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Guanghui Lu and Shuangping Tao*

Commutators of Littlewood-Paley g_{κ}^* -functions on non-homogeneous metric measure spaces

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Abstract: The main purpose of this paper is to prove that the boundedness of the commutator $\mathcal{M}_{\kappa,b}^*$ generated by the Littlewood-Paley operator \mathcal{M}_{κ}^* and RBMO(μ) function on non-homogeneous metric measure spaces satisfying the upper doubling and the geometrically doubling conditions. Under the assumption that the kernel of \mathcal{M}_{κ}^* satisfies a certain Hörmander-type condition, the authors prove that $\mathcal{M}_{\kappa,b}^*$ is bounded on Lebesgue spaces $L^p(\mu)$ for $1 < p < \infty$, bounded from the space $L \log L(\mu)$ to the weak Lebesgue space $L^{1,\infty}(\mu)$, and is bounded from the atomic Hardy spaces $H^1(\mu)$ to the weak Lebesgue spaces $L^{1,\infty}(\mu)$.

Keywords: Non-homogeneous metric measure space, Commutators, g_{κ}^* -functions, RBMO(μ), Hardy space

MSC: 42B25, 42B35, 30L99

1 Introduction

In 1958, Stein [1] firstly introduced and studied Littlewood-Paley g_{κ}^* -functions on \mathbb{R}^n . After that, many authors paid much attention to study the properties of the Littlewood-Paley g_{κ}^* -functions on various function spaces, for example, see [2-7]. With deeper research, the boundedness of Littlewood-Paley operators and their commutators under the cases of non-doubling measures is also widely discussed (see [8-14]).

To solve the unity of the homogeneous type spaces and the metric spaces endowed with measures satisfying the polynomial growth condition, in 2010, Hytönen [15] introduced a new class of metric measure space satisfying the so-called geometrically doubling and the upper doubling conditions (see Definitions 1.1 and 1.3, respectively), which is now called *non-homogeneous metric measure space*. So, it is interesting to generalize and improve the known results to the non-homogeneous metric measure spaces, see [16-24].

In this paper, (\mathcal{X}, d, μ) stands for a non-homogeneous metric measure space in the sense of Hytönen in [15]. In this setting, we will discuss the boundedness of commutators of Littlewood-Paley g_{κ}^* -functions on (\mathcal{X}, d, μ) .

Before stating the main results, we firstly recall some definitions and remarks. The following notion of the geometrically doubling condition was originally introduced by Coifman and Weiss in [25].

Definition 1.1 ([25]). *A metric space (\mathcal{X}, d) is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there is a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0 .*

Guanghui Lu: College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China,

E-mail: lghwmm1989@126.com

***Corresponding Author: Shuangping Tao:** College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China, E-mail: taosp@nwnu.edu.cn

Remark 1.2. Let (\mathcal{X}, d) be a metric space. Hytönen in [15] showed the following statements are mutually equivalent:

- (1) (\mathcal{X}, d) is geometrically doubling.
- (2) For any $\epsilon \in (0, 1)$ and ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$. Here and in what follows, N_0 is as Definition 1.1 and $n := \log_2 N_0$.
- (3) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ can contain at most $N_0 \epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls with radius ϵr .
- (4) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ can contain at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^M$.

Now, we recall the definition of upper doubling conditions given in [15].

Definition 1.3 ([15]). A metric measure space (\mathcal{X}, d, μ) is said to be upper doubling, if μ is Borel measure on \mathcal{X} and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \rightarrow (0, \infty)$ and a positive constant C_λ such that, for each $x \in \mathcal{X}, r \rightarrow \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, \frac{r}{2}). \tag{1}$$

Hytönen et al. proved in [16] that there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda, C_{\tilde{\lambda}} \leq C_\lambda$ and, for all $x, y \in \mathcal{X}$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq C_{\tilde{\lambda}} \tilde{\lambda}(y, r). \tag{2}$$

Based on this, from now on, we always assume that the dominating function λ as in (1) satisfies (2).

The following coefficient $K_{B,S}$ which introduced [15] by Hytönen is analogous to Tolsa’s number in [8,9].

Given any two balls $B \subset S$, set

$$K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(y), \tag{3}$$

where c_B represents the center of the ball B .

Hytönen [15] gave the definition of (α, β) -doubling, that is, a ball $B \subset \mathcal{X}$ is called (α, β) -doubling if $\mu(\alpha B) \leq \beta \mu(B)$ for $\alpha, \beta > 1$. At the same time, Hytönen proved that if a metric measure space (\mathcal{X}, d, μ) is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} =: \alpha^\nu$, then for every ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+$ such that $\alpha^j B$ is (α, β) -doubling. In addition, let (\mathcal{X}, d) be geometrically doubling, $\beta > \alpha^n$ with $n = \log_2 N_0$ and μ Borel measure on \mathcal{X} which is finite on bounded sets. Hytönen also showed that for μ -a.e $x \in \mathcal{X}$, there exist arbitrarily small (α, β) -doubling balls centered at x . Furthermore, the radius of these balls may be chosen to be form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this paper, for any $\alpha \in (1, \infty)$ and ball B , the smallest (α, β_α) -doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is denoted by \tilde{B}^α , where

$$\beta_\alpha := \alpha^{3(\max\{n, \nu\})} + 30^n + 30^\nu. \tag{4}$$

For convenience, we always assume $\alpha = 6$ in this paper and denote \tilde{B}^6 simply by \tilde{B} .

Now we recall the notion of RBMO(μ) from [15].

Definition 1.4 ([15]). Let $\nu > 1$. A function $f \in L^1_{\text{loc}}(\mu)$ is claimed to be in the space RBMO(μ) if there exist a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\nu B)} \int_B |f(x) - f_B| d\mu(x) \leq C \tag{5}$$

and, for any two balls B and R such that $B \subset R$,

$$|f_B - f_R| \leq CK_{B,R}. \tag{6}$$

The infimum of the constants C satisfying (5) and (6) is defined to be the RBMO(μ) norm of f and denoted by $\|f\|_{\text{RBMO}(\mu)}$.

Next, we recall the definition of the Littlewood-Paley g_{κ}^* -function given in [17].

Definition 1.5 ([17]). *Let $K(x, y)$ be a locally integrable function on $(\mathcal{X} \times \mathcal{X}) \setminus \{(x, x) : x \in \mathcal{X}\}$. Assume that there exists a non-negative constant C such that, for all $x, y \in \mathcal{X}$ with $x \neq y$,*

$$|K(x, y)| \leq C \frac{d(x, y)}{\lambda(x, d(x, y))} \tag{7}$$

and, for all $y, y' \in \mathcal{X}$,

$$\int_{d(x, y) \geq 2d(y, y')} \left[|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \right] \frac{d\mu(x)}{d(x, y)} \leq C. \tag{8}$$

The Littlewood-Paley g_{κ}^* -function \mathfrak{M}_{κ}^* is defined by

$$\mathfrak{M}_{\kappa}^*(f)(x) := \left[\iint_{\mathcal{X} \times (0, \infty)} \left(\frac{t}{t + d(x, y)} \right)^{\kappa} \left| \frac{1}{t} \int_{d(y, z) \leq t} K(y, z) f(z) d\mu(z) \right|^2 \frac{d\mu(y) dt}{\lambda(y, t)t} \right]^{\frac{1}{2}}, \tag{9}$$

where $x \in \mathcal{X}$, $\mathcal{X} \times (0, \infty) = \{(y, t) : y \in \mathcal{X}, t > 0\}$ and $\kappa > 1$.

Let $b \in \text{RBMO}(\mu)$ and $K(x, y)$ satisfy (7) and (8). The commutator of Littlewood-Paley g_{κ}^* -function $\mathfrak{M}_{\kappa, b}^*$ is formally defined by

$$\begin{aligned} \mathfrak{M}_{\kappa, b}^*(f)(x) := & \left[\iint_{\mathcal{X} \times (0, \infty)} \left(\frac{t}{t + d(x, y)} \right)^{\kappa} \right. \\ & \left. \times \left| \frac{1}{t} \int_{d(y, z) \leq t} K(y, z) [b(x) - b(z)] f(z) d\mu(z) \right|^2 \frac{d\mu(y) dt}{\lambda(y, t)t} \right]^{\frac{1}{2}}. \end{aligned} \tag{10}$$

The following notion of the atomic Hardy space is from [16].

Definition 1.6 (16). *Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(p, 1)_{\tau}$ -atomic block, if*

- (1) *there exists a ball B such that $\text{supp}(b) \subset B$;*
- (2) *$\int_{\mathcal{X}} b(x) d\mu(x) = 0$;*
- (3) *for any $i \in \{1, 2\}$, there exists a function a_i supported on a ball $B_i \subset B$ and $\tau_i \in \mathbb{C}$ such that*

$$b = \tau_1 a_1 + \tau_2 a_2$$

and

$$\|a_i\|_{L^p(\mu)} \leq [\mu(\rho B_i)]^{\frac{1}{p}-1} K_{B_i, B}^{-1}. \tag{11}$$

Moreover, let $|b|_{H_{\text{atb}}^{1,p}(\mu)} := |\tau_1| + |\tau_2|$.

Definition 1.7 ([16]). *Let $p \in (1, \infty]$. A function $f \in L^1(\mu)$ is said to belong to the atomic Hardy space $H_{\text{atb}}^{1,p}(\mu)$, if there exist $(p, 1)_{\tau}$ -atomic blocks $\{b_i\}_{i=1}^{\infty}$ such that $f = \sum_{i=1}^{\infty} b_i$ in $L^1(\mu)$ and $\sum_{i=1}^{\infty} |b_i|_{H_{\text{atb}}^{1,p}(\mu)} < \infty$. The norm of f in $H_{\text{atb}}^{1,p}(\mu)$ is defined by*

$$\|f\|_{H_{\text{atb}}^{1,p}(\mu)} := \inf \left\{ \sum_{i=1}^{\infty} |b_i|_{H_{\text{atb}}^{1,p}(\mu)} \right\},$$

where the infimum is taken overall the possible decompositions of f as above.

According to [14], the definition of the Hörmander-type condition on (\mathcal{X}, d, μ) is defined by:

$$\sup_{\substack{r > 0 \\ d(y', y) \leq r}} \sum_{i=1}^{\infty} \int_{6^i r < d(x, y) \leq 6^{i+1} r} i \left[|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \right] \frac{d\mu(x)}{d(x, y)} \leq C, \tag{12}$$

which is slightly stronger (8).

Our main results in this paper are formulated as follows.

Theorem 1.8. *Let $b \in \text{RBMO}(\mu)$, $K(x, y)$ satisfy (7) and (12) and $\mathfrak{M}_{\kappa, b}^*(f)$ be as in (10). Suppose that \mathfrak{M}_{κ}^* is bounded on $L^2(\mu)$. Then $\mathfrak{M}_{\kappa, b}^*(f)$ is bounded on $L^p(\mu)$ for $1 < p < \infty$, that is, there exists a constant $C > 0$, such that for all functions f with bounded support, one has*

$$\|\mathfrak{M}_{\kappa, b}^*(f)\|_{L^p(\mu)} \leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)}.$$

Theorem 1.9. *Let $b \in \text{RBMO}(\mu)$, $K(x, y)$ satisfy (7) and (12) and $\mathfrak{M}_{\kappa, b}^*(f)$ be as in (10). Suppose that \mathfrak{M}_{κ}^* is bounded on $L^2(\mu)$. Then there is a positive constant C , such that for all functions f with bounded support,*

$$\mu(\{x \in \mathcal{X} : \mathfrak{M}_{\kappa, b}^*(f)(x) > t\}) \leq C \Phi(\|b\|_{\text{RBMO}(\mu)}) \int_{\mathcal{X}} \Phi\left(\frac{|f(x)|}{t}\right) d\mu(y),$$

where $\Phi_{\alpha}(t) = t \log^{\alpha}(2 + t)$ for $\alpha \geq 1$.

Theorem 1.10. *Let $b \in \text{RBMO}(\mu)$, $K(x, y)$ satisfy (7) and (12) and $\mathfrak{M}_{\kappa, b}^*(f)$ be as in (10). Suppose that \mathfrak{M}_{κ}^* is bounded on $L^2(\mu)$. Then $\mathfrak{M}_{\kappa, b}^*(f)$ is bounded from $H^1(\mu)$ into $L^{1, \infty}(\mu)$, namely, there is a positive constant C , such that for all $f \in H^1(\mu)$ and $t > 0$, one has*

$$\mu(\{x \in \mathcal{X} : \mathfrak{M}_{\kappa, b}^*(f)(x) > t\}) \leq C \|b\|_{\text{RBMO}(\mu)} \frac{\|f\|_{H^1(\mu)}}{t}.$$

2 Preliminaries

In this section, we shall recall some lemmas used in the proofs of our main theorems. Firstly, we recall some useful properties of $K_{B, S}$ as in (3) (see [15]).

Lemma 2.1 ([15]). (1) *For all balls $B \subset R \subset S$, it holds true that $K_{B, R} \leq K_{B, S}$.*

(2) *For any $\xi \in [1, \infty)$, there exists a positive constant C_{ξ} , such that, for all balls $B \subset S$ with $r_S \leq \xi r_B$, $K_{B, S} \leq C_{\xi}$.*

(3) *For any $\varrho \in (1, \infty)$, there exists a positive constant C_{ϱ} , depending on ϱ , such that, for all balls $B, K_{B, \tilde{B}^{\varrho}} \leq C_{\varrho}$.*

(4) *There exists a positive constant c such that, for all balls $B \subset R \subset S, K_{B, S} \leq K_{B, R} + cK_{R, S}$. In particular, if B and R are concentric, then $c = 1$.*

(5) *There exists a positive constant \tilde{c} such that, for all balls $B \subset R \subset S, K_{B, R} \leq \tilde{c}K_{B, S}$; moreover, if B and R are concentric, then $K_{R, S} \leq K_{B, S}$.*

Now, we recall the following conclusion, which is just [18].

Corollary 2.2 ([18]). *If $f \in \text{RBMO}(\mu)$, then there exists a positive constant C such that, for any ball B , $\varpi \in (1, \infty)$ and $r \in [1, \infty)$,*

$$\left(\frac{1}{\mu(\varpi B)} \int_B |f(x) - m_{\tilde{B}}(f)|^r d\mu(x) \right)^{\frac{1}{r}} \leq C \|f\|_{\text{RBMO}(\mu)}, \tag{13}$$

where above and in what follows, $m_{\tilde{B}}(f)$ denotes the mean of f over \tilde{B} , namely,

$$m_{\tilde{B}}(f) := \frac{1}{\mu(\tilde{B})} \int_{\tilde{B}} f(y) d\mu(y).$$

Moreover, the infimum of the positive constants C satisfying $|m_B(f) - m_S(f)| \leq CK_{B, S}$ and (13) is an equivalent $\text{RBMO}(\mu)$ norm of f .

Next, we recall some results from [15, 19].

Lemma 2.3 ([15]). (1) Let $p \in (1, \infty)$, $r \in (1, p)$ and $\phi \in (0, \infty)$. The following maximal operators defined, respectively, be setting, for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$,

$$M_{r,\phi} f(x) := \sup_{Q \ni x} \left(\frac{1}{\mu(\phi Q)} \int_Q |f(y)|^r d\mu(y) \right)^{\frac{1}{r}},$$

$$Nf(x) := \sup_{Q \ni x, Q \text{ doubling}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y)$$

and

$$M_\phi f(x) := \sup_{Q \ni x} \frac{1}{\mu(\phi Q)} \int_Q |f(y)| d\mu(y)$$

are bounded on $L^p(\mu)$ and also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

(2) For all $f \in L^1_{\text{loc}}(\mu)$, it holds true that $|f(x)| \leq Nf(x)$ for μ -almost every $x \in \mathcal{X}$.

The following result is given in [19].

Lemma 2.4 ([19]). Let $f \in L^1_{\text{loc}}(\mu)$ with the extra condition $\int_{\mathcal{X}} f(x) d\mu(x) = 0$ if $\|\mu\| := \mu(\mathcal{X}) < \infty$. Assume that for some p , $1 < p < \infty$, $\inf\{1, Nf\} \in L^p(\mu)$. Then we have

$$\|Nf\|_{L^p(\mu)} \leq C \|M^\# f\|_{L^p(\mu)},$$

where $M^\# f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(x) - m_B(f)| d\mu(x) + \sup_{(Q,R) \in \Delta_x} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}$ for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$, and $\Delta_x := \{(Q, R) : x \in Q \subset R \text{ and } Q, R \text{ are doubling balls}\}$.

Also, we recall the following Calderón-Zygmund decomposition theorem given in [19]. Suppose γ_0 is a fixed positive constant satisfying that $\gamma_0 > \max\{C_\lambda^{3 \log_2 6}, 6^{3n}\}$, where C_λ is as in (1) and n as in Remark 1.2.

Lemma 2.5 ([19]). Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $t \in (0, \infty)$ ($t > \frac{\gamma_0 \|f\|_{L^p(\mu)}}{\mu(\mathcal{X})}$ when $\mu(\mathcal{X}) < \infty$). Then

(1) there exists a family of finite overlapping balls $\{6B_i\}_i$ such that $\{B_i\}_i$ is pairwise disjoint,

$$\frac{1}{\mu(6^2 B_i)} \int_{B_i} |f(x)|^p d\mu(x) > \frac{t^p}{\gamma_0} \text{ for all } i, \tag{14}$$

$$\frac{1}{\mu(6^2 \tau B_i)} \int_{\tau B_i} |f(x)|^p d\mu(x) \leq \frac{t^p}{\gamma_0} \text{ for all } i \text{ and all } \tau \in (2, \infty),$$

and

$$|f(x)| \leq t \text{ for } \mu\text{-almost every } x \in \mathcal{X} \setminus (\cup_i 6B_i); \tag{15}$$

(2) for each i , let S_i be a $(3 \times 6^2, C_\lambda^{\log_2(3 \times 6^2)+1})$ -doubling ball of the family $\{(3 \times 6^2)^k B_i\}_{k \in \mathbb{N}}$, and $\omega_i = \chi_{6B_i} / (\sum_k \chi_{6^k B_i})$. Then there exists a family $\{\varphi_i\}_i$ of functions that for each i , $\text{supp}(\varphi_i) \subset S_i$, φ_i has a constant sign on S_i and

$$\int_{\mathcal{X}} \varphi_i(x) d\mu(x) = \int_{6B_i} f(x) \omega_i(x) d\mu(x), \tag{16}$$

$$\sum_i |\varphi_i(x)| \leq \gamma t \text{ for } \mu\text{-almost every } x \in \mathcal{X}, \tag{17}$$

where γ is some positive constant depending only on (\mathcal{X}, μ) , and there exists a positive constant C , independent of f, t and i , such that, if $p = 1$, then

$$\|\varphi_i\|_{L^\infty(\mu)} \mu(S_i) \leq C \int_{\mathcal{X}} |f(x)\omega_i(x)| d\mu(x), \tag{18}$$

and if $p \in (1, \infty)$,

$$\left(\int_{S_i} |\varphi_i(x)|^p d\mu(x) \right)^{\frac{1}{p}} [\mu(S_i)]^{\frac{1}{p'}} \leq \frac{C}{t^{p-1}} \int_{\mathcal{X}} |f(x)\omega_i(x)|^p d\mu(x). \tag{19}$$

Finally, we recall the following John-Nirenberg inequality from [15].

Lemma 2.6 ([15]). *For every $\zeta > 1$, there exists a positive constant C such that, for every $b \in \text{RBMO}(\mu)$ and every ball B ,*

$$\mu(\{x \in B : |b(x) - m_{\bar{B}}(b)| > t\}) \leq C\mu(\zeta B) \exp\left(\frac{-Ct}{\|b\|_{\text{RBMO}(\mu)}}\right).$$

From Lemma 2.6, it is easy to prove that there are two positive constants B_1 and B_2 such that, for any ball B and $b \in \text{RBMO}(\mu)$,

$$\frac{1}{\mu(\zeta B)} \int_B \exp\left(\frac{|b(x) - m_{\bar{B}}(b)|}{B_1 \|b\|_{\text{RBMO}(\mu)}}\right) d\mu(x) \leq B_2.$$

3 Proofs of Theorems 1.8–1.10

Proof of Theorem 1.8. Let $0 < r < 1$, we firstly claim that, for any $p \in (1, \infty)$, $b \in L^\infty(\mu)$ and all bounded functions f with compact support,

$$\mu(\{x \in \mathcal{X} : M_r^\sharp[\mathfrak{M}_{\kappa,b}^*(f)](x) > t\}) \leq Ct^{-p} \|b\|_{\text{RBMO}(\mu)}^p \|f\|_{L^p(\mu)}^p. \tag{20}$$

Once (20) is established, by the Marcinkiewicz interpolation theorem, it is easy to obtain that

$$\|M_r^\sharp[\mathfrak{M}_{\kappa,b}^*(f)]\|_{L^p(\mu)} \leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)}.$$

By Lemma 2.4, for any $p \in (1, \infty)$, $b \in L^\infty(\mu)$ and all bounded function f with compact support and integral zero,

$$\|\mathfrak{M}_{\kappa,b}^*(f)\|_{L^p(\mu)} \leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)},$$

together with the fact that the bounded function f with compact support and integral zero is dense in $L^p(\mu)$ (see [19, Theorem 6.4]), we finish the proof of Theorem 1.8.

Now, we turn to estimate (20). Without loss of generality, let $\phi = 6$ as in Lemma 2.3 and $\|b\|_{\text{RBMO}(\mu)} = 1$. For each fixed $t > 0$ and bounded function f with compact support and integral zero, applying Lemma 2.5 to f , we see that $f = g + h$, where $g := f\chi_{\mathcal{X} \setminus \cup_j B_j} + \sum_j \varphi_j$ and $h := \sum_j (\omega_j f - \varphi_j) =: \sum_j h_j$. By (15) and (17), we easily get

$$\|g\|_{L^\infty(\mu)} \leq Ct. \tag{21}$$

On the other hand, applying (17), (19) and Hölder inequality, we have

$$\begin{aligned} \|g\|_{L^p(\mu)} &\leq C \|f\|_{L^p(\mu)} + Ct^{1-\frac{1}{p}} \left\| \sum_j \varphi_j \right\|_{L^1(\mu)}^{\frac{1}{p}} \\ &\leq C \|f\|_{L^p(\mu)} + Ct^{1-\frac{1}{p}} \left[\sum_j \left(\int_{S_j} |\varphi_j(x)|^p d\mu(x) \right)^{\frac{1}{p}} [\mu(S_j)]^{\frac{1}{p'}} \right]^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned} &\leq C \|f\|_{L^p(\mu)} + Ct^{1-\frac{1}{p}}t^{\frac{1}{p}-1} \left(\int_{\mathcal{X}} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\leq C \|f\|_{L^p(\mu)}. \end{aligned}$$

Further, for $1 < s < p$, by applying the result of the Claim and (21), we deduce

$$\begin{aligned} &\mu(\{x \in \mathcal{X} : M_r^\sharp[\mathfrak{M}_{\kappa,b}^*(g)](x) > 2Ct\}) \\ &\leq \mu(\{x \in \mathcal{X} : M_{s,6}[\mathfrak{M}_{\kappa}^*(g)](x) > Ct\}) + \mu(\{x \in \mathcal{X} : \|g\|_{L^\infty(\mu)} > Ct\}) \\ &\leq Ct^{-p} \|M_{s,6}[\mathfrak{M}_{\kappa}^*(g)]\|_{L^p(\mu)}^p \\ &\leq Ct^{-p} \|g\|_{L^p(\mu)}^p \leq Ct^{-p} \|f\|_{L^p(\mu)}^p. \end{aligned} \tag{22}$$

From this, we can write

$$\begin{aligned} &\mu(\{x \in \mathcal{X} : M_r^\sharp[\mathfrak{M}_{\kappa,b}^*(h)](x) > t\}) \\ &\leq \mu(\{x \in \mathcal{X} : CM_{r,6}[\mathfrak{M}_{\kappa,b}^*(h)](x) > t\}) \\ &\leq \mu(\{x \in \mathcal{X} : CM_{r,6}[\mathfrak{M}_{\kappa}^*(\sum_j (b - m_{\widetilde{6B_j}}(b))h_j)](x) > \frac{t}{2}\}) \\ &\quad + \mu(\{x \in \mathcal{X} : CM_{r,6}[\sum_j |b(x) - m_{\widetilde{6B_j}}(b)|\mathfrak{M}_{\kappa}^*(h_j)](x) > \frac{t}{2}\}) \\ &=: H_1 + H_2, \end{aligned}$$

where we have used the fact that $M_r^\sharp f(x) \leq CM_{r,6}f(x)$ (see [20]).

By applying the $(L^1(\mu), L^{1,\infty}(\mu))$ -boundedness of M_ρ , for any $\sigma > 0$, we get

$$\sigma\mu(\{x \in \mathcal{X} : M_{r,6}(u)(x) > \sigma\}) \leq C \sup_{\tau > C\sigma} \tau\mu(\{x \in \mathcal{X} : |u(x)| > \tau\}). \tag{23}$$

Choosing $1 < p_1 < p$, by $h_j := \sum_j (f\omega_j - \varphi_j)$ and (23), we have

$$\begin{aligned} H_1 &\leq Ct^{-1} \sup_{\tau > Ct} \tau \times \frac{1}{\tau^{p_1}} \|\mathfrak{M}_{\kappa}^*[\sum_j (b - m_{\widetilde{6B_j}}(b))h_j]\|_{L^{p_1}(\mu)}^{p_1} \\ &\leq Ct^{-p_1} \|\sum_j (b - m_{\widetilde{6B_j}}(b))f\omega_j\|_{L^{p_1}(\mu)}^{p_1} + Ct^{-p_1} \|\sum_j (b - m_{\widetilde{6B_j}}(b))\varphi_j\|_{L^{p_1}(\mu)}^{p_1} \\ &=: H_{11} + H_{12}. \end{aligned}$$

For H_{11} . By Hölder inequality, (13) and (14), we have

$$\begin{aligned} H_{11} &\leq Ct^{-p_1} \sum_j \left(\int_{6B_j} |f(x)|^p d\mu(x) \right)^{\frac{p_1}{p}} \left(\int_{6B_j} |b(x) - m_{\widetilde{6B_j}}(b)|^{\frac{pp_1}{p-p_1}} d\mu(x) \right)^{\frac{p-p_1}{p}} \\ &\leq Ct^{-p_1} \sum_j \left(\int_{6B_j} |f(x)|^p d\mu(x) \right)^{\frac{p_1}{p}} [\mu(6^2 B_j)]^{1-\frac{p_1}{p}} \\ &\leq Ct^{-p_1} \sum_j \left(\int_{6B_j} |f(x)|^p d\mu(x) \right)^{\frac{p_1}{p}} \left(t^{-p} \int_{B_j} |f(x)|^p d\mu(x) \right)^{1-\frac{p_1}{p}} \\ &\leq Ct^{-p} \|f\|_{L^p(\mu)}^p. \end{aligned}$$

With a way similar to that used in the proof of D_{12} in [20], it is easy to obtain

$$H_{12} \leq Ct^{-p} \|f\|_{L^p(\mu)}^p.$$

Now, we turn to estimate H_2 . By (23) and $h_j := \sum_j (f\omega_j - \varphi_j)$, write

$$\begin{aligned} H_2 &\leq C t^{-1} \sup_{\tau > C \frac{1}{2}} \tau \mu(\{x \in \mathcal{X} : \sum_j |b - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(h_j)(x) > C \tau\}) \\ &\leq C t^{-1} \sum_j \int_{\mathcal{X}} |b(x) - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(h_j)(x) d\mu(x) \\ &\leq C t^{-1} \sum_j \int_{\mathcal{X} \setminus 6S_j} |b(x) - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(h_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_j \int_{6S_j} |b(x) - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(\varphi_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_j \int_{\frac{6}{5}6B_j} |b(x) - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(f\omega_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_j \int_{6S_j \setminus \frac{6}{5}6B_j} |b(x) - m_{\widetilde{6B_j}}(b)| \mathfrak{M}_\kappa^*(f\omega_j)(x) d\mu(x) \\ &=: H_{21} + H_{22} + H_{23} + H_{24}. \end{aligned}$$

An argument similar to that used in the proof of E_1 in [17, Theorem 1.10] shows that

$$H_{21} \leq C t^{-1} \|h_j\|_{L^1(\mu)}, \quad H_{24} \leq C t^{-1} \|h_j\|_{L^1(\mu)},$$

which, together with the fact that $\|h_j\|_{L^1(\mu)} \leq C t^{1-p} \|f\|_{L^p(\mu)}^p$, thus, $H_{21} + H_{24} \leq C t^{-p} \|f\|_{L^p(\mu)}^p$.

For H_{22} . By Hölder inequality, the $L^p(\mu)$ -boundedness of \mathfrak{M}_κ^* , (13) and (19), we have

$$\begin{aligned} H_{22} &\leq C t^{-1} \sum_j \int_{6S_j} |b(x) - m_{\widetilde{6S_j}}(b)| \mathfrak{M}_\kappa^*(\varphi_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_j |m_{\widetilde{6S_j}}(b) - m_{\widetilde{6B_j}}(b)| \int_{6S_j} \mathfrak{M}_\kappa^*(\varphi_j)(x) d\mu(x) \\ &\leq C t^{-1} \sum_j \|\mathfrak{M}_\kappa^*(\varphi_j)\|_{L^p(\mu)} \left[\left(\int_{6S_j} |b(x) - m_{\widetilde{6S_j}}(b)|^{p'} d\mu(x) \right)^{\frac{1}{p'}} + \mu(6S_j)^{\frac{1}{p'}} \right] \\ &\leq C t^{-1} \sum_j \|\varphi_j\|_{L^p(\mu)} \mu(6S_j)^{\frac{1}{p'}} \\ &\leq C t^{-1} t^{1-p} \sum_j \int_{6B_j} |f(x)|^p d\mu(x) \\ &\leq C t^{-p} \|f\|_{L^p(\mu)}^p. \end{aligned}$$

Similar to the estimate of the H_{22} , we conclude

$$\begin{aligned} H_{23} &\leq C t^{-1} \sum_j \left(\int_{\frac{6}{5}6B_j} |b(x) - m_{\widetilde{6B_j}}(b)|^{p'} d\mu(x) \right)^{\frac{1}{p'}} \|\mathfrak{M}_\kappa^*(f\omega_j)\|_{L^p(\mu)} \\ &\leq C t^{-1} \sum_j \mu(6^2 B_j)^{\frac{1}{p'}} \|f\omega_j\|_{L^p(\mu)} \\ &\leq C t^{-p} \sum_j \int_{6B_j} |f(x)|^p d\mu(x) \leq C t^{-p} \|f\|_{L^p(\mu)}^p. \end{aligned}$$

Combining the estimates for H_{21} , H_{22} , H_{23} and H_{24} , we get

$$H_2 \leq Ct^{-p} \|f\|_{L^p(\mu)}^p,$$

which, together with H_1 and (22), imply (20) and hence complete the proof of Theorem 1.8. □

Next, we come to prove Theorem 1.9. In order to do this, we need the following claim.

Claim. Let $K(x, y)$ satisfy (7) and (12), $s \in (1, \infty)$, $p_0 \in (1, \infty)$ and $b \in L^\infty(\mu)$. If \mathfrak{M}_{κ}^* is bounded on $L^2(\mu)$, then there exists a positive constant C such that, for all $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ and for all $x \in \mathcal{X}$,

$$M^\sharp[\mathfrak{M}_{b,\kappa}^*(f)](x) \leq C \{ \|b\|_{\text{RBMO}(\mu)} M_{s,6}[\mathfrak{M}_{\kappa}^*(f)](x) + \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^\infty(\mu)} \}. \tag{24}$$

Proof. Without loss generality, we may assume $\|b\|_{\text{RBMO}(\mu)} = 1$. Let B be an arbitrary ball and S be a doubling ball with $B \subset S$, denote

$$h_B := m_B[\mathfrak{M}_{\kappa}^*([b - m_{\bar{B}}(b)]f\chi_{\mathcal{X} \setminus \frac{6}{5}B})]$$

and

$$h_S := m_S[\mathfrak{M}_{\kappa}^*([b - m_S(b)]f\chi_{\mathcal{X} \setminus \frac{6}{5}B})]$$

To prove (24), it only needs to prove

$$\frac{1}{\mu(6B)} \int_B |\mathfrak{M}_{\kappa,b}^*(f)(y) - h_B| d\mu(y) \leq CM_{s,6}[\mathfrak{M}_{\kappa}^*(f)](x) + \|f\|_{L^\infty(\mu)} \tag{25}$$

and

$$|h_B - h_S| \leq CK_{B,S}^2 \{ M_{s,6}[\mathfrak{M}_{\kappa}^*(f)](x) + \|f\|_{L^\infty(\mu)} \}. \tag{26}$$

To prove (25), for a fixed ball, $x \in B$ and $f \in L^\infty(\mu)$, we decompose f as

$$f(y) = f\chi_{\frac{6}{5}B}(y) + f\chi_{\mathcal{X} \setminus \frac{6}{5}B}(y) =: f_1(y) + f_2(y).$$

Thus, we write

$$\begin{aligned} & \frac{1}{\mu(6B)} \int_B |\mathfrak{M}_{\kappa,b}^*(f)(y) - h_B| d\mu(y) \\ & \leq \frac{1}{\mu(6B)} \int_B \left| |b(y) - m_{\bar{B}}(b)| \mathfrak{M}_{\kappa}^*(f)(y) + \mathfrak{M}_{\kappa}^*([b - m_{\bar{B}}(b)]f_1)(y) \right. \\ & \quad \left. + \mathfrak{M}_{\kappa}^*([b - m_{\bar{B}}(b)]f_2)(y) - h_B \right| d\mu(y) \\ & \leq \frac{1}{\mu(6B)} \int_B |b(y) - m_{\bar{B}}(b)| \mathfrak{M}_{\kappa}^*(f)(y) d\mu(y) + \frac{1}{\mu(6B)} \int_B \mathfrak{M}_{\kappa}^*([b - m_{\bar{B}}(b)]f_1)(y) d\mu(y) \\ & \quad + \frac{1}{\mu(6B)} \int_B |\mathfrak{M}_{\kappa}^*([b - m_{\bar{B}}(b)]f_2)(y) - h_B| d\mu(y) \\ & =: E_1 + E_2 + E_3. \end{aligned}$$

For E_1 , by Hölder inequality and (13), we have

$$\begin{aligned} E_1 & \leq \frac{1}{\mu(6B)} \left(\int_B |b(y) - m_{\bar{B}}(b)|^{s'} d\mu(y) \right)^{\frac{1}{s'}} \left(\int_B [\mathfrak{M}_{\kappa}^*(f)(y)]^s d\mu(y) \right)^{\frac{1}{s}} \\ & \leq \left(\frac{1}{\mu(6B)} \int_B |b(y) - m_{\bar{B}}(b)|^{s'} d\mu(y) \right)^{\frac{1}{s'}} \left(\frac{1}{\mu(6B)} \int_B [\mathfrak{M}_{\kappa}^*(f)(y)]^s d\mu(y) \right)^{\frac{1}{s}} \end{aligned}$$

$$\leq CM_{s,6}[\mathfrak{M}_\kappa^*(f)](x).$$

Applying Hölder inequality, the $L^2(\mu)$ -boundedness of \mathfrak{M}_κ^* and (13), we can conclude

$$\begin{aligned} E_2 &\leq \frac{[\mu(B)]^{\frac{1}{2}}}{\mu(6B)} \left(\int_B [\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_1)(y)]^2 d\mu(y) \right)^{\frac{1}{2}} \\ &\leq C \frac{[\mu(B)]^{\frac{1}{2}}}{\mu(6B)} \left(\int_{\frac{6}{5}B} |b(y) - m_{\tilde{B}}(b)|^2 |f(y)|^2 d\mu(y) \right)^{\frac{1}{2}} \\ &\leq C \|f\|_{L^\infty(\mu)} \frac{[\mu(B)]^{\frac{1}{2}} [\mu(2 \times \frac{6}{5}B)]^{\frac{1}{2}}}{\mu(6B)} \left(\frac{1}{\mu(2 \times \frac{6}{5}B)} \int_{\frac{6}{5}B} |b(y) - m_{\frac{6}{5}B}(b)|^2 d\mu(y) \right)^{\frac{1}{2}} \\ &\quad + C \|f\|_{L^\infty(\mu)} \frac{[\mu(B)]^{\frac{1}{2}} [\mu(\frac{6}{5}B)]^{\frac{1}{2}}}{\mu(6B)} |m_{\frac{6}{5}B}(b) - m_{\tilde{B}}(b)| \\ &\leq C \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Where we use the fact that $|m_{\tilde{B}}(b) - m_{\frac{6}{5}B}(b)| \leq CK_{B, \frac{6}{5}B} \leq C$.

For E_3 , it follows that

$$\begin{aligned} E_3 &= \frac{1}{\mu(6B)} \int_B |\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(y) - h_B| d\mu(y) \\ &= \frac{1}{\mu(6B)} \int_B |\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(y) - \frac{1}{\mu(B)} \int_B \mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(x) d\mu(x)| d\mu(y) \\ &\leq \frac{1}{\mu(6B)} \frac{1}{\mu(B)} \int_B \int_B |\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(y) - \mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(x)| d\mu(x) d\mu(y), \end{aligned}$$

For any $x, y \in B$, by (7) and Minkowski inequality, we have

$$\begin{aligned} &|\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(x) - \mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f_2)(y)| \\ &\leq \left[\iint_{\mathcal{X} \times (0, \infty)} \left| \left(\frac{t}{t + d(x, \tilde{x})} \right)^{\frac{\kappa}{2}} - \left(\frac{t}{t + d(y, \tilde{x})} \right)^{\frac{\kappa}{2}} \right|^2 \right. \\ &\quad \left. \times \left| \int_{d(\tilde{x}, z) \leq t} K(\tilde{x}, z) [b(z) - m_{\tilde{B}}(b)] f_2(z) d\mu(z) \right|^2 \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} \\ &\leq C \int_{\tilde{x}} |b(z) - m_{\tilde{B}}(b)| |f_2(z)| \left[\iint_{d(\tilde{x}, z) \leq t} \left[\frac{d(x, y)}{d(x, \tilde{x})} \right]^2 \frac{[d(\tilde{x}, z)]^2}{[\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} d\mu(z) \\ &\leq C \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} |b(z) - m_{\tilde{B}}(b)| \left[\iint_{\substack{d(\tilde{x}, z) \leq t \\ 2d(\tilde{x}, z) > d(x, z)}} \left[\frac{d(x, y)}{d(x, \tilde{x})} \right]^2 \frac{[d(\tilde{x}, z)]^2}{[\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \right. \\ &\quad \left. \times \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} d\mu(z) \\ &\quad + C \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} |b(z) - m_{\tilde{B}}(b)| \left[\iint_{\substack{d(\tilde{x}, z) \leq t, d(x, \tilde{x}) < t \\ 2d(\tilde{x}, z) \leq d(x, z)}} \left[\frac{d(x, y)}{d(x, \tilde{x})} \right]^2 \frac{[d(y, z)]^2}{[\lambda(y, d(y, z))]^2} \right. \\ &\quad \left. \times \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} d\mu(z) \end{aligned}$$

$$\begin{aligned}
 &+ C \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} |b(z) - m_{\bar{B}}(b)| \left[\iint_{\substack{d(\tilde{x}, z) \leq t, d(x, \tilde{x}) > t \\ 2d(\tilde{x}, z) \leq d(x, z)}} \left[\frac{d(x, y)}{d(x, \tilde{x})} \right]^2 \frac{[d(y, z)]^2}{[\lambda(y, d(y, z))]^2} \right. \\
 &\quad \left. \times \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} d\mu(z) \\
 &=: F_1 + F_2 + F_3.
 \end{aligned}$$

For F_1 , we have

$$\begin{aligned}
 F_1 &\leq C r_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} |b(z) - m_{\bar{B}}(b)| \left[\int_{d(\tilde{x}, z)}^\infty \int_{2d(\tilde{x}, z) > d(x, z)} \frac{[d(\tilde{x}, z)]^2}{[d(x, z)]^2 [\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \right. \\
 &\quad \left. \times \frac{d\mu(\tilde{x}) dt}{\lambda(\tilde{x}, t) t^3} \right]^{\frac{1}{2}} d\mu(z) \\
 &\leq C r_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\bar{B}}(b)|}{d(x, z)} \left[\int_{2d(\tilde{x}, z) > d(x, z)} \frac{[d(\tilde{x}, z)]^2}{[\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \right. \\
 &\quad \left. \times \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))} \left(\int_{d(\tilde{x}, z)}^\infty \frac{dt}{t^3} \right) d\mu(\tilde{x}) \right]^{\frac{1}{2}} d\mu(z) \\
 &\leq C r_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\bar{B}}(b)|}{d(x, z)} \left[\int_{2d(\tilde{x}, z) > d(x, z)} \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \right. \\
 &\quad \left. \times \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))} d\mu(\tilde{x}) \right]^{\frac{1}{2}} d\mu(z) \\
 &\leq C r_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\bar{B}}(b)|}{d(x, z)} \left[\int_{2d(\tilde{x}, z) > d(x, z)} \frac{1}{\lambda(\tilde{x}, \frac{1}{2}d(x, z))]^2} \right. \\
 &\quad \left. \times \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))} d\mu(\tilde{x}) \right]^{\frac{1}{2}} d\mu(z) \\
 &\leq C r_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\bar{B}}(b)|}{d(x, z)} \left[\sum_{k=0}^\infty \int_{2^k B(z, d(c_B, z)) \setminus 2^{k-1} B(z, d(c_B, z))} \right. \\
 &\quad \left. \times \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))} d\mu(\tilde{x}) \right]^{\frac{1}{2}} \frac{1}{\lambda(z, d(x, z))} d\mu(z) \\
 &\leq C \|f\|_{L^\infty(\mu)} \sum_{k=1}^\infty 6^{-(k-1)} \int_{6^k \frac{6}{5}B \setminus 6^{k-1} \frac{6}{5}B} \frac{|b(z) - m_{\bar{B}}(b)|}{\lambda(c_B, d(c_B, z))} d\mu(z) \\
 &\leq C \|f\|_{L^\infty(\mu)} \sum_{k=1}^\infty \frac{6^{-(k-1)}}{\lambda(c_B, 6^{k-1} \frac{6}{5}r_B)} \left[\int_{6^k \frac{6}{5}B} |b(z) - m_{\widetilde{6^k \frac{6}{5}B}}(b)| d\mu(z) + m_{\widetilde{6^k \frac{6}{5}B}}(b) - m_{\bar{B}}(b) \right] \\
 &\leq C \|f\|_{L^\infty(\mu)}.
 \end{aligned}$$

Next we estimate F_2 . For any $\tilde{x} \in \mathcal{X}$ and $x \in \mathcal{X} \setminus \frac{6}{5}B$ satisfying $d(\tilde{x}, x) < t$, $2d(\tilde{x}, z) \leq d(x, z)$ and $\frac{1}{2}d(x, z) < t$, we can conclude

$$\begin{aligned} F_2 &\leq Cr_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\tilde{B}}(b)|}{d(x, z)} \left[\int_{2d(\tilde{x}, z) \leq d(x, z)} \int_{\frac{1}{2}d(x, z)}^\infty \frac{[d(\tilde{x}, z)]^2}{[\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \right. \\ &\quad \left. \times \frac{d\mu(\tilde{x})dt}{\lambda(\tilde{x}, t)t^3} \right]^{\frac{1}{2}} d\mu(z) \\ &\leq Cr_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\tilde{B}}(b)|}{d(x, z)} \left[\int_{2d(\tilde{x}, z) \leq d(x, z)} \frac{[d(\tilde{x}, z)]^2}{[\lambda(\tilde{x}, d(\tilde{x}, z))]^2} \frac{1}{\lambda(\tilde{x}, d(\tilde{x}, z))} \right. \\ &\quad \left. \times \left(\int_{\frac{1}{2}d(x, z)}^\infty \frac{dt}{t^3} \right) d\mu(y) \right]^{\frac{1}{2}} d\mu(z) \\ &\leq Cr_B \|f\|_{L^\infty(\mu)} \int_{\mathcal{X} \setminus \frac{6}{5}B} \frac{|b(z) - m_{\tilde{B}}(b)|}{\lambda(c_B, d(c_B, z))} \frac{1}{d(x, z)} d\mu(z) \\ &\leq C \|f\|_{L^\infty(\mu)} \sum_{k=1}^\infty 6^{-(k-1)} \int_{6^k \frac{6}{5}B \setminus 6^{k-1} \frac{6}{5}B} \frac{|b(z) - m_{\tilde{B}}(b)|}{\lambda(c_B, d(c_B, z))} d\mu(z) \\ &\leq C \|f\|_{L^\infty(\mu)}. \end{aligned}$$

With a way similar to that used in the proof of F_2 , it follows that

$$F_3 \leq C \|f\|_{L^\infty(\mu)},$$

which, together with the estimates of F_1 and F_2 , it is easy to see that

$$E_3 \leq C \|f\|_{L^\infty(\mu)},$$

thus, the proof of (25) is finished.

Now, we estimate (26). For any two balls $B \subset S$ with $x \in B$ and assume $N := N_{B,S} + 1$, where S is a doubling ball. Write

$$\begin{aligned} |h_B - h_S| &= |m_B[\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f\chi_{\mathcal{X} \setminus \frac{6}{5}B})] - m_S[\mathfrak{M}_\kappa^*([b - m_S(b)]f\chi_{\mathcal{X} \setminus \frac{6}{5}B})]| \\ &\leq |m_B[\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f\chi_{\mathcal{X} \setminus 6^N B})] - m_S[\mathfrak{M}_\kappa^*([b - m_S(b)]f\chi_{\mathcal{X} \setminus 6^N B})]| \\ &\quad + |m_S[\mathfrak{M}_\kappa^*([b - m_S(b)]f\chi_{\mathcal{X} \setminus 6^N B})] - m_S[\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f\chi_{\mathcal{X} \setminus 6^N B})]| \\ &\quad + |m_B[\mathfrak{M}_\kappa^*([b - m_{\tilde{B}}(b)]f\chi_{6^N B \setminus \frac{6}{5}B})]| + |m_S[\mathfrak{M}_\kappa^*([b - m_S(b)]f\chi_{6^N B \setminus \frac{6}{5}B})]| \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Following the proof of E_3 , it is not difficult to see that

$$I_1 + I_4 \leq C \|f\|_{L^\infty(\mu)}.$$

Now, we estimate I_2 , for any $y \in \mathcal{X}$, applying Hölder inequality, we deduce

$$\begin{aligned} I_2 &\leq \frac{1}{\mu(S)} \int_S |m_S(b) - m_{\tilde{B}}(b)| \mathfrak{M}_\kappa^*(f\chi_{\mathcal{X} \setminus 6^N B})(y) d\mu(y) \\ &\leq C \frac{K_{B,S}}{\mu(S)} \left(\int_S [\mathfrak{M}_\kappa^*(f\chi_{\mathcal{X} \setminus 6^N B})(y)]^s d\mu(y) \right)^{\frac{1}{s}} \mu(S)^{1-\frac{1}{s}} \\ &\leq CK_{B,S} M_{s,6}[\mathfrak{M}_\kappa^*(f)](x), \end{aligned}$$

where we have used the fact that $|m_S(b) - m_{\tilde{B}}(b)| \leq CK_{B,S}$.

Finally, we estimate for I_3 . For $x \in B$, we have

$$\begin{aligned} I_3 &= |m_B[\mathfrak{M}_{\kappa}^*([b - m_{\tilde{B}}(b)]f\chi_{6^N B \setminus \frac{6}{5}B})]| \\ &\leq |m_B[\mathfrak{M}_{\kappa}^*([b - m_{\tilde{B}}(b)]f\chi_{6^N B \setminus 6B})]| + |m_B[\mathfrak{M}_{\kappa}^*([b - m_{\tilde{B}}(b)]f\chi_{6B \setminus \frac{6}{5}B})]| \\ &=: I_{31} + I_{32}. \end{aligned}$$

With a way similar to that used in the proof of E_2 , it follows that

$$I_{32} \leq C \|f\|_{L^\infty(\mu)}.$$

Meanwhile, following the proof of E_3 , we have

$$I_{31} \leq CK_{B,S}^2 \|f\|_{L^\infty(\mu)}.$$

Combining the estimates for I_{31} , I_{32} , I_1 , I_2 and I_4 , we obtain (26). Thus, we complete the proof of (24). □

Proof of Theorem 1.9. For convenience, we assume $\|b\|_{\text{RBMO}(\mu)} = 1$. For each fixed $t > 0$ and functions f with bounded support, applying Lemma 2.5 to $|f|$ with $p = 1$, and letting B_j, S_j, φ_j and ω_j as the same as Lemma 2.5. We see that $f = g + h$, where

$$g(x) := f\chi_{\mathcal{X} \setminus \cup_j(6B_j)} + \sum_j \varphi_j, \quad h(x) := \sum_j [f(x)\omega_j(x) - \varphi_j(x)] := \sum_j h_j(x). \tag{27}$$

Noticing that $\|g\|_{L^1(\mu)} \leq C \|f\|_{L^1(\mu)}$. Applying the $L^2(\mu)$ -boundedness of $\mathfrak{M}_{b,\kappa}^*$ in Theorem 1.8 and the fact that $|g(x)| \leq Ct$, it is not difficult to obtain that

$$\mu(\{x \in \mathcal{X} : \mathfrak{M}_{\kappa,b}^*(g)(x) > t\}) \leq Ct^{-1} \int_{\mathcal{X}} |f(y)| d\mu(y).$$

From (14), we have $\mu(\cup_j 6^2 B_j) \leq \frac{C}{t} \|f\|_{L^1(\mu)}$, so the proof of Theorem 1.9 is reduced to prove

$$\mu\left(\left\{x \in \mathcal{X} \setminus \cup_j(6^2 B_j) : \mathfrak{M}_{\kappa,b}^*(h)(x) > t\right\}\right) \leq C \int_{\mathcal{X}} \frac{|f(y)|}{t} \log\left(2 + \frac{|f(y)|}{t}\right) d\mu(y). \tag{28}$$

For each fixed j and $x \in \mathcal{X} \setminus \cup_j(6^2 B_j)$, let $b_j(x) := b(x) - m_{\tilde{B}_j}(b)$ and write

$$\mathfrak{M}_{b,\kappa}^*(h)(x) \leq \sum_j |b_j(x)| \mathfrak{M}_{\kappa}^*(h_j)(x) + \mathfrak{M}_{\kappa}^*(\sum_j b_j h_j)(x) := \text{I}(x) + \text{II}(x).$$

With a way similar to that used in the estimate of H_{21} , H_{22} and H_{23} in Theorem 1.8, we have

$$\mu\left(\left\{x \in \mathcal{X} \setminus \cup_j(6^2 B_j) : \text{I}(x) > t\right\}\right) \leq \frac{C}{t} \|f\|_{L^1(\mu)}. \tag{29}$$

By $h_j := f\omega_j - \varphi_j$, write

$$\begin{aligned} &\mu(\{x \in \mathcal{X} \setminus \cup_j(6^2 B_j) : |\text{II}(x)| > t\}) \\ &\leq \mu(\{\mathcal{X} \setminus \cup_j(6^2 B_j) : \mathfrak{M}_{\kappa}^*(\sum_j b_j f\omega_j)(x) > \frac{t}{2}\}) + \mu(\{\mathcal{X} \setminus \cup_j(6^2 B_j) : \mathfrak{M}_{\kappa}^*(\sum_j b_j \varphi_j)(x) > \frac{t}{2}\}) \\ &\leq C \sum_j \int_{B_j} \Phi\left(\frac{|b(y) - m_{\tilde{B}_j}| |f(y)|}{t}\right) d\mu(y) + C \int_{\mathcal{X}} \Phi\left(\sum_j \frac{|b(y) - m_{\tilde{B}_j}| |\varphi(y)|}{t}\right) d\mu(y) \\ &=: \text{II}_1 + \text{II}_2. \end{aligned}$$

For all $\alpha \geq 1$, let $\Phi_\alpha(t) = t \log^\alpha(2 + t)$, $\Psi_{\frac{1}{\alpha}}(t) = \exp t^{\frac{1}{\alpha}}$. For any $s, t > 0$, we have the following facts

$$\Phi(st) \leq C[\Phi(s) + \Psi(t)],$$

and for any $s > 0$ and $t_1, t_2 > 0$, we have

$$\Phi_s(t_1 t_2) \leq C\Phi_s(t_1)\Phi_s(t_2),$$

For Π_1 , by (14) and Lemma 2.6, we have

$$\begin{aligned} \Pi_1 &\leq C \sum_j \int_{B_j} \left[\Psi\left(\frac{|b(y) - m_{B_j}(b)|}{B_1}\right) + \Phi\left(\frac{|f(y)|}{t} B_1\right) \right] d\mu(y) \\ &\leq C \sum_j \int_{B_j} \left[\exp\left(\frac{|b(y) - m_{B_j}(b)|}{B_1}\right) + \Phi\left(\frac{|f(y)|}{t} B_1\right) \right] d\mu(y) \\ &\leq C \sum_j \int_{B_j} \left[\mu(6^2 B_j) + \Phi\left(\frac{|f(y)|}{t} B_1\right) \right] d\mu(y) \\ &\leq \frac{C}{t} \sum_j \int_{B_j} |f(y)| d(y) + \int_{\mathcal{X}} \Phi\left(\frac{|f(y)|}{t}\right) d\mu(y) \\ &\leq C \int_{\mathcal{X}} \Phi\left(\frac{|f(y)|}{t}\right) d\mu(y). \end{aligned}$$

In order to estimate Π_2 , we assume that $\Lambda \subset N$ is a set of finite index, $r_j(y) := \frac{1}{t}|\varphi_j(y)|$. By applying the convex property of Φ , we get

$$\Phi\left(\sum_{j \in \Lambda} r_j(y)|b(y) - m_{\tilde{B}_j}(b)|\right) \leq C \sum_{j \in \Lambda} r_j(y)\Phi(|b(y) - m_{\tilde{B}_j}(b)|).$$

On the other hand, if we take $\Lambda = N$, the above inequality also holds by the property of Φ . With a way similar to H_{12} in the proof of Theorem 1.8, we have

$$\begin{aligned} \Pi_2 &\leq C \frac{1}{t} \sum_j \|\varphi_j\|_{L^\infty(\mu)} \int_{S_j} \Phi(|b(y) - m_{\tilde{B}_j}(b)|) d\mu(y) \\ &\leq C \frac{1}{t} \sum_j \|\varphi_j\|_{L^\infty(\mu)} \int_{S_j} |b(y) - m_{\tilde{B}_j}(b)| [1 + |b(y) - m_{\tilde{B}_j}(b)|] d\mu(y) \\ &\leq C \frac{1}{t} \sum_j \|\varphi_j\|_{L^\infty(\mu)} \mu(S_j) \\ &\leq C \int_{\mathcal{X}} \frac{|f(y)|}{t} d\mu(y), \end{aligned}$$

which, together with Π_1 and (29), imply (28), and hence the proof of Theorem 1.9 is finished. □

Proof of Theorem 1.10. Without loss generality, we may assume that $\|b\|_{\text{RBMO}(\mu)} = 1$ and $\rho = 2$ as in Definition 1.6. It suffices to prove that, for any $(\infty, 1)_\tau$ -atomic block h ,

$$\mu(\{x \in \mathcal{X} : \mathfrak{M}_{\kappa,b}^*(h)(x) > t\}) \leq C \frac{|h|_{H_{\text{ab}}^{1,\infty}(\mu)}}{t}. \tag{30}$$

Assume that $\text{supp}(h) \subset R$ and $h = \sum_{j=1}^2 \tau_j a_j$, where a_j is a function supported in $B_j \subset R$ such that $\|a_j\|_{L^\infty(\mu)} \leq [\mu(4B_j)]^{-1} K_{B_j,R}^{-1}$ and $|\tau_1| + |\tau_2| \sim |h|_{H_{\text{ab}}^{1,\infty}(\mu)}$. Write

$$\mathfrak{M}_{\kappa,b}^*(h)(x) \leq |b(x) - m_{\tilde{R}}(b)| \mathfrak{M}_{\kappa}^*(h)(x) + \mathfrak{M}_{\kappa}^*([m_{\tilde{R}}(b) - b]h)(x)$$

$$=: J_1(x) + J_2(x).$$

By the $(L^1(\mu), L^{1,\infty}(\mu))$ -boundedness of \mathfrak{M}_{κ}^* (see [17]), we have

$$\begin{aligned} \mu(\{x \in \mathcal{X} : J_2(x) > \frac{t}{2}\}) &\leq C \frac{1}{t} \int_R |b(x) - m_{\widetilde{R}}(b)| |h(x)| d\mu(x) \\ &\leq C \frac{|\tau_1|}{t} \int_R |b(x) - m_{\widetilde{R}}(b)| |a_1(x)| d\mu(x) \\ &\quad + C \frac{|\tau_2|}{t} \int_R |b(x) - m_{\widetilde{R}}(b)| |a_2(x)| d\mu(x) \\ &=: J_{21} + J_{22}. \end{aligned}$$

Now, we estimate J_{21} , applying Hölder inequality and (13), we have

$$\begin{aligned} J_{21} &\leq C \frac{|\tau_1|}{t} \|a_1\|_{L^\infty(\mu)} \left[\int_{B_j} |b(x) - m_{\widetilde{B}_j}(b)| d\mu(x) + |m_{\widetilde{B}_j}(b) - m_{\widetilde{R}}(b)| \mu(B_j) \right] \\ &\leq C \frac{|\tau_1|}{t} \mu(4B_j)^{-1} K_{B_j,R}^{-1} K_{B_j,R} \mu(B_j) \leq C \frac{|\tau_1|}{t}. \end{aligned}$$

Similar to the estimate of J_{21} , it is easy to obtain that

$$J_{22} \leq C \frac{|\tau_2|}{t}.$$

Thus, we can conclude that

$$\mu(\{x \in \mathcal{X} : J_2(x) > \frac{t}{2}\}) \leq C t^{-1} (|\tau_1| + |\tau_2|) \leq C t^{-1} |h|_{H_{\text{ab}}^{1,\infty}(\mu)}. \tag{31}$$

Now, we turn to estimate J_1 . Write

$$\begin{aligned} \mu(\{x \in \mathcal{X} : J_1(x) > \frac{t}{2}\}) &\leq C t^{-1} \int_{\mathcal{X} \setminus 2R} |b(x) - m_{\widetilde{R}}(b)| \mathfrak{M}_{\kappa}^*(h)(x) d\mu(x) \\ &\quad + C t^{-1} \int_{2R} |b(x) - m_{\widetilde{R}}(b)| \mathfrak{M}_{\kappa}^*(h)(x) d\mu(x) \\ &=: J_{11} + J_{12} \end{aligned}$$

An argument similar to H_{21} that used Theorem 1.8 and $\|a_j\|_{L^\infty(\mu)} \leq [\mu(4B_j)]^{-1} K_{B_j,R}^{-1}$, we have

$$J_{11} \leq C t^{-1} |h|_{H_{\text{ab}}^{1,\infty}(\mu)}.$$

It remains to estimate J_{12} . Write

$$\begin{aligned} J_{12} &\leq C t^{-1} \sum_{j=1}^2 |\tau_j| \int_{2R} |b(x) - m_{\widetilde{R}}(b)| \mathfrak{M}_{\kappa}^*(a_j)(x) d\mu(x) \\ &\leq C t^{-1} \sum_{j=1}^2 |\tau_j| \int_{2R \setminus 6^2 B_j} |b(x) - m_{\widetilde{R}}(b)| \mathfrak{M}_{\kappa}^*(a_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_{j=1}^2 |\tau_j| \int_{6^2 B_j} |b(x) - m_{\widetilde{6^2 B_j}}(b)| \mathfrak{M}_{\kappa}^*(a_j)(x) d\mu(x) \\ &\quad + C t^{-1} \sum_{j=1}^2 |\tau_j| |m_{\widetilde{R}}(b) - m_{\widetilde{6^2 B_j}}(b)| \int_{6^2 B_j} \mathfrak{M}_{\kappa}^*(a_j)(x) d\mu(x) \end{aligned}$$

$$=: U_1 + U_2 + U_3.$$

With an argument similar to that used in the proof of V_{11} in [17, Theorem 1.11], it is easy to get

$$U_3 \leq Ct^{-1}|b|_{H_{\text{atb}}^1(\mu)}.$$

For U_2 , applying Hölder inequality, the $L^2(\mu)$ -boundedness of \mathfrak{M}_κ^* , (13) and $\|a_j\|_{L^\infty(\mu)} \leq [\mu(4B_j)]^{-1}K_{B_j,R}^{-1}$, we have

$$\begin{aligned} U_2 &\leq Ct^{-1} \sum_{j=1}^2 |\tau_j| K_{B_j,R} \|\mathfrak{M}_\kappa^*(a_j)\|_{L^2(\mu)} \mu(6^3 B_j)^{\frac{1}{2}} \\ &\leq Ct^{-1} \sum_{j=1}^2 |\tau_j| K_{B_j,R} \|a_j\|_{L^\infty(\mu)} \mu(B_j)^{\frac{1}{2}} \mu(6^3 B_j)^{\frac{1}{2}} \\ &\leq Ct^{-1}|b|_{H_{\text{atb}}^1(\mu)}. \end{aligned}$$

By the $L^2(\mu)$ -boundedness of \mathfrak{M}_κ^* and an argument similar to H_{24} in Theorem 1.8,

$$U_1 \leq Ct^{-1}|b|_{H_{\text{atb}}^1(\mu)}.$$

Combining the whole estimates as above, we finish the proof of Theorem 1.10. \square

4 Conclusions

In this work we proved that the commutators $\mathcal{M}_{\kappa,b}^*$ generated by the Littlewood-Paley operators \mathcal{M}_κ^* and $\text{RBMO}(\mu)$ functions were bounded on $L^p(\mu)$ with $1 < p < \infty$, and bounded from the spaces $L \log L(\mu)$ to the weak Lebesgue spaces over non-homogeneous metric measure spaces in the sense of Hytönen. Also, we obtained the boundedness of the commutators $\mathcal{M}_{\kappa,b}^*$ on Hardy spaces.

With the results of the commutators given herein, we shall consider the boundedness of the $\mathcal{M}_{\kappa,b}^*$ on Morrey spaces and generalized Morrey spaces over non-homogeneous metric measure spaces in the follow-up work.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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