(3)}\}$ are determined from the linear problem

$$\begin{cases}
-\pi\dfrac{d}{dx_i}\left(h_i^2(x_i)\dfrac{d\omega_1^{(i)}}{dx_i}(x_i)\right) & \\
+\pi h_i^2(x_i)\kappa_0'\Big(\omega_0^{(i)}(x_i)\Big)\omega_1^{(i)}(x_i) & \\
+2\pi h_i(x_i)\kappa_i'\Big(\omega_0^{(i)}(x_i)\Big)\omega_1^{(i)}(x_i)=\widehat{F}_1^{(i)}(x_i), \qquad x_i\in I_i, & i=1,2,3,\\
\omega_1^{(i)}(1)=0, & i=1,2,3,\\
\omega_1^{(1)}(0)\;=\;\omega_1^{(2)}(0)-\delta_k^{(2)}=\omega_1^{(3)}(0)-\delta_k^{(3)}, & \\
\sum\limits_{i=1}^{3}\pi h_i^2(0)\dfrac{d\omega_1^{(i)}}{dx_i}(0)=d_1^*, &
\end{cases} \tag{41}$$

where

$$\widehat{F}_1^{(i)}(x_i) = \int\limits_{\Upsilon_i(x_i)} f_1^{(i)}(x_i, \overline{\xi}_i)\, d\overline{\xi}_i, \quad x \in I_i, \quad i = 1,2,3.$$

The constants $\delta_1^{(2)}$ and $\delta_1^{(3)}$ are uniquely determined (see Remark 2.4) by formula

$$\delta_1^{(i)} = \int\limits_{\Xi} \mathfrak{N}_i \sum_{j=1}^{3} \Big(\xi_j \frac{d\omega_0^{(j)}}{dx_j}(0) \chi_j''(\xi_j) + 2 \frac{d\omega_0^{(j)}}{dx_j}(0) \chi_j'(\xi_j) \Big)\, d\xi, \quad i = 2,3. \tag{42}$$

With the help of the substitutions

$$\phi_1^{(1)}(x_1) = \omega_1^{(1)}(x_1), \quad \phi_1^{(2)}(x_2) = \omega_1^{(2)}(x_2) - \delta_1^{(2)}(1-x_2), \quad \phi_1^{(3)}(x_3) = \omega_1^{(3)}(x_3) - \delta_1^{(3)}(1-x_3),$$

we reduce the problem (41) to the respective integral identity in the space $\mathcal{H}_0$ and then the existence and uniqueness of a solution of this identity (and hence the problem (41)) follows from the Riesz representation theorem.

3 Justification

With the help of the coefficients $\{\omega_0^{(i)}\}$, $\{\omega_1^{(i)}\}$, N_1 and smooth cut-off functions defined by formulas

$$\chi_\ell^{(i)}(x_i) = \begin{cases} 1, \text{ if } x_i \ge 3\,\ell, \\ 0, \text{ if } x_i \le 2\,\ell, \end{cases} \qquad i = 1,2,3, \tag{43}$$

we construct the following asymptotic approximation:

$$U_\varepsilon^{(1)}(x) = \sum_{i=1}^{3} \chi_\ell^{(i)}\Big(\frac{x_i}{\varepsilon^\alpha}\Big)\Big(\omega_0^{(i)}(x_i) + \varepsilon\,\omega_1^{(i)}(x_i)\Big) + \Bigg(1 - \sum_{i=1}^{3} \chi_\ell^{(i)}\Big(\frac{x_i}{\varepsilon^\alpha}\Big)\Bigg)\Big(\omega_0^{(1)}(0) + \varepsilon N_1\Big(\frac{x}{\varepsilon}\Big)\Big), \quad x \in \Omega_\varepsilon, \tag{44}$$

where α is a fixed number from the interval $(\frac{2}{3}, 1)$.

Theorem 3.1. *Let assumptions made in the statement of the problem* (1) *be satisfied. Then the sum* (44) *is the asymptotic approximation for the solution* u_ε *to the boundary-value problem* (1) *in the Sobolev space* $H^1(\Omega_\varepsilon)$, *i.e.,*

$$\exists\, C_0 > 0 \;\; \exists\, \varepsilon_0 > 0 \;\; \forall\, \varepsilon \in (0, \varepsilon_0): \qquad \| u_\varepsilon - U_\varepsilon^{(1)} \|_{H^1(\Omega_\varepsilon)} \le C_0\, \varepsilon^{1+\frac{\alpha}{2}}. \tag{45}$$

Proof. Substituting $U_\varepsilon^{(0)}$ in the equations and the boundary conditions of problem (1), we find

$$\begin{cases} -\Delta U_\varepsilon^{(1)} + \kappa_0\Big(U_\varepsilon^{(1)}\Big) - f = \widehat{R}_\varepsilon & \text{in } \Omega_\varepsilon, \\ -\partial_\nu U_\varepsilon^{(1)} - \varepsilon\kappa_i\Big(U_\varepsilon^{(1)}\Big) - \varphi_\varepsilon = \check{R}_{\varepsilon,(i)} & \text{on } \Gamma_\varepsilon^{(i)}, \quad i = 1,2,3, \\ U_\varepsilon^{(1)} = 0 & \text{on } \Upsilon_\varepsilon^{(i)}(1),\ i = 1,2,3, \\ \partial_\nu U_\varepsilon^{(1)} = 0 & \text{on } \Gamma_\varepsilon^{(0)}, \end{cases} \tag{46}$$

where

$$\begin{aligned} \widehat{R}_\varepsilon(x) = -\sum_{i=1}^{3} \Bigg(& 2\varepsilon^{-\alpha} \frac{d\chi_\ell^{(i)}}{d\zeta_i}(\zeta_i)\Big|_{\zeta_i = \frac{x_i}{\varepsilon^\alpha}} \Bigg(\frac{d\omega_0^{(i)}}{dx_i}(x_i) - \frac{d\omega_0^{(i)}}{dx_i}(0) + \varepsilon \frac{d\omega_1^{(i)}}{dx_i}(x_i) - \Big(\frac{\partial N_1}{\partial \xi_i}(\xi) - \frac{\partial G_1}{\partial \xi_i}(\xi) \Big)\Big|_{\xi = \frac{x}{\varepsilon}} \Bigg) \\ & + \varepsilon^{-2\alpha} \frac{d^2\chi_\ell^{(i)}}{d\zeta_i^2}(\zeta_i)\Big|_{\zeta_i = \frac{x_i}{\varepsilon^\alpha}} \Bigg(\omega_0^{(i)}(x_i) - \omega_0^{(i)}(0) - x_i \frac{d\omega_0^{(i)}}{dx_i}(0) + \varepsilon\omega_1^{(i)}(x_i) - \varepsilon\omega_1^{(i)}(0) - \varepsilon N_1\Big(\frac{x}{\varepsilon}\Big) + \varepsilon G_1\Big(\frac{x}{\varepsilon}\Big) \Bigg) \\ & + \chi_\ell^{(i)}\Big(\frac{x_i}{\varepsilon^\alpha}\Big) \Bigg(\frac{d^2\omega_0^{(i)}}{dx_i^2}(x_i) + \varepsilon \frac{d^2\omega_1^{(i)}}{dx_i^2}(x_i) \Bigg) \Bigg) + \kappa_0\Big(U_\varepsilon^{(1)}(x)\Big) - f(x), \end{aligned} \tag{47}$$

and

$$\breve{R}_{\varepsilon,(i)}(x)=\frac{\varepsilon h_i'(x_i)}{\sqrt{1+\varepsilon^2|h_i'(x_i)|^2}}\chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\left(\frac{d\omega_0^{(i)}}{dx_i}(x_i)+\varepsilon\frac{d\omega_1^{(i)}}{dx_i}(x_i)\right)-\varepsilon\kappa_i\left(U_\varepsilon^{(1)}(x)\right)-\varphi_\varepsilon(x). \tag{48}$$

From (46) we derive the following integral relation:

$$\int_{\Omega_\varepsilon}\nabla U_\varepsilon^{(1)}\cdot\nabla v\,dx+\int_{\Omega_\varepsilon}\kappa_0(U_\varepsilon^{(1)})v\,dx+\varepsilon\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}}\kappa_i(U_\varepsilon^{(1)})v\,d\sigma_x$$
$$-\int_{\Omega_\varepsilon}f\,v\,dx+\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}}\varphi_\varepsilon\,v\,d\sigma_x=R_\varepsilon(v)\qquad\forall v\in\mathcal{H}_\varepsilon, \tag{49}$$

where

$$R_\varepsilon(v)=\int_{\Omega_\varepsilon}\widehat{R}_\varepsilon\,v\,dx-\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}}\breve{R}_{\varepsilon,(i)}\,v\,d\sigma_x.$$

From (12) and (14) we deduce that integral identities

$$\int_{\Upsilon_i(x_i)}\frac{d^2\omega_0^{(i)}}{dx_i^2}\eta\,d\overline{\xi}_i=\int_{\Upsilon_i(x_i)}\nabla_{\overline{\xi}_i}u_2^{(i)}\cdot\nabla_{\overline{\xi}_i}\eta\,d\overline{\xi}_i-\int_{\partial\Upsilon_i(x_i)}h_i'\frac{d\omega_0^{(i)}}{dx_i}\eta\,dl_{\overline{\xi}_i}$$
$$+\int_{\Upsilon_i(x_i)}\kappa_0\big(\omega_0^{(i)}\big)\eta\,d\overline{\xi}_i+\int_{\partial\Upsilon_i(x_i)}\kappa_i\big(\omega_0^{(i)}\big)\eta\,dl_{\overline{\xi}_i}-\int_{\Upsilon_i(x_i)}f_0^{(i)}\eta\,d\overline{\xi}_i+\int_{\partial\Upsilon_i(x_i)}\varphi^{(i)}\eta\,dl_{\overline{\xi}_i} \tag{50}$$

and

$$\int_{\Upsilon_i(x_i)}\frac{d^2\omega_1^{(i)}}{dx_i^2}\eta\,d\overline{\xi}_i=\int_{\Upsilon_i(x_i)}\nabla_{\overline{\xi}_i}u_3^{(i)}\cdot\nabla_{\overline{\xi}_i}\eta\,d\overline{\xi}_i-\int_{\partial\Upsilon_i(x_i)}h_i'\frac{d\omega_1^{(i)}}{dx_i}\eta\,dl_{\overline{\xi}_i}$$
$$+\int_{\Upsilon_i(x_i)}\kappa_0'\big(\omega_0^{(i)}\big)\omega_1^{(i)}\eta\,d\overline{\xi}_i+\int_{\partial\Upsilon_i(x_i)}\kappa_i'\big(\omega_0^{(i)}\big)\omega_1^{(i)}\eta\,dl_{\overline{\xi}_i}-\int_{\Upsilon_i(x_i)}f_1^{(i)}\eta\,d\overline{\xi}_i \tag{51}$$

hold for all $\eta\in H^1(\Upsilon_i(x_i))$ and for all $x_i\in I_\varepsilon^{(i)},\quad i=1,2,3.$

Using (50) and (51), we rewrite R_ε in the form

$$R_\varepsilon(v)=\sum_{j=1}^{10}R_{\varepsilon,j}(v),$$

where

$$R_{\varepsilon,1}(v)=\int_{\Omega_\varepsilon}\left(\kappa_0\big(U_\varepsilon^{(1)}(x)\big)-\sum_{i=1}^{3}\chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\left(\kappa_0\big(\omega_0^{(i)}(x_i)\big)+\varepsilon\kappa_0'\big(\omega_0^{(i)}(x_i)\big)\omega_1^{(i)}(x_i)\right)\right)v(x)\,dx,$$

$$R_{\varepsilon,2}(v)=\varepsilon\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}}\left(\kappa_i\big(U_\varepsilon^{(1)}(x)\big)-\chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\left(\kappa_i\big(\omega_0^{(i)}(x_i)\big)+\varepsilon\kappa_i'\big(\omega_0^{(i)}(x_i)\big)\omega_1^{(i)}(x_i)\right)\right)v(x)\,d\sigma_x,$$

$$R_{\varepsilon,3}(v)=-\int_{\Omega_\varepsilon}\left(f(x)-\sum_{i=1}^{3}\chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\left(f_0^{(i)}(x_i)+\varepsilon f_1^{(i)}\Big(x_i,\frac{\overline{x}_i}{\varepsilon}\Big)\right)\right)v(x)\,dx,$$

$$R_{\varepsilon,4}(v)=\sum_{i=1}^{3}\int_{\Gamma_\varepsilon^{(i)}}\left(1-\chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\right)\varphi_\varepsilon(x)\,v(x)\,d\sigma_x,$$

$$R_{\varepsilon,5}(v) = \varepsilon \sum_{i=1}^{3} \int_{\Gamma_\varepsilon^{(i)}} h_i'(x_i) \left(\frac{d\omega_0^{(i)}}{dx_i}(x_i) + \varepsilon \frac{d\omega_1^{(i)}}{dx_i}(x_i) \right) \left(1 - \frac{1}{\sqrt{1+\varepsilon^2 |h_i'(x_i)|^2}} \right) \chi_\ell^{(i)} \left(\frac{x_i}{\varepsilon^\alpha} \right) v(x)\, d\sigma_x,$$

$$R_{\varepsilon,6}(v) = -2\varepsilon^{-\alpha} \sum_{i=1}^{3} \int_{\Omega_\varepsilon} \frac{d\chi_\ell^{(i)}}{d\zeta_i}(\zeta_i) \Big|_{\zeta_i = \frac{x_i}{\varepsilon^\alpha}} \left(\frac{d\omega_0^{(i)}}{dx_i}(x_i) - \frac{d\omega_0^{(i)}}{dx_i}(0) + \varepsilon \frac{d\omega_1^{(i)}}{dx_i}(x_i) \right) v(x)\, dx,$$

$$R_{\varepsilon,7}(v) = -\varepsilon^{-2\alpha} \sum_{i=1}^{3} \int_{\Omega_\varepsilon} \frac{d^2\chi_\ell^{(i)}}{d\zeta_i^2}(\zeta_i) \Big|_{\zeta_i = \frac{x_i}{\varepsilon^\alpha}} \left(\omega_0^{(i)}(x_i) - \omega_0^{(i)}(0) - x_i \frac{d\omega_0^{(i)}}{dx_i}(0) + \varepsilon\omega_1^{(i)}(x_i) - \varepsilon\omega_1^{(i)}(0) \right) v(x)\, dx,$$

$$R_{\varepsilon,8}(v) = -\varepsilon^2 \sum_{i=1}^{3} \int_{I_\varepsilon^{(i)}} \int_{\Upsilon_i(x_i)} \chi_\ell^{(i)} \left(\frac{x_i}{\varepsilon^\alpha} \right) \nabla_{\overline{\xi}_i} u_2^{(i)}(x_i, \overline{\xi}_i) \cdot \nabla_{\overline{\xi}_i} v(x)\, d\overline{\xi}_i\, dx_i,$$

$$R_{\varepsilon,9}(v) = -\varepsilon^3 \sum_{i=1}^{3} \int_{I_\varepsilon^{(i)}} \int_{\Upsilon_i(x_i)} \chi_\ell^{(i)} \left(\frac{x_i}{\varepsilon^\alpha} \right) \nabla_{\overline{\xi}_i} u_3^{(i)}(x_i, \overline{\xi}_i) \cdot \nabla_{\overline{\xi}_i} v(x)\, d\overline{\xi}_i\, dx_i,$$

$$R_{\varepsilon,10}(v) = -\sum_{i=1}^{3} \int_{\Omega_\varepsilon} \Bigg(2\varepsilon^{-\alpha} \frac{d\chi_\ell^{(i)}}{d\zeta_i}(\zeta_i) \Big(\frac{\partial N_1}{\partial \xi_i}(\xi) - \frac{\partial G_1}{\partial \xi_i}(\xi) \Big) + \varepsilon^{1-2\alpha} \frac{d^2\chi_\ell^{(i)}}{d\zeta_i^2}(\zeta_i) \Big(N_1(\xi) - G_1(\xi) \Big) \Bigg) \Bigg|_{\zeta_i = \frac{x_i}{\varepsilon^\alpha},\, \xi = \frac{x}{\varepsilon}} v(x)\, dx.$$

Let us estimate the value R_ε. Using (5) and (6), we deduce the following estimates:

$$|R_{\varepsilon,1}(v)| \le \check{C} \sqrt{|\Xi^{(0)}| + 3\pi\ell \sum_{i=1}^{3} h_i^2(0)}\ \varepsilon^{1+\frac{\alpha}{2}} \, \|v\|_{L^2(\Omega_\varepsilon)}, \tag{52}$$

$$|R_{\varepsilon,j}(v)| \le \check{C} \sum_{i=1}^{3} \sqrt{6\pi\ell h_i(0)}\, \varepsilon^{1+\frac{\alpha}{2}} \, \|v\|_{H^1(\Omega_\varepsilon)}, \quad j = 2, 4, \tag{53}$$

$$|R_{\varepsilon,3}(v)| \le \check{C} \left(\sum_{i=1}^{3} \sqrt{\pi \max_{x_i \in I_i} h_i^2(x_i)}\, \varepsilon^2 + \sqrt{|\Xi^{(0)}| + 2\pi\ell \sum_{i=1}^{3} h_i^2(0)}\ \varepsilon^{1+\frac{\alpha}{2}} \right) \|v\|_{L^2(\Omega_\varepsilon)}, \tag{54}$$

$$|R_{\varepsilon,5}(v)| \le \check{C} \sum_{i=1}^{3} \sqrt{2\pi \max_{x_i \in I_i} h_i(x_i)}\, \varepsilon^3 \, \|v\|_{H^1(\Omega_\varepsilon)}, \tag{55}$$

$$|R_{\varepsilon,j}(v)| \le \check{C} \sum_{i=1}^{3} \sqrt{\pi\ell h_i^2(0)}\ \varepsilon^{1+\frac{\alpha}{2}} \, \|v\|_{L^2(\Omega_\varepsilon)}, \quad j = 6, 7, \tag{56}$$

$$|R_{\varepsilon,8}(v)| \le \check{C}\varepsilon^2 \, \|\nabla_x v\|_{L^2(\Omega_\varepsilon)}, \qquad |R_{\varepsilon,9}(v)| \le \check{C}\varepsilon^3 \, \|\nabla_x v\|_{L^2(\Omega_\varepsilon)}. \tag{57}$$

Due to the exponential decreasing of function $N_1 - G_1$ (see Remark 2.6) and the fact that the support of the derivative of $\chi_\ell^{(i)}$ belongs to the set $\{x_i : 2\ell\varepsilon^\alpha \le x_i \le 3\ell\varepsilon^\alpha\}$, we arrive at

$$|R_{\varepsilon,10}(v)| \le \check{C} \sum_{i=1}^{3} \sqrt{\pi\ell h_i^2(0)}\, \varepsilon^{1-\frac{\alpha}{2}} \exp\left(-\frac{2\ell}{\varepsilon^{1-\alpha}} \min_{i=1,2,3} \gamma_i \right) \|v\|_{L^2(\Omega_\varepsilon)}. \tag{58}$$

Subtracting the integral identity (3) from (49), we obtain

$$\int_{\Omega_\varepsilon} \nabla \left(U_\varepsilon^{(1)} - u_\varepsilon\right) \cdot \nabla v \, dx + \int_{\Omega_\varepsilon} \left(\kappa_0(U_\varepsilon^{(1)}) - \kappa_0(u_\varepsilon)\right) v \, dx$$

$$+ \varepsilon \sum_{i=1}^{3} \int_{\Gamma_\varepsilon^{(i)}} \left(\kappa_i(U_\varepsilon^{(1)}) - \kappa_i(u_\varepsilon)\right) v \, d\sigma_x = R_\varepsilon(v) \qquad \forall v \in \mathcal{H}_\varepsilon. \quad (59)$$

Now set $v = U_\varepsilon^{(1)} - u_\varepsilon$ in (59). Then, taking into account (2) and (52)–(58), we arrive at the inequality

$$\int_{\Omega_\varepsilon} \left|\nabla \left(U_\varepsilon^{(1)} - u_\varepsilon\right)\right|^2 dx \le C\varepsilon^{1+\frac{\alpha}{2}} \left\|U_\varepsilon^{(1)} - u_\varepsilon\right\|_{H^1(\Omega_\varepsilon)}, \quad (60)$$

whence thanks to (4) it follows (45). □

Corollary 3.2. *The differences between the solution u_ε of problem* (1) *and the sum*

$$U_\varepsilon^{(0)}(x) = \sum_{i=1}^{3} \chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\omega_0^{(i)}(x_i) + \left(1 - \sum_{i=1}^{3} \chi_\ell^{(i)}\left(\frac{x_i}{\varepsilon^\alpha}\right)\right)\omega_0^{(1)}(0), \quad x \in \Omega_\varepsilon$$

admit the following asymptotic estimates:

$$\| u_\varepsilon - U_\varepsilon^{(0)}\|_{H^1(\Omega_\varepsilon)} \le \widetilde{C}_0\, \varepsilon^{1+\frac{\alpha}{2}}, \qquad \| u_\varepsilon - U_\varepsilon^{(0)}\|_{L^2(\Omega_\varepsilon)} \le \widetilde{C}_0\, \varepsilon^{1+\frac{\alpha}{2}}, \quad (61)$$

where α is a fixed number from the interval $(\frac{2}{3}, 1)$.

In thin cylinders $\Omega_{\varepsilon,\alpha}^{(i)} := \Omega_\varepsilon^{(i)} \cap \{x \in \mathbb{R}^3 : \ x_i \in I_{\varepsilon,\alpha}^{(i)} := (3\ell\varepsilon^\alpha, 1)\}$, $i = 1,2,3$, the following estimates hold:

$$\| u_\varepsilon - \omega_0^{(i)}\|_{H^1(\Omega_{\varepsilon,\alpha}^{(i)})} \le \widetilde{C}_1\, \varepsilon^{1+\frac{\alpha}{2}}, \ \ i = 1,2,3, \quad (62)$$

where $\{\omega_0^{(i)}\}_{i=1}^3$ is the solution of the limit problem (33).

In the neighbourhood $\Omega_{\varepsilon,\ell}^{(0)} := \Omega_\varepsilon \cap \{x : \ x_i < 2\ell\varepsilon, \ i = 1,2,3\}$ of the aneurysm $\Omega_\varepsilon^{(0)}$, we get estimates

$$\| \nabla_x u_\varepsilon - \nabla_\xi N_1\|_{L^2(\Omega_{\varepsilon,\ell}^{(0)})} \le \| u_\varepsilon - \omega_0^{(i)}(0) - \varepsilon N_1\|_{H^1(\Omega_{\varepsilon,\ell}^{(0)})} \le \widetilde{C}_4\, \varepsilon^{1+\frac{\alpha}{2}}, \quad (63)$$

Proof. Denote by $\chi_{\ell,\alpha,\varepsilon}^{(i)}(\cdot) := \chi_\ell^{(i)}(\frac{\cdot}{\varepsilon^\alpha})$ (the function $\chi_\ell^{(i)}$ is determined in (43)). Using the smoothness of the functions $\{\omega_1^{(i)}\}$ and the exponential decay of the functions $\{N_1 - G_1\}$, $i = 1,2,3$, at infinity, we deduce the inequalities (61) from estimate (45):

$$\left\| u_\varepsilon - U_\varepsilon^{(0)}\right\|_{H^1(\Omega_\varepsilon)} \le \left\| u_\varepsilon - U_\varepsilon^{(1)}\right\|_{H^1(\Omega_\varepsilon)} + \varepsilon \left\| \sum_{i=1}^{3} \chi_{\ell,\alpha,\varepsilon}^{(i)}\, \omega_1^{(i)} + \left(1 - \sum_{i=1}^{3} \chi_{\ell,\alpha,\varepsilon}^{(i)}\right) N_1 \right\|_{H^1(\Omega_\varepsilon)}$$

$$\le C_1\, \varepsilon^{1+\frac{\alpha}{2}} + \varepsilon \sum_{i=1}^{3} \left\| \left(\chi_{\ell,\alpha,\varepsilon}^{(i)}\omega_1^{(i)} + (1 - \chi_{\ell,\alpha,\varepsilon}^{(i)}) N_1\right) \right\|_{H^1(\Omega_\varepsilon^{(i)})} + \varepsilon \, \|N_1\|_{H^1(\Omega_\varepsilon^{(0)})}$$

$$\le C_1\, \varepsilon^{1+\frac{\alpha}{2}} + \sum_{i=1}^{3} \left\|\left(1 - \chi_{\ell,\alpha,\varepsilon}^{(i)}\right) x_i \frac{d\omega_0^{(i)}}{dx_i}(0)\right\|_{H^1(\Omega_\varepsilon^{(i)})} + \varepsilon \sum_{i=1}^{3} \left\|\left(1 - \chi_{\ell,\alpha,\varepsilon}^{(i)}\right)\left(\omega_1^{(i)}(0) - \omega_1^{(i)}\right)\right\|_{H^1(\Omega_\varepsilon^{(i)})}$$

$$+\varepsilon \sum_{i=1}^{3} \left\| \omega_1^{(i)} \right\|_{H^1(\Omega_\varepsilon^{(i)})} + \varepsilon \sum_{i=1}^{3} \left\| \left(1 - \chi_{\ell,\alpha,\varepsilon}^{(i)}\right)(N_1 - G_1)\right\|_{H^1(\Omega_\varepsilon^{(i)})} + \varepsilon^{\frac{3}{2}} \, \|N_1\|_{H^1(\Xi^{(0)})} \le \widetilde{C}_0\, \varepsilon^{1+\frac{\alpha}{2}}.$$

With the help of estimate (45), we deduce

$$\left\| u_\varepsilon - \omega_0^{(i)}\right\|_{H^1(\Omega_{\varepsilon,\alpha}^{(i)})} \le \left\| u_\varepsilon - U_\varepsilon^{(1)}\right\|_{H^1(\Omega_\varepsilon)} + \varepsilon \left\| \omega_1^{(i)}\right\|_{H^1(\Omega_{\varepsilon,\alpha}^{(i)})} \le \widetilde{C}_2\, \varepsilon^{1+\frac{\alpha}{2}},$$

whence we get (62).

The energetic estimate (63) in a neighbourhood of the aneurysm $\Omega_\varepsilon^{(0)}$ follows directly from (45). □

Using the Cauchy-Buniakovskii-Schwarz inequality and the continuously embedding of the space $H^1(I^{(i)}_{\varepsilon,\alpha})$ in $C\big(\overline{I^{(i)}_{\varepsilon,\alpha}}\big)$, from (62) we get the following corollary.

Corollary 3.3. *If* $h_i(x_i) \equiv h_i \equiv const,\ (i = 1,2,3)$, *then*

$$\| E^{(i)}_\varepsilon(u_\varepsilon) - \omega^{(i)}_0 \|_{H^1(I^{(i)}_{\varepsilon,\alpha})} \le \widetilde{C}_2\, \varepsilon^{\frac{\alpha}{2}}, \tag{64}$$

$$\max_{x_i \in \overline{I^{(i)}_{\varepsilon,\alpha}}} \Big| E^{(i)}_\varepsilon(u_\varepsilon)(x_i) - \omega^{(i)}_0(x_i) \Big| \le \widetilde{C}_3\, \varepsilon^{\frac{\alpha}{2}}, \quad i = 1,2,3, \tag{65}$$

where

$$\big(E^{(i)}_\varepsilon u_\varepsilon\big)(x_i) = \frac{1}{\pi \varepsilon^2 h_i^2} \int\limits_{\Upsilon^{(i)}_\varepsilon(0)} u_\varepsilon(x)\, d\overline{x}_i, \quad i = 1,2,3.$$

4 Conclusions

1. An important problem of existing multi-scale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that has been applied to the analysis of the efficiency of a multi-scale method. In our paper, we have done this for the solution to the problem (1).

The results of Theorem 3.1 and Corollary 3.2 showed the possibility to replace the complex boundary-value problem (1) with the corresponding one-dimensional boundary-value problem (33) in the graph $I = \cup_{i=1}^3 I_i$ with sufficient accuracy measured by the parameter ε characterizing the thickness and the local geometrical irregularity. In this regard, the uniform pointwise estimates (65), which are important for applied problems, also confirm this conclusion.

2. In [16], the authors considered the boundary-value problem

$$\begin{cases} \Delta u_\varepsilon(x) = f(x_1) & \text{in } Q_\varepsilon, \\ \partial_{\boldsymbol{\nu}} u_\varepsilon(x) = 0 & \text{on the lateral side of } Q_\varepsilon, \\ u_\varepsilon(x) = \pm t & \text{as } x_1 = \pm a, \end{cases}$$

where Q_ε is a thin $2D$ rod with a small local geometric irregularity in the middle.

The energetic estimate (61) partly confirms the first formal result of [16] (see p. 296) that the local geometric irregularity of the analyzed structure does not significantly affect the global-level properties of the framework, which are described by the limit problem (33) and its solution $\{\omega^{(i)}_0\}_{i=1}^3$ (the first terms of the asymptotics). But thanks to estimates (45) and (63) it has become possible to identify the impact of the geometric irregularity and material characteristics of the aneurysm on the global level through the second terms $\{\omega^{(i)}_1\}_{i=1}^3$ of the regular asymptotics (8). They depend on the constants d_1^*, $\delta_1^{(2)}$ and $\delta_1^{(3)}$ that take into account all those factors (see (38) and (42)). This conclusion does not coincide with the second main result of [16] (see p. 296) that "*the joints of normal type manifest themselves on the local level only*".

In addition, in [16] the authors stated that the main idea of their approach "*is to use a local perturbation corrector of the form* $\varepsilon N(\mathbf{x}/\varepsilon)\frac{du_0}{dx_1}$ *with the condition that the function* $N(\mathbf{y})$ *is localized near the joint* ", i.e., $N(\mathbf{y}) \to 0$ as $|\mathbf{y}| \to +\infty$, and the main assumption of this approach is that $\nabla_y N \in L_1(Q_\infty)$.

As shown the coefficients $\{N_k\}$ of the inner asymptotics (9) behave as polynomials at infinity and do not decrease exponentially (see (40)). Therefore, they influence directly the terms of the regular asymptotics beginning with the second terms. Thus, the main assumption made in [16] is not correct.

3. From the first estimate in (61) it follows that the gradient ∇u_ε is equivalent to $\{\frac{d\omega^{(i)}_0}{dx_i}\}_{i=1}^3$ in the L^2-norm over whole junction Ω_ε as $\varepsilon \to 0$. Obviously, this estimate is not informative in the neighbourhood $\Omega^{(0)}_{\varepsilon,\ell}$ of the aneurysm $\Omega^{(0)}_\varepsilon$.

Thanks to estimates (45) and (63), we get the approximation of the gradient (flux) of the solution both in the curvilinear cylinders $\Omega^{(i)}_{\varepsilon,\alpha}$, $i = 1, 2, 3$:

$$\nabla u_\varepsilon(x) \sim \frac{d\omega_0^{(i)}}{dx_i}(x_i) + \varepsilon \frac{d\omega_1^{(i)}}{dx_i}(x_i) \quad \text{as} \quad \varepsilon \to 0$$

and in the neighbourhood $\Omega^{(0)}_{\varepsilon,\ell}$ of the aneurysm:

$$\nabla u_\varepsilon(x) \sim \nabla_\xi \big(N_1(\xi)\big)\Big|_{\xi=\frac{x}{\varepsilon}} \quad \text{as} \quad \varepsilon \to 0.$$

4. We hope that this asymptotic approach can be applied to the study of the blood flow in vessels with a local geometric heterogeneity what we are going to do in our further studies. Nevertheless, the results obtained in this article can be considered as the first steps in this direction, since it is known that for the incompressible flow it is possible in some cases to couple pressure and velocity through the Poisson equation ($\kappa_0 \equiv 0$) for pressure. Also the pressure Poisson equation with Neumann boundary conditions is encountered in the time-discretization of the incompressible Navier-Stokes equations.

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