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Numerical methods for the multiplicative partial differential equations<https://doi.org/10.1515/math-2017-0113>

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Abstract: We propose the multiplicative explicit Euler, multiplicative implicit Euler, and multiplicative Crank-Nicolson algorithms for the numerical solutions of the multiplicative partial differential equation. We also consider the truncation error estimation for the numerical methods. The stability of the algorithms is analyzed by using the matrix form. The result reveals that the proposed numerical methods are effective and convenient.

Keywords: Finite difference methods, Multiplicative calculus, Partial differential equation, Stability analysis

MSC: 65M06, 35K05

1 Introduction

Mathematical models sometimes described by differential equations are developed to understand physical phenomena. Different types of calculus are proposed for approaching the solutions of these differential equations rather than classical or Newtonian calculus. Additionally, these types of calculus simplify the analysis in some cases. Bashirov et al. [1] showed that some ordinary differential equations in the multiplicative calculus are more suitable for scrutinizing some problems in economics, biology, etc. Englehardt, Swartout and Loewinstine [2] used this calculus to investigate the distribution of microbial counts in environmental sciences. Moreover, Filip and Piatecki [3] applied the multiplicative calculus to the growth in economics. It was also used for the exponential type signals by Ozyapici and Bilgehan [4].

In this paper, we consider the multiplicative calculus, defined by Grossman and Katz [5]. The basic operation for so-called the multiplicative derivative of a positive function f on an open interval A of the real line \mathbb{R} ,

$$f^*(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} \quad (1)$$

is defined in [5], and some qualitative properties of the operation are revealed in Bashirov, Kurpinar and Özyapıcı [6]. In addition, it was shown in [6] that if $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a positive definite function and its classical n th order derivative $f^{(n)}$ with respect to x exists, then the n th multiplicative derivative of f also exists and

$$\ln f^{*(n)}(x) = (\ln f)^{(n)}(x). \quad (2)$$

Based on (2), the function f is said to be * -differentiable if it is differentiable in the Newtonian calculus.

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However, we are interested in the initial value problem for the multiplicative heat equation,

$$\begin{aligned} u_t^* &= \Delta^* u, & \text{for } (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ u(x, 0) &= f(x), & \text{for } x \in \mathbb{R}^d, \end{aligned} \quad (3)$$

where the multiplicative Laplacian is defined as $\Delta^* = \exp \circ \Delta \circ \ln$. Here, we remark in the commutative case that this implies $\Delta^* u = \partial_{x^1 x^1}^* u \dots \partial_{x^d x^d}^* u$. The equation in (3) is a special case of a so-called pseudo-linear scale space in the Newtonian calculus [7]. Indeed if we put (2) into the equation in (3), we obtain the special case of the following equation

$$\partial_t u = \Delta u + \mu \|\nabla u\|^2, \quad (4)$$

where $\mu := (\ln \gamma'(u))'$ with $\gamma'(u) > 0$. More precisely, when we set $\gamma'(u) = \frac{1}{u}$, we obtain the following nonlinear equation

$$\partial_t u = \Delta u - \frac{1}{u} \|\nabla u\|^2, \quad (5)$$

which coincides the equation in (3). The solution of (5) with the initial condition in (3) is nontrivial. On the contrary, the solution of the problem for the multiplicative heat equation (3) is written explicitly as

$$u(x, t) = \exp \left(\int_{\mathbb{R}^d} \frac{1}{\sqrt{(4\pi t)^d}} \exp \left(-\frac{\|x - y\|^2}{4t} \right) (\ln f)(y) dy \right), \quad (6)$$

in which $\ln u$ is the solution of the heat equation with the initial condition $\ln f$ in the Newtonian calculus. So, the multiplicative calculus provides an important advantage to find the solutions of some differential equations.

On the other hand, the numerical solutions for the multiplicative differential equations are studied extensively. The multiplicative algorithms of Runge-Kutta methods for the Volterra's multiplicative differential equations were showed by Aniszewska [8]. Furthermore, Riza, Özayapıcı and Mısırlı [9] presented the multiplicative finite difference algorithms and gave some applications by using these methods. In addition to iterative and discretization methods, multistep methods are also studied. The multiplicative algorithms of Adams-Basforth and Adams-Moulton methods were investigated by Mısırlı and Gurefe [10].

Turning back to the initial value problem (3), Florack and Assen [11] used this equation in (3) for the image analysis. Our aim in the present article is to develop the Crank-Nicolson, explicit Euler and implicit Euler algorithms in the multiplicative sense for the numerical approximation of the multiplicative heat equation.

This paper is organized as follows. In Section 2, we give the multiplicative finite difference formulas and develop the explicit Euler, implicit Euler and Crank-Nicolson methods in the multiplicative case. Truncation errors of these methods are also discussed. The stability for each method is analyzed in Section 2. The last section is devoted to a numerical example to illustrate our result.

Throughout this article, the multiplicative partial derivative is just denoted by

$$u_{x^i t^{m-i}}^{*(m)}(x, t) = \frac{\partial^{*(m)} u(x, t)}{\partial x^i \partial t^{m-i}}.$$

2 Numerical methods

2.1 Derivation of the multiplicative methods

We give the multiplicative Taylor theorem for two variables given in [6] to derive the multiplicative finite differences scheme.

Theorem 2.1 (Multiplicative Taylor theorem). *Let $(a, b) \times (c, d) \subset \mathbb{R}^2$. Assume that $u : (a, b) \times (c, d) \rightarrow \mathbb{R}$ has all partial *derivatives of order $n + 1$ times on $(a, b) \times (c, d)$. If $x_0 \in [a, b]$, and $t_0 \in [c, d]$ then for every $x \in [a, b]$*

and $t \in [c, d]$ with $x \neq x_0$, and $t \neq t_0$, there exist $x_1 \in (x, x_0)$ and $t_1 \in (t, t_0)$ such that

$$u(x, t) = \prod_{m=0}^n \prod_{i=0}^m \left(u_{x^i t^{m-i}}^*(x_0, t_0) \right)^{\frac{h^i k^{m-i}}{i!(m-i)!}} \prod_{i=0}^{n+1} \left(u_{x^i t^{n+1-i}}^*(x_1, t_1) \right)^{\frac{h^i k^{n+1-i}}{i!(n+1-i)!}}, \quad (7)$$

where $h = x - x_0$ and $k = t - t_0$.

From (7), the multiplicative Taylor expansion of $u(x + \epsilon h, t + k)$ and $u(x + \epsilon h, t)$ about the virtual node $(x, t + \frac{k}{2})$ for $\epsilon \in \{-1, 0, 1\}$ are written as follows:

$$\begin{aligned} u(x + \epsilon h, t + k) &= u(x, t + \frac{k}{2}) \left[u_x^*(x, t + \frac{k}{2}) \right]^{\epsilon h} \left[u_t^*(x, t + \frac{k}{2}) \right]^{\frac{k}{2}} \\ &\quad \times \left[u_{x^2}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(\epsilon h)^2}{2!}} \left[u_{t^2}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(k/2)^2}{2!}} \\ &\quad \times \left[u_{xt}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(2\epsilon h(k/2))}{2!}} \left[u_{x^3}^{***}(x, t + \frac{k}{2}) \right]^{\frac{(\epsilon h)^3}{3!}} \\ &\quad \times \left[u_{x^2 t}^{***}(x, t + \frac{k}{2}) \right]^{\frac{3(k/2)(\epsilon h)^2}{3!}} \left[u_{xt^2}^{***}(x, t + \frac{k}{2}) \right]^{\frac{3(k/2)^2 \epsilon h}{3!}} \\ &\quad \times \left[u_{t^3}^{***}(x, t + \frac{k}{2}) \right]^{\frac{(k/2)^3}{3!}} \dots, \end{aligned} \quad (8)$$

$$\begin{aligned} u(x + \epsilon h, t) &= u(x, t + \frac{k}{2}) \left[u_x^*(x, t + \frac{k}{2}) \right]^{\epsilon h} \left[u_t^*(x, t + \frac{k}{2}) \right]^{\frac{-k}{2}} \\ &\quad \times \left[u_{x^2}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(\epsilon h)^2}{2!}} \left[u_{t^2}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(-k/2)^2}{2!}} \\ &\quad \times \left[u_{xt}^{**}(x, t + \frac{k}{2}) \right]^{\frac{(2\epsilon h(-k/2))}{2!}} \left[u_{x^3}^{***}(x, t + \frac{k}{2}) \right]^{\frac{(\epsilon h)^3}{3!}} \\ &\quad \times \left[u_{x^2 t}^{***}(x, t + \frac{k}{2}) \right]^{\frac{3(-k/2)(\epsilon h)^2}{3!}} \left[u_{xt^2}^{***}(x, t + \frac{k}{2}) \right]^{\frac{3(-k/2)^2 \epsilon h}{3!}} \\ &\quad \times \left[u_{t^3}^{***}(x, t + \frac{k}{2}) \right]^{\frac{(-k/2)^3}{3!}} \dots \end{aligned} \quad (9)$$

The approximation of the first order multiplicative partial derivative with respect to t is obtained by dividing (8) to (9) with $\epsilon = 0$

$$u_t^*(x, t + \frac{k}{2}) \approx \left[\frac{u(x, t + k)}{u(x, t)} \right]^{\frac{k}{2}}. \quad (10)$$

Analogously, to approximate the second order multiplicative partial derivative $u_{x^2}^{**}(x, t + \frac{k}{2})$, we use the geometric mean of the second multiplicative centered differences for $u_{x^2}^{**}(x, t)$ and $u_{x^2}^{**}(x, t + k)$ in (8),

$$u_{x^2}^{**}(x, t + \frac{k}{2}) \approx \left[\frac{u(x + h, t + k)u(x - h, t + k)}{u(x, t + k)^2} \frac{u(x + h, t)u(x - h, t)}{u(x, t)^2} \right]^{\frac{1}{2h^2}}. \quad (11)$$

We consider the initial value problem (3) with some boundary conditions in one space dimension,

$$\begin{aligned} u_t^* &= u_{x^2}^{**}, & \text{for } 0 \leq x \leq L, \quad t \geq 0, \\ u(x, 0) &= f(x), & \text{for } 0 \leq x \leq L, \\ u(0, t) &= g_0(t), \quad u(L, t) = g_1(t) & \text{for } t \geq 0 \end{aligned} \quad (12)$$

We remark that the length between two consecutive points need not be equal but for simplicity we denote $x_i = ih$, for $i = 0, 1, 2, \dots, n$ with the space step size $h = \frac{L}{n}$ and $t_j = jk$, for $j = 0, 1, \dots, m$ with the time step size $k = \frac{t_{max}}{m}$ where t_{max} is the maximum time for the desired solution.

The multiplicative methods for (12) is proposed in the form:

$$\left(\frac{u_{i,j+1}}{u_{i,j}} \right)^{\frac{1}{k}} = \left[\left(\frac{u_{i+1,j}u_{i-1,j}}{(u_{i,j})^2} \right)^{\frac{1-\theta}{h^2}} \left(\frac{u_{i+1,j+1}u_{i-1,j+1}}{(u_{i,j+1})^2} \right)^{\frac{\theta}{h^2}} \right], \quad (13)$$

where $u_{i,j} = u(x_i, t_j)$, and $\theta \in [0, 1]$. If we set $r = \frac{k}{h^2}$ which is called parabolic mesh ratio, then the finite difference equation (13) is rewritten as

$$(u_{i+1,j+1})^{-r\theta} (u_{i,j+1})^{1+2r\theta} (u_{i-1,j+1})^{-r\theta} = (u_{i+1,j})^{r(1-\theta)} (u_{i,j})^{1-2r(1-\theta)} (u_{i-1,j})^{r(1-\theta)}. \quad (14)$$

It is noted that the numerical scheme (14) for $\theta = 0$ and $\theta = 1$ yields the multiplicative explicit Euler method and multiplicative implicit Euler method respectively. When $\theta = 1/2$, (14) yields the multiplicative Crank-Nicolson method. If we set $\vec{U}^j = [\ln u_{1,j}, \ln u_{2,j}, \dots, \ln u_{n-1,j}]^T$, we rewrite (14) in the matrix form

$$(I - r\theta A) \vec{U}^{j+1} = (I + r(1-\theta)A) \vec{U}^j + \vec{b}, \quad (15)$$

where I is the unit matrix,

$$A = \begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad \vec{b} = r \begin{bmatrix} \ln(g_0(t_m)) + \ln(g_0(t_{m+1})) \\ 0 \\ \vdots \\ 0 \\ \ln(g_1(t_m)) + \ln(g_1(t_{m+1})) \end{bmatrix}.$$

2.2 Truncation error estimation

Recalling (2) that $u(x, t)$ must be a positive function and we may assume $u(x, t) = \exp(y(x, t))$, where $y(x, t)$ is a real-valued function (see e.g. [9, 10]), then the partial multiplicative derivatives of $u(x, t)$ can be written in terms of partial derivatives of $y(x, t)$,

$$u_{x^n t^m}^{*(n+m)}(x, t) = \exp \left\{ \frac{\partial^{(n+m)} y(x, t)}{\partial x^n \partial t^m} \right\}. \quad (16)$$

Substituting the expressions (8) and (9) with $\epsilon = 0$ into the left-hand side of (13), we obtain

$$\text{Error} \left(\left(\frac{u_{i,j+1}}{u_{i,j}} \right)^{\frac{1}{k}} \right) = \exp \left\{ \frac{k^2}{2^3 3!} \frac{\partial^{(3)} y(x, t)}{\partial t^3} \right\}. \quad (17)$$

Besides, when we substitute the expressions (8) and (9) into the right-hand side of (13), we get

$$\left[\delta_x^{*2} u_{i,j} \right]^{\frac{1-\theta}{h^2}} \left[\delta_x^{*2} u_{i,j+1} \right]^{\frac{\theta}{h^2}} = u_{x^2 t}^{**} \left(u_{x^2 t}^{*(3)} \right)^{k(\theta-1/2)} \left(u_{x^4}^{*(4)} \right)^{h^2/12} \left(u_{x^2 t^2}^{*(4)} \right)^{\frac{\theta(k/2)+(1-\theta)(k/2)^2}{2}} \dots, \quad (18)$$

where we set $\delta_x^{*2} u_{i,j} = u_{i+1,j} u_{i-1,j} (u_{i,j})^{-2}$. By using the heat equation in (12), we write

$$u_{x^2 t}^{*(3)} = \frac{\partial^{**}}{\partial x^2} u_t^* = u_{x^4}^{*(4)}. \quad (19)$$

Thus, from (19), we obtain

$$\text{Error} \left(\left[\delta_x^{*2} u_{i,j} \right]^{\frac{1-\theta}{h^2}} \left[\delta_x^{*2} u_{i,j+1} \right]^{\frac{\theta}{h^2}} \right) = \exp \left\{ \left(k \left(\theta - \frac{1}{2} \right) + \frac{h^2}{12} \right) \frac{\partial y^{(4)}(x, t)}{\partial x^4} \right\}. \quad (20)$$

2.3 Stability analysis

The stability analysis arises in all problems where time is an independent variable in numerical algorithms of partial differential equations. Since the determinant of the matrix $(2I - rA) \neq 0$, we rewrite (16) with $\tilde{b} = 0$,

$$\vec{U}^{j+1} = (I - r\theta A)^{-1}(I + r(1-\theta)A)\vec{U}^j. \quad (21)$$

We remark that the inhomogeneous part does not alter the stability condition. Therefore, we only consider the homogeneous part in order to find the stability condition by analyzing the eigenvalues of $(I - r\theta A)^{-1}(I + r(1-\theta)A)$ for the numerical scheme. If the magnitudes of all eigenvalues do not exceed 1, it is said that the numerical scheme is stable.

As indicated above for the stability of the numerical scheme, we require that the eigenvalues satisfy

$$\left| \frac{1 - 4r(1-\theta) \sin^2 \left(\frac{\pi j}{2n} \right)}{1 + 4r\theta \sin^2 \left(\frac{\pi j}{2n} \right)} \right| \leq 1, \quad j = 1, 2, \dots, n-1. \quad (22)$$

When $\theta = 1/2$, the numerical scheme (21) becomes multiplicative Crank-Nicolson method. Then, it is clearly seen that the condition (22) is satisfied for all $j = 1, 2, \dots, n-1$, without any restriction on $r > 0$. The numerical scheme for so called multiplicative implicit Euler method is also unconditionally stable for $r > 0$ while $\theta = 1$. From (22), we obtain the following inequality,

$$r(1-2\theta) \sin^2 \left(\frac{\pi j}{2n} \right) \leq \frac{1}{2}. \quad (23)$$

Concerning the inequality (23) for $\theta = 0$, the numerical scheme denoted by multiplicative explicit Euler method is stable under the condition $r \leq \frac{1}{2}$.

3 Application

In this section, the multiplicative numerical methods are applied to solve the multiplicative heat equation with the initial and the boundary conditions in one space dimension. In order to illustrate the accuracy of the numerical results, we compare them with the exact solution. We have used Fortran programming language for the numerical results.

Example. We consider the initial boundary value problem

$$\begin{aligned} u_t^* &= u_{x^2}^{**}, \quad \text{for } 0 \leq x \leq 1, \quad t \geq 0, \\ u(x, 0) &= \exp(\sin(\pi x)), \quad \text{for } 0 \leq x \leq 1, \\ u(0, t) &= u(1, t) = 1 \quad \text{for } t \geq 0. \end{aligned} \quad (24)$$

By using (6), the analytical solution of (24) is written as,

$$u(x, t) = \exp \left(\sin(\pi x) e^{-\pi^2 t} \right). \quad (25)$$

The multiplicative initial boundary value problem is also solved by using the multiplicative explicit Euler, multiplicative implicit Euler and multiplicative Crank-Nicolson methods. These methods are performed with $r = 1/2$, for which $k = 0.0002$ and $h = 0.01$.

The results of the algorithm are presented for $t = 0.05$ in Table 1 and for $t = 1$ in Table 2 where the solutions obtained by using M-E-E (the multiplicative explicit Euler) method, M-I-E (the multiplicative implicit Euler) method, and M-C-N (the multiplicative Crank-Nicolson) method are compared with the analytical solution. The range of the relative errors of the multiplicative methods are also showed for $t = 0.05$ and $t = 1$ in Table 1 and Table 2, respectively.

Table 1. Relative error estimation of the numerical results for $t = 0.05$

x	Exact Result	Method	Result	Error(%)
0.1	1.2076233624	M-E-E	1.2075493252	7.403×10^{-7}
		M-I-E	1.2077711429	1.477×10^{-6}
		M-C-N	1.2076603114	3.694×10^{-7}
0.2	1.4316702015	M-E-E	1.4315032517	1.669×10^{-6}
		M-I-E	1.4320034658	3.332×10^{-6}
		M-C-N	1.4317535228	8.332×10^{-7}
0.3	1.6387000539	M-E-E	1.6384370442	2.630×10^{-6}
		M-I-E	1.6392251071	5.250×10^{-6}
		M-C-N	1.6388313212	1.312×10^{-6}
0.4	1.7871427672	M-E-E	1.7868055777	3.371×10^{-6}
		M-I-E	1.7878159361	6.731×10^{-6}
		M-C-N	1.7873110610	1.682×10^{-6}
0.5	1.8413482083	M-E-E	1.8409829146	3.652×10^{-6}
		M-I-E	1.8420774956	7.292×10^{-6}
		M-C-N	1.8415305305	1.823×10^{-6}

Table 2. Relative error estimation of the numerical results for $t = 1$

x	Exact Result	Method	Result	Error(%)
0.1	1.0000159834	M-E-E	1.0000158799	1.035×10^{-9}
		M-I-E	1.0000161921	2.086×10^{-9}
		M-C-N	1.0000160353	5.192×10^{-10}
0.2	1.0000304025	M-E-E	1.0000302055	1.970×10^{-9}
		M-I-E	1.0000307994	3.968×10^{-9}
		M-C-N	1.0000305013	9.876×10^{-10}
0.3	1.0000418458	M-E-E	1.0000415746	2.711×10^{-9}
		M-I-E	1.0000423920	5.462×10^{-9}
		M-C-N	1.0000419817	1.359×10^{-9}
0.4	1.0000491928	M-E-E	1.0000488741	3.187×10^{-9}
		M-I-E	1.0000498349	6.421×10^{-9}
		M-C-N	1.0000493526	1.598×10^{-9}
0.5	1.0000517245	M-E-E	1.0000513893	3.351×10^{-9}
		M-I-E	1.0000523996	6.751×10^{-9}
		M-C-N	1.0000518925	1.680×10^{-9}

4 Conclusion

In this article, explicit Euler method, implicit Euler method and Crank-Nicolson method based on multiplicative calculus have been developed for numerical solutions of the multiplicative heat equation with the initial and boundary conditions. The algorithms are tested and the numerical results compared with the exact solution are quite satisfactory. The present methods with some modifications might be applied to many multiplicative partial differential equations arising in engineering and sciences.

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