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Liping Sun and Wende Liu*

Hom-Lie superalgebra structures on exceptional simple Lie superalgebras of vector fields

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Abstract: According to the classification by Kac, there are eight Cartan series and five exceptional Lie superalgebras in infinite-dimensional simple linearly compact Lie superalgebras of vector fields. In this paper, the Hom-Lie superalgebra structures on the five exceptional Lie superalgebras of vector fields are studied. By making use of the $\mathbb{Z}$-grading structures and the transitivity, we prove that there is only the trivial Hom-Lie superalgebra structures on exceptional simple Lie superalgebras. This is achieved by studying the Hom-Lie superalgebra structures only on their 0-th and (-1)-th $\mathbb{Z}$-components.

Keywords: Exceptional Lie superalgebra, Hom-Lie superalgebra structure, Gradation, Transitivity

MSC: 17B40, 17B65, 17B66

1 Introduction

The motivations to study Hom-Lie algebras and related algebraic structures come from physics and deformations of Lie algebras, in particular, Lie algebras of vector fields. Recently, this kind of investigation has become rather popular [1–10], partially due to the prospect of having a general framework in which one can produce many types of natural deformations of Lie algebras, in particular, q-deformations. The notion of Hom-Lie algebras was introduced by J. T. Hartwig, D. Larsson and S. D. Silvestrov in [1] to describe the structures on certain deformations of Witt algebras and Virasoro algebras, which are applied widely in the theoretical physics [11–14]. Later, W. J. Xie and Q. Q. Jin gave a description of Hom-Lie algebra structures on semi-simple Lie algebras [9, 10]. In 2010, F. Ammar and A. Makhlouf generalized Hom-Lie algebras to Hom-Lie superalgebras [15]. In 2013 and 2015, B. T. Cao, L. Luo, J. X. Yuan and W. D. Liu studied the Hom-structures on finite dimensional simple Lie superalgebras [16, 17].

In the well-known paper [18], Kac classified the infinite-dimensional simple linearly compact Lie superalgebras of vector fields, including eight Cartan series and five exceptional simple Lie superalgebras. In 2014, the Hom-Lie superalgebra structures on the eight infinite-dimensional Cartan series were investigated by J. X. Yuan, L. P. Sun and W. D. Liu [19]. In the present paper, we characterize the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields. Taking advantage of the $\mathbb{Z}$-grading structures and the transitivity, we analyse the Hom-Lie superalgebra structures through computing the Hom-Lie superalgebra structures on the 0-th and (-1)-th $\mathbb{Z}$-components. The main result of this study shows that the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields must be trivial.

Liping Sun: Department of Applied sciences, Harbin University of Science and Technology, Harbin 150080, China
***Corresponding Author: Wende Liu:** School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China, E-mail: wendeliu@ustc.edu.cn

2 Preliminaries

Throughout the paper, we will use the following notations. $\mathbb{F}$ denotes an algebraically closed field of characteristic zero. $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of two-elements. The symbols $|x|$ and zd(x) denote the $\mathbb{Z}_2$-degree of a $\mathbb{Z}_2$-homogeneous element x and the $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element x, respectively. Notation $\langle v_1, \dots, v_k \rangle$ is used to represent the linear space spanned by $v_1, \dots, v_k$ over $\mathbb{F}$. Now, let us review the five infinite dimensional exceptional simple Lie superalgebras and their $\mathbb{Z}$-grading structures adopted in this paper. More detailed descriptions can be found in [18, 20].

We construct a Weisfeiler filtration of L by using open subspaces:

$$L = L_{-h} \supset L_{-h+1} \supset \dots \supset L_0 \supset L_1 \supset \dots$$

where L is a simple linearly compact Lie superalgebra. Let $\mathfrak{g}_i = L_i / L_{i+1}$, $\mathrm{Gr}L = \oplus_{i=-h}^{\infty} \mathfrak{g}_i$. In [18], all possible choices for nonpositive part $\mathfrak{g}_{\leq 0} = \oplus_{i=-h}^{0} \mathfrak{g}_i$ of the associated graded Lie superalgebra of L were derived. The five exceptional simple Lie superalgebras are as follows (the subalgebra $\mathfrak{g}_{\leq 0}$ is written below as the $h+1$-tuple$(\mathfrak{g}_{-h}, \mathfrak{g}_{-h+1}, \dots \mathfrak{g}_{-1}, \mathfrak{g}_0)$):

$$\begin{aligned}
E(5,10):&\ (\mathbb{F}^{5^*},\ \wedge^2(\mathbb{F}^5),\ \mathfrak{sl}(5))\\
E(3,6):&\ (\mathbb{F}^{3^*},\ \mathbb{F}^3 \otimes \mathbb{F}^2,\ \mathfrak{gl}(3) \oplus \mathfrak{sl}(2))\\
E(3,8):&\ (\mathbb{F}^2,\ \mathbb{F}^{3^*},\ \mathbb{F}^3 \otimes \mathbb{F}^2,\ \mathfrak{gl}(3) \oplus \mathfrak{sl}(2))\\
E(1,6):&\ (\mathbb{F},\ \mathbb{F}^n,\ \mathfrak{cso}(n), n \geq 1, n \neq 2)\\
E(4,4):&\ (\mathbb{F}^{4|4},\ \widehat{P}(4))
\end{aligned}$$

As we know, the five exceptional simple Lie superalgebras are transitive and irreducible. In particular, $\mathfrak{g}_{-i} = \mathfrak{g}_{-1}^{i}$ for $i \geq 1$ and the transitivity will be used frequently in this paper:

$$\textbf{transitivity}: \text{ if } x \in \mathfrak{g}_i, i \geq 0, \text{ then } [x, \mathfrak{g}_{-1}] = 0 \text{ implies } x = 0. \tag{1}$$

To study the Hom-Lie superalgebras structure, we recall the following definitions [16].

A Hom-Lie superalgbra is a triple $(\mathfrak{g}, [\cdot,\cdot], \sigma)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g}$, an even bilinear mapping $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and an even linear mapping: $\sigma : \mathfrak{g} \to \mathfrak{g}$ satisfying:

$$\sigma[x, y] = [\sigma(x), \sigma(y)], \tag{2}$$

$$[x, y] = -(-1)^{|x||y|}[y, x], \tag{3}$$

$$(-1)^{|x||z|}[\sigma(x), [y, z]] + (-1)^{|y||x|}[\sigma(y), [z, x]] + (-1)^{|z||y|}[\sigma(z), [x, y]] = 0. \tag{4}$$

for all homogeneous elements $x, y, z \in \mathfrak{g}$.

An even linear mapping σ on a Lie superalgebra $\mathfrak{g}$ is called a Hom-Lie superalgebra structure on $\mathfrak{g}$ if $(\mathfrak{g}, [\cdot,\cdot], \sigma)$ is a Hom-Lie superalgebra. In particular, σ is called trivial if $\sigma = \mathrm{id}|_g$.

It is obvious that σ is graded and if σ is a Hom-Lie superalgebra structure on a simple Lie superalgebra $\mathfrak{g}$, then σ must be a monomorphism.

Lemma 2.1. *Let $\mathfrak{g} = \oplus_{i \geq -h} \mathfrak{g}_i$ be a transitive Lie superalgebra. If σ is a Hom-Lie superalgebra structure on $\mathfrak{g}$ and $\sigma|_{\mathfrak{g}_{-1}} = \mathrm{id}|_{\mathfrak{g}_{-1}}$, then $\sigma(x) - x \in \mathfrak{g}_{-k}$, where $x \in \mathfrak{g}_0$, $k \geq 1$.*

Proof. For any $y \in \mathfrak{g}_{-1}$, from $\sigma|_{\mathfrak{g}_{-1}} = \mathrm{id}|_{\mathfrak{g}_{-1}}$ and (2), one can deduce

$$[\sigma(x), y] = [\sigma(x), \sigma(y)] = \sigma[x, y] = [x, y].$$

Then $[\sigma(x) - x, y] = 0$. So, $[\sigma(x) - x, \mathfrak{g}_{-1}] = 0$. The transitivity(1) of $\mathfrak{g}$ and the gradation of σ imply that $\sigma(x) - x \in \mathfrak{g}_{-k}$, $k \geq 1$. □

Lemma 2.2. *Let $\mathfrak{g} = \oplus_{i\geq -h}\mathfrak{g}_i$ be a transitive and irreducible Lie superalgebra. If σ is a Hom-Lie superalgebra structure on $\mathfrak{g}$, and satisfies*

$$\sigma|_{\mathfrak{g}_0\oplus\mathfrak{g}_{-1}} = \mathrm{id}|_{\mathfrak{g}_0\oplus\mathfrak{g}_{-1}},$$

then

$$\sigma = \mathrm{id}|_{\mathfrak{g}}.$$

Proof. Let $i \geq 1$. Equation (2) and $\mathfrak{g}_{-i} = \mathfrak{g}_{-1}^i$ imply

$$\sigma\ |_{\mathfrak{g}_{-i}} = \mathrm{id}\ |_{\mathfrak{g}_{-i}}\ . \tag{5}$$

Suppose $x \in \mathfrak{g}_i$, $y, z \in \mathfrak{g}_{\leq 0} = \oplus_{i=-h}^0\mathfrak{g}_i$, by (4) and (5), one can deduce

$$[\sigma(x), [y, z]] = [x, [y, z]].$$

Since $\mathfrak{g}$ is transitive and irreducible, then $\mathfrak{g}_{-1} = [\mathfrak{g}_0, \mathfrak{g}_{-1}]$. Together with Equation (4), it implies

$$[\sigma(x) - x, \mathfrak{g}_{\leq 0}] = 0.$$

Then $\sigma(x) - x = 0$, $\sigma|_{\mathfrak{g}_i} = \mathrm{id}|_{\mathfrak{g}_i}$. Thus, we have $\sigma = \mathrm{id}|_{\mathfrak{g}}$, that is, σ is trivial. □

According to Lemma 2.2, to study the Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras, we can begin with their 0-th and (-1)-th components. In the remaining of this paper, we will directly use Equation (4) without further remarks.

3 Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras

3.1 E(4,4)

From [20], there is a unique irreducible gradation over $E(4, 4)$, such that $E(4, 4)$ is the only inconsistent gradated algebra (its (-1)-th component is not purely odd) of the five exceptional simple Lie superalgebras. The 0-th component $E(4, 4)_0$ is isomorphic to $\widehat{\mathrm{P}}(4)$, which is the unique nontrivial central extension of P(4). As $E(4, 4)_0$-module, $E(4, 4)_{-1}$ is isomorphic to the natural module $\mathbb{F}^{4|4}$. Throughout the rest of the paper, we will use the notations which are introduced in [21] for $E(4, 4)$,.

Let $C = (c_{ij}) \in \mathfrak{gl}_4(\mathbb{F})$ be a skew-symmetric matrix. $\widetilde{C} = (\widetilde{c}_{ij}) = (\varepsilon_{ijkl}c_{kl})$ stands for the Hodge dual of C, where ε_{ijkl} is the symbol of permutation $(1234) \mapsto (ijkl)$. For short, write $i' := i + 4$, $i = 1, 2, 3, 4$. Fix a basis of $\widehat{P}(4) : \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{I}$, where

$$\begin{aligned}
&\mathcal{A} := \{E_{ij} - E_{j'i'}, \quad 1 \leq i \neq j \leq 4\},\\
&\mathcal{B} := \{E_{ij'} + E_{ji'}, \quad 1 \leq i \leq j \leq 4\},\\
&\mathcal{C} := \{E_{i'j} - E_{j'i} - (\widetilde{E}_{i'j} - \widetilde{E}_{j'i}), \quad 1 \leq i < j \leq 4\},\\
&\mathcal{I} := \{I,\ E_{ii} - E_{i+1,i+1} - (E_{i'i'} - E_{i'+1,i'+1}), \quad 1 \leq i \leq 3\},
\end{aligned}$$

I is the unit matrix.

Suppose $E(4, 4)_{-1} = \langle v_i, v_{i'} | i = 1, \ldots, 4\rangle$, where $|v_i| = \bar{0}$, $|v_{i'}| = \bar{1}$.

Proposition 3.1. *If σ is a Hom-Lie superalgebra structure on $E(4, 4)$, then*

$$\sigma|_{E(4,4)_{-1}} = \mathrm{id}|_{E(4,4)_{-1}}.$$

Proof. Let $i = 1, \ldots, 8$, $k, j = 1, \ldots, 4$ and $i \neq k'$, $j \neq i$. By using Equation (4), one can deduce

$$[\sigma(v_i), v_k] = [\sigma(v_i), [E_{kj} - E_{j'k'}, v_j]] = 0. \tag{6}$$

Similarly, one has

$$[\sigma(v_i), v_{k'}] = -[\sigma(v_i), [E_{jk} - E_{k'j'}, v_{j'}]] = 0 \tag{7}$$

and

$$[\sigma(v_k), v_{k'}] = -[\sigma(v_k), [E_{k'j} - E_{j'k} - (\widetilde{E}_{k'j} - \widetilde{E}_{j'k}), v_j]] = -[\sigma(v_j), v_{j'}].$$

The arbitrariness of k, j in the last equation shows

$$[\sigma(v_i), v_{i'}] = 0, \quad i = 1, \ldots, 4. \tag{8}$$

It follows from Equations (6)-(8) that

$$[\sigma(E(4,4)_{-1}), E(4,4)_{-1}] = 0.$$

Using the transitivity (1), one gets $\sigma(E(4,4)_{-1}) \subseteq E(4,4)_{-1}$. Recall that $|\sigma| = \bar{0}$, one may suppose

$$\sigma(v_i) = \sum_{m=1}^{8} \lambda_m v_m, \lambda_m, \in \mathbb{F}.$$

Now, for distinct $i, j, k, l = 1, 2, 3, 4$, suppose $(1234) \mapsto (ijkl)$ is an even permutation and

$$c = E_{k'j} - E_{j'k} - (E_{li'} - E_{il'}) \in \mathcal{C}, \ a = E_{jl} - E_{l'j'} \in \mathcal{A}.$$

Obviously, $[a, v_i] = 0$, $[v_i, c] = 0$. Then

$$0 = [\sigma(v_i), [c, a]] = [\sigma(v_i), E_{k'l} - E_{l'k} - (E_{ij'} - E_{ji'})] = -\lambda_l v_{k'} + \lambda_k v_{l'}.$$

Hence, $\lambda_k = \lambda_l = 0$, and then $\sigma(v_i) = \lambda_i v_i$, $i = 1, 2, 3, 4$.

The equation

$$0 = [\sigma(v_{i'}), [E_{kj} - E_{j'k'}, E_{jl'} + E_{lj'}]] = [\sigma(v_{i'}), E_{kl'} + E_{lk'}] = \lambda_{l'} v_k + \lambda_{k'} v_l$$

implies $\lambda_{k'} = \lambda_{l'} = 0$. One has $\sigma(v_{i'}) = \lambda_{i'} v_{i'}$.

At last, let us prove $\sigma(v_i) = v_i$ and $\sigma(v_{i'}) = v_{i'}$. For distinct i, j, k, put

$$h = E_{jj} - E_{j+1,j+1} - (E_{kk} - E_{k+1,k+1}) \in \mathcal{I}, \ a = E_{ji} - E_{j'i'} \in \mathcal{A}.$$

Clearly,

$$[\sigma(h), v_j] = [\sigma(h), [a, v_i]] = -[\sigma(v_i), a] = \lambda_i v_j. \tag{9}$$

Recall that σ is monomorphic. Let σ^{-1} denote a left inverse of σ . Then

$$\sigma^{-1}[\sigma(h), v_j] = [h, \sigma^{-1}(v_j)] = \lambda_j^{-1}[h, v_j] = \lambda_j^{-1} v_j.$$

It implies that

$$[\sigma(h), v_j] = \sigma(\lambda_j^{-1} v_j) = v_j. \tag{10}$$

Comparing (9) with (10), one gets $\lambda_i = 1$. Analogously, one gets $\sigma(v_{i'}) = v_{i'}$.

Summing up the above, we have proved $\sigma|_{E(4,4)_{-1}} = \mathrm{id}|_{E(4,4)_{-1}}$. □

Proposition 3.2. *If σ is a Hom-Lie superalgebra structure on $E(4,4)$, then*

$$\sigma|_{E(4,4)_0} = \mathrm{id}|_{E(4,4)_0}.$$

Proof. Put $x \in E(4,4)_0$, $v \in E(4,4)_{-1}$. Using Propersition 3.1 and Lemma 2.1, one can deduce $\sigma(x) - x \in E(4,4)_{-1}$. Note that $\mathcal{A}$ and $\mathcal{I}$ are generated (Lie product) by $\mathcal{B}$ and $\mathcal{C}$, it is sufficient to prove the cases for $x \in \mathcal{B}$ and $x \in \mathcal{C}$.

Case 1 : Let $x = E_{ij'} + E_{ji'} \in \mathcal{B}$. Noting that $|\sigma| = \bar{0}$, one may suppose

$$\sigma(x) = x + \sum_{m=1}^{4} \lambda_{m'} v_{m'}, \quad \lambda_{m'} \in \mathbb{F}.$$

Put

$$c = E_{k'l} - E_{l'k} - (\widetilde{E}_{k'l} - \widetilde{E}_{l'k}) \in \mathcal{C}, \quad b = E_{kk'} \in \mathcal{B},$$

where $k, l \neq i, j$. Then

$$0 = [\sigma(x), [b, c]] = \left[E_{ij'} + E_{ji'} + \sum_{m=1}^{4} \lambda_{m'} v_{m'}, E_{kl} - E_{l'k'} \right] = \lambda_{k'} v_{l'}.$$

It implies that $\lambda_{k'} = 0$, $k \neq i, j$. Then,

$$\sigma(x) = x + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}. \tag{11}$$

Now, put

$$c = E_{i'k} - E_{k'i} - (\widetilde{E}_{i'k} - \widetilde{E}_{k'i'}) \in \mathcal{C}, \quad b = E_{ii'} \in \mathcal{B}, \quad k \neq i, j.$$

Using Equation (11), one may suppose $\sigma(b) = b + \mu_{i'} v_{i'}$, $\mu_{i'} \in \mathbb{F}$, then

$$0 = [\sigma(x), a] = [E_{ij'} + E_{ji'} + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}, E_{ik} - E_{k'i'}] = \lambda_{i'} v_{k'}.$$

So $\lambda_{i'} = 0$. That is, $\sigma(x) = x + \lambda_{j'} v_{j'}$. Similarly, put

$$c = E_{j'k} - E_{k'j} - (\widetilde{E}_{j'k} - \widetilde{E}_{k'j'}) \in \mathcal{C}, \quad b = E_{jj'} \in \mathcal{B}.$$

One obtains $\lambda_{j'} = 0$. Thus, we have $\sigma(x) = x$ for any $x \in \mathcal{B}$.

Case 2 : Let $x = E_{i'j} - E_{j'i} - (E_{kl'} - E_{lk'}) \in \mathcal{C}$, where $\varepsilon_{ijkl} = 1$. Noting that $|\sigma| = \bar{0}$, one may assume $\sigma(x) = x + \sum_{m=1}^{4} \mu_{m'} v_{m'}$, $\mu_{m'} \in \mathbb{F}$.

Firstly, put

$$c = E_{k'l} - E_{l'k} - (E_{ji'} - E_{ij'}) \in \mathcal{C}, \quad b = E_{kk'} \in \mathcal{B},$$

where $\varepsilon_{ijkl} = 1$. Equation (4) and the result $\sigma(b) = b$ obtained in Case 1 imply

$$0 = [\sigma(x), [c, b]] = [x + \sum_{m=1}^{4} \mu_{m'} v_{m'}, E_{kl} - E_{l'k'}] = c_0 + \mu_{k'} v_{l'}, \quad c_0 \in \mathcal{C}.$$

The equation above shows $\mu_{k'} = 0$. By the arbitrariness of $k \neq i, j$, one has

$$\sigma(x) = x + \mu_{i'} v_{i'} + \mu_{j'} v_{j'}. \tag{12}$$

Secondly, put

$$c = E_{l'j} - E_{j'l} - (E_{ik'} - E_{k'i}) \in \mathcal{C}, \quad b = E_{jj'}.$$

Using Equation (12), one may suppose

$$\sigma(c) = c + \gamma_{l'} v_{l'} + \gamma_{j'} v_{j'}, \quad \gamma_{l'}, \gamma_{j'} \in \mathbb{F}.$$

On one hand,

$$[\sigma(x), [b, c]] = -[\sigma(b), [c, x]] - [\sigma(c), [x, b]] = E_{l'i} - E_{i'l} - (E_{kj'} - E_{jk'}) - \gamma_{j'} v_{i'}.$$

On the other hand,

$$\begin{aligned}[\sigma(x),[b,c]] &= [E_{i'j}-E_{j'i}-(E_{kl'}-E_{lk'})+\mu_{i'}v_{i'}+\mu_{j'}v_{j'}, E_{l'j'}-E_{jl}]\\ &= E_{l'i}-E_{i'l}-(E_{kj'}-E_{jk'})-\mu_{j'}v_{l'}.\end{aligned}$$

Since $i\neq l$, one has $\gamma_{j'}=\mu_{j'}=0$. Then $\sigma(x)=x+\lambda_{i'}v_{i'}$.

At last, put

$$c=E_{i'k}-E_{k'i}-(E_{lj'}-E_{jl'})\in\mathcal{C},\quad b=E_{ii'}\in\mathcal{B}.$$

We can obtain $\lambda_{i'}=0$ in the same way. Thus, $\sigma(x)=x$ is proved.

Summing up, we have proved $\sigma(x)=x$ for any $x\in E(4,4)_0$, that is, $\sigma|_{E(4,4)_0}=\mathrm{id}|_{E(4,4)_0}$. □

3.2 E(3,6), E(5,10) and E(3,8)

Before studying the Hom-Lie superalgebra structures on $E(3,6)$, $E(5,10)$ and $E(3,8)$, we would like to review their algebraic structures briefly [18, 20, 22, 23]. From [18] we know that the even part $E(5,10)_{\bar{0}}$ of $E(5,10)$ is isomorphic to the Lie algebra S_5, which consists of divergence 0 polynomial vector fields on $\mathbb{F}^5$, i.e., polynomial vector fields annihilating the volume form $\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\ldots\wedge\mathrm{d}x_5$. As S_5-module, the odd part $E(5,10)_{\bar{1}}$ of $E(5,10)$ is isomorphic to $\mathrm{d}\Omega^1(5)$, the space of closed polynomial differential 2-forms on $\mathbb{F}^5$. In the following, we keep the notations from [22]:

$$\mathrm{d}_{ij}:=\mathrm{d}x_i\wedge\mathrm{d}x_j,\quad \partial_i:=\partial/\partial x_i.$$

We write each element D of $E(5,10)_{\bar{0}}$ as

$$D=\sum_{i=1}^{5}f_i\partial_i,\quad\text{where } f_i\in\mathbb{F}[[x_1,x_2,\ldots,x_5]],\quad \sum_{i=1}^{5}\partial_i(f_i)=0,$$

and denote $E\in E(5,10)_{\bar{1}}$ through

$$E=\sum_{i,j=1}^{5}f_{ij}\mathrm{d}_{ij},\quad\text{where } f_{ij}\in\mathbb{F}[[x_1,x_2,\ldots,x_5]],\quad \mathrm{d}E=0. \tag{13}$$

The bracket in $E(5,10)_{\bar{1}}$ is defined by

$$[f\mathrm{d}_{ij},g\mathrm{d}_{kl}]=\varepsilon_{tijkl}fg\partial_t,$$

where ε_{tijkl} is the sign of permutation $(tijkl)$ when $\{tijkl\}=\{12345\}$ and zero otherwise. The bracket of $E(5,10)_{\bar{0}}$ with $E(5,10)_{\bar{1}}$ is defined by the usual action of vector fields on differential forms. The irreducible consistent $\mathbb{Z}$-gradation over $E(5,10)$ is defined by letting (see [22])

$$\mathrm{zd}(x_i)=2,\quad \mathrm{zd}(\mathrm{d})=-\frac{5}{2},\quad \mathrm{zd}(\mathrm{d}x_i)=-\frac{1}{2}.$$

Then $E(5,10)=\oplus_{i\geq-2}\mathfrak{g}_i$, where

$$\begin{aligned}&\mathfrak{g}_0\simeq\mathfrak{sl}(5)=\langle x_i\partial_j,\ x_k\partial_k-x_{k+1}\partial_{k+1}\mid i,j=1,\ldots,5, i\neq j,\ k=1,\ldots,4\rangle,\\ &\mathfrak{g}_{-1}\simeq\wedge^2\mathbb{F}^5=\langle \mathrm{d}_{ij}\mid 1\leq i<j\leq 5\rangle,\\ &\mathfrak{g}_{-2}\simeq\mathbb{F}^{5^*}=\langle\partial_i\mid i=1,\ldots,5\rangle.\end{aligned}$$

The exceptional simple Lie superalgebra $E(3,6)$ is a subalgebra of $E(5,10)$. The irreducible consistent $\mathbb{Z}$-gradation over $E(3,6)$ is induced by the $\mathbb{Z}$-gradation of $E(5,10)$ above. Let $h_1=x_1\partial_1-x_2\partial_2$, $h_2=x_2\partial_2-x_3\partial_3$, $h_3=x_4\partial_4-x_5\partial_5$, $h_4=-x_2\partial_2-x_3\partial_3+2x_5\partial_5$. Then $E(3,6)=\oplus_{i\geq-2}\mathfrak{g}_i$, where

$$\mathfrak{g}_0\simeq\mathfrak{gl}(3)\oplus\mathfrak{sl}(2)$$

$$= \langle x_i\partial_j,\ x_k\partial_l,\ h_m \mid i,j=1,2,3, i\neq j,\ k,l=4,5,\ k\neq l,\ m=1,\dots,4\rangle,$$
$$\mathfrak{g}_{-1} \simeq \mathbb{F}^3\otimes\mathbb{F}^2 = \langle \mathrm{d}_{ij} \mid i=1,2,3,\ j=4,5\rangle,$$
$$\mathfrak{g}_{-2} \simeq \mathbb{F}^{3^*} = \langle \partial_i \mid i=1,2,3\rangle.$$

The exceptional simple Lie superalgebra $E(3,8)$, which is strikingly similar to $E(3,6)$, carries a unique irreducible consistent $\mathbb{Z}$-gradation [23] defined by

$$\mathrm{zd}(x_i) = -\mathrm{zd}(\partial_i) = \mathrm{zd}(\mathrm{d}x_i) = 2,\ i=1,2,3; \quad \mathrm{zd}(x_4)=\mathrm{zd}(x_5)=-3.$$

Then $E(3,8)=\oplus_{i\geq -3}\mathfrak{g}_i$, where

$$\mathfrak{g}_j = E(3,6)_j, \quad j=0,-1,-2; \quad \mathfrak{g}_{-3}\simeq\mathbb{F}^2\simeq\langle \mathrm{d}x_4,\mathrm{d}x_5\rangle. \tag{14}$$

Hereafter, $\mathfrak{g}$ denotes $E(5,10)$, $E(3,6)$ or $E(3,8)$ unless otherwise noted. We establish a technical lemma, which can be verified directly.

Lemma 3.3. *If σ is a Hom-Lie superalgebra structure on $\mathfrak{g}$, then*

(1) $[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{il}] = \begin{cases} 0, & \text{if } \mathrm{d}_{ij},\mathrm{d}_{il}\in E(5,10),\\ -[\sigma(\mathrm{d}_{il}),\mathrm{d}_{ij}], & \text{if } \mathrm{d}_{ij},\mathrm{d}_{il}\in E(3,6) \text{ or } E(3,8);\end{cases}$

(2) $[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{kj}]=0$, *in particular,* $[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{ij}]=0$;

(3) $[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{kl}] = [\mathrm{d}_{ij},\sigma(\mathrm{d}_{kl})]$ *for distinct* i,j,k,l.

Proposition 3.4. *Let $\mathfrak{g}$ be Lie superalgebra $E(5,10)$, $E(3,6)$ or $E(3,8)$. If σ is a Hom-Lie superalgebra structure on $\mathfrak{g}$, then*

$$\sigma|_{\mathfrak{g}_{-1}} = \mathrm{id}|_{\mathfrak{g}_{-1}}.$$

Proof. First, let us prove $\sigma|_{\mathfrak{g}_{-1}}=\lambda\mathrm{id}|_{\mathfrak{g}_{-1}}$, where $\lambda\in\mathbb{F}$.

Case 1 : $\mathfrak{g}=E(5,10)$. Noting that the gradation over $E(5,10)$ is consistent and $|\sigma|=\bar{0}$, $|\mathrm{d}_{ij}|=\bar{1}$, one may suppose

$$\sigma(\mathrm{d}_{ij}) = \sum_{1\leq m<n\leq 5} f_{mn}\mathrm{d}_{mn}, \quad \text{where } f_{mn}\in\mathbb{F}[[x_1,\dots,x_5]].$$

By Lemma 3.3 (1), one has

$$[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{il}] = \left[\sum_{1\leq m<n\leq 5} f_{mn}\mathrm{d}_{mn},\mathrm{d}_{il}\right] = \left[\sum_{m,n\neq i,l} f_{mn}\mathrm{d}_{mn},\mathrm{d}_{il}\right]$$
$$= \sum_{m,n\neq i,l}[f_{mn}\mathrm{d}_{mn},\mathrm{d}_{il}] = \sum_{m,n\neq i,l}\varepsilon_{qmnil}f_{mn}\partial_q = 0.$$

The arbitrariness of l shows

$$f_{mn}=0, \quad \text{where } m,n\neq i.$$

Similarly, by Lemma 3.3 (2), one gets

$$f_{mn}=0, \quad \text{where } m,n\neq j.$$

Thus,

$$\sigma(\mathrm{d}_{ij}) = f_{ij}\mathrm{d}_{ij}.$$

In view of $\sigma(\mathrm{d}_{ij})\in E(5,10)_{\bar{1}}\simeq \mathrm{d}\Omega^1(5)$ and (13), one has

$$0=\mathrm{d}(\sigma(\mathrm{d}_{ij})) = \mathrm{d}(f_{ij}\mathrm{d}_{ij}) = \sum_{m=1}^{5}\partial_m(f_{ij})\mathrm{d}x_m\wedge\mathrm{d}x_i\wedge\mathrm{d}x_j.$$

Therefore, $\partial_m(f_{ij})=0$ for $m\neq i,j$, that is $f_{ij}\in\mathbb{F}[[x_i,x_j]]$. So one may suppose $\sigma(\mathrm{d}_{kl})=f_{kl}\mathrm{d}_{kl}$, where $f_{kl}\in\mathbb{F}[[x_k,x_l]]$. Then

$$[\sigma(\mathrm{d}_{ij}),\mathrm{d}_{kl}] = [f_{ij}\mathrm{d}_{ij},\mathrm{d}_{kl}] = \varepsilon_{qijkl}f_{ij}\partial_q,$$

$$[\sigma(\mathrm{d}_{kl}), \mathrm{d}_{ij}] = [f_{kl}\mathrm{d}_{kl}, \mathrm{d}_{ij}] = \varepsilon_{qklij} f_{kl}\partial_q.$$

For distinct i, j, k, l, Lemma 3.3 (3) implies $f_{ij} = f_{kl} \in \mathbb{F}$. Noting that there exists t such that $t \neq i, j, k, l$, one may obtain $f_{kt} = f_{ij}$ in the same way. Then $f_{ij} = \lambda \in \mathbb{F}$ for any $i, j = 1, 2, 3, 4, 5$ and $i \neq j$. Thus, we proved $\sigma(\mathrm{d}_{ij}) = \lambda \mathrm{d}_{ij}$.

Case 2 : $\mathfrak{g} = E(3, 6)$ or $E(3, 8)$. Noting that the gradation over $\mathfrak{g}$ is consistent and $|\sigma| = \bar{0}$, $|\mathrm{d}_{ij}| = \bar{1}$, one may suppose

$$\sigma(\mathrm{d}_{ij}) = \sum_{m,n} f_{mn}\mathrm{d}_{mn}, \quad \text{where } m = 1, 2, 3, n = 4, 5, \ f_{mn} \in \mathbb{F}[[x_1, \ldots, x_5]].$$

By Lemma 3.3 (2), one can deduce

$$\left[\sigma(\mathrm{d}_{ij}), \mathrm{d}_{kj}\right] = \left[\sum_{m,n} f_{mn}\mathrm{d}_{mn}, \mathrm{d}_{kj}\right] = \left[\sum_{m\neq k, n\neq j} f_{mn}\mathrm{d}_{mn}, \mathrm{d}_{kj}\right] = 0.$$

It implies $f_{mt} = 0$, $m \neq k$, $t \neq j$. By the arbitrariness of k, one may suppose

$$\sigma(\mathrm{d}_{ij}) = \sum_{m=1}^{3} f_{mj}\mathrm{d}_{mj}, \ \sigma(\mathrm{d}_{il}) = \sum_{m=1}^{3} f_{ml}\mathrm{d}_{ml}, \quad f_{mj}, f_{ml} \in \mathbb{F}[[x_1, \ldots, x_5]].$$

Noting that $[\sigma(\mathrm{d}_{il}), \mathrm{d}_{kj}] = [\sigma(\mathrm{d}_{il}), [x_k\partial_i, \mathrm{d}_{ij}]] = -[\sigma(\mathrm{d}_{ij}), \mathrm{d}_{kl}]$, the simple computations show $f_{mj} = f_{ml}$ for $m = 1, 2, 3$. In view of $\sigma(\mathrm{d}_{ij}) \in \mathfrak{g}_{\bar{1}} \subseteq \mathrm{d}\Omega^1(5)$ and (13), one could further deduce

$$f_{mj} = f_{ml} \in \mathbb{F}[[x_1, x_2, x_3]]. \tag{15}$$

For distinct $i, k, q = 1, 2, 3$,

$$\begin{aligned} 0 &= [\sigma(\mathrm{d}_{ij}), [x_i\partial_q, x_q\partial_k]] = [\sigma(\mathrm{d}_{ij}), x_i\partial_k] = \left[\sum_{m=1}^{3} f_{mj}\mathrm{d}_{mj}, x_i\partial_k\right] \\ &= -\sum_{m=1}^{3} \left(x_i\partial_k(f_{mj})\mathrm{d}_{mj} + f_{mj}x_i\partial_k(\mathrm{d}_{mj})\right) \\ &= -(x_i\partial_k(f_{ij}) + f_{kj})\mathrm{d}_{ij} - x_i\partial_k(f_{kj})\mathrm{d}_{kj} - x_i\partial_k(f_{qj})\mathrm{d}_{qj}. \end{aligned}$$

Therefore,

$$f_{kj} = -x_i\partial_k(f_{ij}). \tag{16}$$

$$\partial_k(f_{kj}) = \partial_k(f_{qj}) = 0, \tag{17}$$

From (15)-(17) and the arbitrariness of k, one has $f_{mj} = 0$, $m \neq i$. Thus, one may suppose

$$\sigma(\mathrm{d}_{ij}) = f_i\mathrm{d}_{ij}, \ \sigma(\mathrm{d}_{kl}) = f_k\mathrm{d}_{kl}, \quad \text{where } f_i \in \mathbb{F}[[x_i]], \ f_k \in \mathbb{F}[[x_k]].$$

Then, for distinct i, j, k, l, q,

$$[\sigma(\mathrm{d}_{ij}), \mathrm{d}_{kl}] = [f_i\mathrm{d}_{ij}, \mathrm{d}_{kl}] = \varepsilon_{qijkl} f_i\partial_q,$$

$$[\mathrm{d}_{ij}, \sigma(\mathrm{d}_{kl})] = [\mathrm{d}_{ij}, f_k\mathrm{d}_{kl},] = \varepsilon_{qijkl} f_k\partial_q.$$

By Lemma 3.3 (3), $f_i = f_k \in \mathbb{F}$. Thus, we have proved $\sigma(\mathrm{d}_{ij}) = \lambda\mathrm{d}_{ij}$ for some $\lambda \in \mathbb{F}$.

Now, let us prove $\lambda = 1$ for $\sigma|_{\mathfrak{g}_{-1}} = \lambda \mathrm{id}|_{\mathfrak{g}_{-1}}$. Suppose $x = x_i\partial_l$, $y = x_l\partial_q$, $z = \mathrm{d}_{qj}$. Then

$$[\sigma(x), \mathrm{d}_{lj}] = [\sigma(x), [y, z]] = -[\sigma(y), [z, x]] - [\sigma(z), [x, y]] = \lambda\mathrm{d}_{ij}.$$

Noting that σ is a monomorphism, one may suppose σ^{-1} is a left linear inverse of σ. Then the equation above implies

$$\sigma^{-1}[\sigma(x), \mathrm{d}_{lj}] = [x, \lambda^{-1}\mathrm{d}_{lj}] = \lambda^{-1}\mathrm{d}_{ij} = \mathrm{d}_{ij}.$$

Thus, $\lambda = 1$, that is, $\sigma(\mathrm{d}_{ij}) = \mathrm{d}_{ij}$. The proof is complete. □

Proposition 3.5. *Let $\mathfrak{g}$ be Lie superalgebra $E(5,10)$, $E(3,6)$ or $E(3,8)$. If σ is a Hom-Lie superalgebra structure on $\mathfrak{g}$, then*

$$\sigma|_{\mathfrak{g}_0} = \mathrm{id}|_{\mathfrak{g}_0}.$$

Proof. **Case 1** : $\mathfrak{g} = E(5,10)$. Note that $E(5,10)_0$ is generated (Lie product) by $\{x_i\partial_j | i,j = 1,\dots,5,\ i \neq j\}$, it is sufficient to prove $\sigma(x) = x$ only for $x = x_i\partial_j$. By Proposition 3.4, Lemma 2.1 and $|\sigma| = \bar{0}$, we have $\sigma(x_i\partial_j) - x_i\partial_j \in \mathfrak{g}_{-2}$. Then one may suppose

$$\sigma(x_i\partial_j) = x_i\partial_j + \sum_{m=1}^{5} \lambda_m\partial_m, \quad \lambda_m \in \mathbb{F}.$$

For distinct i,j,q,k,l,

$$0 = [\sigma(x_i\partial_j), [x_q\partial_k, x_k\partial_l]] = \left[x + \sum_{m=1}^{5} \lambda_m\partial_m, x_q\partial_l\right] = \lambda_q\partial_l.$$

Hence, $\lambda_q = 0$. The arbitrariness of $q \neq i,j$ shows

$$\sigma(x_i\partial_j) = x_i\partial_j + \lambda_i\partial_i + \lambda_j\partial_j.$$

Similarly,

$$0 = [\sigma(x_i\partial_j), [x_i\partial_l, x_l\partial_k]] = [x + \lambda_i\partial_i + \lambda_j\partial_j, x_i\partial_k] = \lambda_i\partial_k$$

implies $\lambda_i = 0$. Therefore,

$$\sigma(x_i\partial_j) = x_i\partial_j + \lambda_j\partial_j. \tag{18}$$

On one hand,

$$[\sigma(x_i\partial_j), [x_j\partial_i, x_i\partial_k]] = [x_i\partial_j + \lambda_j\partial_j, x_j\partial_k] = x_i\partial_k + \lambda_j\partial_k. \tag{19}$$

On the other hand, using (4) and (18), one can derive

$$[\sigma(x_i\partial_j), [x_j\partial_i, x_i\partial_k]] = -[\sigma(x_i\partial_k), x_i\partial_i - x_j\partial_j] = x_i\partial_k. \tag{20}$$

Comparing (19) with (20), one gets $\lambda_j = 0$. By (18), we have $\sigma(x_i\partial_j) = x_i\partial_j$.

Case 2 : $\mathfrak{g} = E(3,6)$ or $E(3,8)$. Firstly, let us show $\sigma(x_i\partial_j) = x_i\partial_j$ for any $x_i\partial_j \in \mathfrak{g}_0$. As before, one may suppose

$$\sigma(x_i\partial_j) = x_i\partial_j + \sum_{m=1}^{3} \lambda_m\partial_m, \quad \lambda_m \in \mathbb{F}.$$

Case 2.1 : Let $i,j = 1,2,3$. For distinct i,j,k, by (4),

$$0 = [\sigma(x_i\partial_j), [x_i\partial_k, x_k\partial_j]] = [\sigma(x_i\partial_j), x_i\partial_j] = \left[x_i\partial_j + \sum_{m=1}^{3} \lambda_m\partial_m, x_i\partial_j\right] = \lambda_i\partial_j.$$

Consequently, $\lambda_i = 0$. One may suppose $\sigma(x_k\partial_j) = x_k\partial_j + \sum_{m\neq k}^{3} \mu_m\partial_m$, On one hand,

$$[\sigma(x_i\partial_j), x_k\partial_i] = \left[x_i\partial_j + \sum_{m\neq i} \lambda_m\partial_m, x_k\partial_i\right] = -x_k\partial_j + \lambda_k\partial_i.$$

On the other hand, from (4) one deduces

$$\begin{aligned}[\sigma(x_i\partial_j), x_k\partial_i] &= [\sigma(x_i\partial_j), [x_k\partial_j, x_j\partial_i]] \\ &= -[\sigma(x_k\partial_j), [x_j\partial_i, x_i\partial_j]] - [\sigma(x_j\partial_i), [x_i\partial_j, x_k\partial_j]]\end{aligned}$$

$$= -x_k \partial_j - \mu_j \partial_j + \mu_i \partial_i.$$

Comparing the two equations above, one gets $\mu_j = 0$ and $\lambda_k = \mu_i$. In view of the arbitrariness of i, j, k, one may assume

$$\sigma(x_i \partial_j) = x_i \partial_j + \lambda \partial_k,\ \sigma(x_k \partial_j) = x_k \partial_j + \lambda \partial_i, \quad i, j, k \text{ are distinct.}$$

Then

$$\sigma(x_i \partial_j) = \sigma[x_i \partial_k, x_k \partial_j] = [\sigma(x_i \partial_k), \sigma(x_k \partial_j)] = x_i \partial_j - \lambda \partial_k.$$

The two equations above imply $\lambda = 0$, and then $\sigma(x_i \partial_j) = x_i \partial_j$, $i, j = 1, 2, 3$.

Case 2.2 : Let $i, j = 4, 5$. For distinct $k, l, q = 1, 2, 3$,

$$0 = [\sigma(x_i \partial_j), [x_k \partial_q, x_q \partial_l]] = [\sigma(x_i \partial_j), x_k \partial_l] = \left[x_i \partial_j + \sum_{m=1}^{3} \lambda_m \partial_m, x_k \partial_l \right] = \lambda_k \partial_l.$$

Therefore, $\lambda_k = 0$. The arbitrariness of k implies $\sigma(x_i \partial_j) = x_i \partial_j$.

Next, we will prove $\sigma(h_m) = h_m$ for $m = 1, 2, 3, 4$. Similarly, suppose $\sigma(h_m) = h_m + \sum_{k=1}^{3} \lambda_k \partial_k$. Noting that h_m is an element of the basis of $\mathfrak{g}_0$'s Cartan subalgebra, one may assume $[h_m, x_i \partial_j] = \gamma x_i \partial_j$, $\gamma \in \mathbb{F}$, $i, j = 1, 2, 3$. By the result obtained in Case 2.1, one can deduce

$$\sigma[h_m, x_i \partial_j] = \gamma \sigma(x_i \partial_j) = \gamma x_i \partial_j.$$

On the other hand,

$$\sigma[h_m, x_i \partial_j] = [\sigma(h_m), \sigma(x_i \partial_j)] = \left[h_m + \sum_{k=1}^{3} \lambda_k \partial_k, x_i \partial_j \right] = \gamma x_i \partial_j + \lambda_i \partial_j.$$

The two equations above imply that $\lambda_i = 0$ for $i = 1, 2, 3$. Thus, we have $\sigma(h_m) = h_m$ for $m = 1, 2, 3, 4$.

Summing up, we have proved $\sigma(x) = x$ for any $x \in \mathfrak{g}_0$, that is, $\sigma|_{\mathfrak{g}_0} = \mathrm{id}|_{\mathfrak{g}_0}$. □

3.3 E(1,6)

The exceptional simple Lie superalgebra $E(1,6)$ is a subalgebra of $K(1,6)$. The principal gradation over $K(1,6)$ (see [20]) induces an irreducible consistent $\mathbb{Z}$-gradation over $E(1,6)$. Moreover, $E(1,6)$ has the same non-positive $\mathbb{Z}$-graded components as $K(1,6)$: $E(1,6)_j = K(1,6)_j$, $j \leq 0$. Let x be even and ξ_i, $i = 1, \ldots, 6$, be odd indeterminates. Then

$$\begin{aligned} &E(1,6)_0 \simeq \mathfrak{cspo}(6) = \langle x, \xi_i \xi_j \mid i, j = 1, \ldots, 6, i \neq j \rangle, \\ &E(1,6)_{-1} \simeq \mathbb{F}^6 = \langle \xi_i \mid i = 1, \ldots, 6 \rangle, \\ &E(1,6)_{-2} \simeq \mathbb{F}. \end{aligned}$$

For $f, g \in E(1,6)$, the bracket product is defined as

$$\begin{aligned}[f, g] = &\left(2f - \sum_{i=1}^{6} \xi_i \partial_i(f)\right) \partial_x(g) - (-1)^{|f||g|} \left(2g - \sum_{i=1}^{6} \xi_i \partial_i(g)\right) \partial_x(f) \\ &+ (-1)^{|f|} \sum_{i=1}^{6} \partial_i(f) \partial_i(g). \end{aligned}$$

Proposition 3.6. *If σ is a Hom-Lie superalgebra structure on $E(1,6)$, then*

$$\sigma|_{E(1,6)_{-1}} = \mathrm{id}|_{E(1,6)_{-1}}.$$

Proof. For $i, j, k = 1, \ldots, 6$ and $j \neq i, k$, by (4),

$$[\sigma(\xi_i), \xi_k] = [\sigma(\xi_i), [\xi_j \xi_k, \xi_j]] = \delta_{k,i}[\sigma(\xi_j), \xi_j].$$

It implies that $[\sigma(\xi_i), \xi_k] = 0$ for any $k, i = 1, \ldots, 6$. From $|\sigma| = \bar{0}$ and the transitivity of $E(1,6)$, one can deduce $\sigma(E(1,6)_{-1}) \subseteq E(1,6)_{-1}$. So one may assume $\sigma(\xi_i) = \sum_{k=1}^{6} \lambda_k \xi_k$, $\lambda_k \in \mathbb{F}$. By using Equation (4), for distinct $i, j, s, t = 1, \ldots, 6$, one can deduce

$$0 = [\sigma(\xi_i), [\xi_j \xi_t, \xi_s \xi_t]] = [\sigma(\xi_i), \xi_j \xi_s] = \lambda_j \xi_s - \lambda_s \xi_j.$$

Then $\lambda_k = 0$ for any $k \neq i$. That is $\sigma(\xi_i) = \lambda_i \xi_i$. By using Equation (4) again, one can deduce

$$-\lambda_j = [\sigma(\xi_i), [\xi_j \xi_i, \xi_j]] = [\sigma(\xi_i), \xi_i] = [\lambda_i \xi_i, \xi_i] = -\lambda_i.$$

Moreover, one knows $\sigma(\xi_i) = \lambda \xi_i$ for any $i = 1, \ldots, 6$. The remaining work is to prove $\lambda = 1$. For distinct i, j, t,

$$[\sigma(\xi_t \xi_j), \xi_t] = [\sigma(\xi_t \xi_j), [\xi_i \xi_t, \xi_i]] = -[\sigma(\xi_i), [\xi_t \xi_j, \xi_i \xi_t]] = [\lambda \xi_i, \xi_j \xi_i] = \lambda \xi_j. \tag{21}$$

Suppose σ^{-1} is a left inverse of the monomorphism σ. Then

$$\sigma^{-1}([\sigma(\xi_t \xi_j), \xi_t]) = [\xi_t \xi_j, \sigma^{-1}(\xi_t)] = \lambda^{-1} \xi_j.$$

Hence

$$[\sigma(\xi_t \xi_j), \xi_t] = \sigma(\lambda^{-1} \xi_j) = \xi_j. \tag{22}$$

Equations (21) and (22) imply $\lambda = 1$. The proof is completed. □

Proposition 3.7. *If σ is a Hom-Lie superalgebra structure on $E(1,6)$, then*

$$\sigma|_{E(1,6)_0} = \mathrm{id}|_{E(1,6)_0.}$$

Proof. For any $y \in E(1,6)_0$, by Proposition 3.6 and Lemma 2.1, $\sigma(y) - y \in E(1,6)_{-2} \simeq \mathbb{F}$. So, one may suppose $\sigma(y) = y + \lambda$, $\lambda \in \mathbb{F}$. Then

$$[\sigma(y), x] = [y + \lambda, x] = [\lambda, x] = 2\lambda. \tag{23}$$

If $y = x$, noting that x is an element of a basis of $E(1,6)_0$'s Cartan subalgebra, one may assume

$$[\sigma(x), x] = \mu\sigma(x) = \mu(x + \lambda), \quad \mu \in \mathbb{F}. \tag{24}$$

Comparing (23) and (24), one has $\mu = 0$, and then $\lambda = 0$, that is $\sigma(x) = x$.

If $y = \xi_i \xi_j$,

$$[\sigma(\xi_i \xi_j), x] = [\sigma(\xi_i \xi_j), \sigma(x)] = \sigma[\xi_i \xi_j, x] = 0. \tag{25}$$

From (23) and (25), one gets $\lambda = 0$ again. The proof is complete. □

3.4 The main result

Propositions 3.1, 3.2, 3.4-3.7 show that all the Hom-Lie superalgebra structures on the 0-th and (−1)-th $\mathbb{Z}$-components of each infinite-dimensional exceptional simple Lie superalgebra are trivial. Then combining this with Lemma 2.2, we immediately have:

Theorem 3.8. *There is only the trivial Hom-Lie superalgebra structure on each exceptional simple Lie superalgebra.*

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