

## Open Mathematics

## Research Article

Joanna Jureczko\*

# $\kappa$ -strong sequences and the existence of generalized independent families

<https://doi.org/10.1515/math-2017-0108>

Received April 27, 2017; accepted July 31, 2017.

**Abstract:** In this paper we will show some relations between generalized versions of strong sequences introduced by Efimov in 1965 and independent families. We also show some inequalities between cardinal invariants associated with these both notions.

**Keywords:** Strong sequences, Generalized independent families, Cardinal invariants

**MSC:** 03E20, 54A05, 54A25

## 1 Introduction

The strong sequences method was introduced by Efimov in 1965 [1] as a useful tool for proving some famous theorems in dyadic spaces, (i.e. continuous images of Cantor cube). Among others Efimov proved that strong sequences do not exist in the subbase of Cantor cube. The problem of the existence of strong sequences in the other spaces was raised in 90's of the last century (see [2] or [3] for more historical details).

The problem of the existence of independent families was raised among others by Fichtenholz and Kantorovitch [4] and Hausdorff [5]. There was shown that for each cardinal  $\kappa$  there exists an independent family of size  $2^\kappa$ . Engelking and Karłowicz introduced in [6] a generalization of the definition of this notion by considering partitions of a given set instead of a pair of sets. However, the name "generalized independent family" was used for the first time by Hu in [7]. In [8] Elser proved among others that under some assumptions there exists a generalized independent family (see Corollary 2.6.), using the result obtained by Hu (see Theorem 2.4 in [7]).

The aim of this paper is to show relations between generalized independent families and generalized strong sequences. There will be also shown relations between cardinal invariants concerning both mentioned notions.

The notation used in the paper is standard for the field and can be found in e. g. [9] and [10].

## 2 Generalized strong sequences

Let  $(X, r)$  be a set with an arbitrary relation  $r$  and let  $\kappa$  be a cardinal. By  $P(X)$  we denote the power set of  $X$ , i. e. the family of all subsets of  $X$ . We say that a set  $A \subset X$  has a bound iff there exists  $b \in X$  such that for all  $a \in A$  we have  $(a, b) \in r$ . (If elements  $a, b \in X$  have not a bound we say that they are *incompatible*). We say that a set  $A \subset X$  is  $\kappa$ -directed iff each subset of  $A$  of cardinality less than  $\kappa$  has a bound.

**Definition 2.1.** Let  $(X, r)$  be a set with a relation  $r$ . A sequence  $(H_\phi)_{\phi < \alpha}$ , where  $H_\phi \subset X$ , is called a  $\kappa$ -strong sequence if:

\*Corresponding Author: Joanna Jureczko: Wrocław University of Science and Technology, Poland,  
E-mail: joanna.jureczko@pwr.edu.pl

$1^\circ H_\phi$  is  $\kappa$ -directed for all  $\phi < \alpha$

$2^\circ H_\psi \cup H_\phi$  is not  $\kappa$ -directed for all  $\phi < \psi < \alpha$ .

Let  $\beta$  and  $\tau$  be cardinals. By  $\beta \ll \tau$  we denote:  $\tau$  is strongly  $\beta$ -inaccessible, i. e.  $\beta < \tau$  and  $\alpha^\lambda < \tau$  whenever  $\alpha < \beta, \lambda < \tau$ .

The following theorem is the generalization of Theorem 3 in [3]).

**Theorem 2.2** (on  $\kappa$ -strong sequences). *Let  $\beta, \kappa, \mu, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau, \mu < \beta, \kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a set of cardinality  $\tau$ . If there exists a  $\kappa$ -strong sequence  $\{H_\alpha \subset X : \alpha < \tau\}$  with  $|H_\alpha| \leq 2^\mu$  for all  $\alpha < \tau$ , then there exists a  $\kappa$ -strong sequence  $\{T_\alpha : \alpha < \beta\}$  with  $|T_\alpha| < \kappa$  for all  $\alpha < \beta$ .*

*Proof.* Consider a  $\kappa$ -strong sequence  $\{H_\alpha : \alpha < \tau\}$  as was done in the theorem. We define families  $\{P_{\alpha_\gamma} \in \mathcal{P}(X) : \gamma < \beta\}$  with the following properties:

- (i)  $\{\alpha_\gamma : \gamma < \beta\}$  is an increasing subsequence of  $\tau$ ;
- (ii)  $P_{\alpha_\gamma} = \{T_{\alpha_\gamma}^g \in [H_{\alpha_\gamma}]^{<\kappa} : |f_{\alpha_\gamma, h}^{-1}(T_{\alpha_\gamma}^g)| = \tau, \text{ for some } h \in {}^\delta \tau, g \in {}^\gamma \tau, \delta < \gamma\}$ ;
- (iii)  $f_{\alpha_\gamma, h} : A_{\alpha_\delta}^h \setminus \{\alpha_\gamma\} \rightarrow [H_{\alpha_\gamma}]^{<\kappa}$  is a function given by the formula  $f_{\alpha_\gamma, h}(\xi) = T_{\alpha_\gamma}^g, h \in {}^\delta \tau, g \in {}^\gamma \tau, \delta < \gamma$ , with the property  $T_{\alpha_\gamma}^g \cup H_\alpha$  and  $T_{\alpha_\gamma}^g \cup T_{\alpha_\delta}^h$  are not  $\kappa$ -directed for  $\alpha_\delta < \alpha_\gamma < \alpha$ , where  $A_{\alpha_\delta}^h = f_{\alpha_\delta, h}^{-1}(T_{\alpha_\delta}^g)$  and  $|A_{\alpha_\delta}^h| = \tau$  for  $\delta < \gamma, T_{\alpha_\delta}^g \in P_{\alpha_\delta}$ .

We proceed by transfinite recursion. Take  $\alpha_0 < \tau$  and  $H_{\alpha_0}$ . (Without the loss of generality we can assume that  $\alpha_0 = 0$ ). By Definition 1 for each  $\alpha > \alpha_0$  there exists  $T \in [H_{\alpha_0}]^{<\kappa}$  such that  $T \cup H_\alpha$  is not  $\kappa$ -directed. Consider a function

$$f_{\alpha_0} : \tau \setminus \{\alpha_0\} \rightarrow [H_{\alpha_0}]^{<\kappa}$$

given by the formula  $f_{\alpha_0}(\xi) = T_{\alpha_0}^\xi$ , where  $T_{\alpha_0}^\xi \in [H_{\alpha_0}]^{<\kappa}$  is such that  $T_{\alpha_0}^\xi \cup H_\alpha$  is not  $\kappa$ -directed for any  $\alpha_0 < \alpha$ . Since  $|H_{\alpha_0}| \leq 2^\mu$  we have  $|[H_{\alpha_0}]^{<\kappa}| \leq (2^\mu)^{<\kappa} = 2^\mu$ . Then the function  $f_{\alpha_0}$  determines a partition of  $\tau \setminus \{\alpha_0\}$  into at most  $2^\mu < \tau$  elements. Since  $\tau$  is regular the family

$$P_{\alpha_0} = \{T_{\alpha_0}^\xi \in [H_{\alpha_0}]^{<\kappa} : |f_{\alpha_0}^{-1}(T_{\alpha_0}^\xi)| = \tau, \text{ for some } \xi < \tau\}$$

is non-empty.

Next for  $\eta < \beta$  we define  $P_{\alpha_{\eta+1}}$ . By previous steps for  $\eta < \beta$  we have constructed  $P_{\alpha_\eta} = \{T_{\alpha_\eta}^g \in [H_{\alpha_\eta}]^{<\kappa} : |f_{\alpha_\eta, h}^{-1}(T_{\alpha_\eta}^g)| = \tau, \text{ for some } h \in {}^\delta \tau, g \in {}^\eta \tau, \delta < \eta\}$ . For each  $T_{\alpha_\eta}^g \in P_{\alpha_\eta}$  let  $A_{\alpha_\eta}^g = f_{\alpha_\eta, h}^{-1}(T_{\alpha_\eta}^g)$ . For each such a set  $A_{\alpha_\eta}^g$  consider a function

$$f_{\alpha_{\eta+1}, g} : A_{\alpha_\eta}^g \setminus \{\alpha_{\eta+1}\} \rightarrow [H_{\alpha_{\eta+1}}]^{<\kappa}$$

given by the formula  $f_{\alpha_{\eta+1}, g}(\xi) = T_{\alpha_{\eta+1}}^{g \wedge \xi}$  for  $T_{\alpha_{\eta+1}}^{g \wedge \xi} \in [H_{\alpha_\eta}]^{<\kappa}$  such that  $T_{\alpha_{\eta+1}}^{g \wedge \xi} \cup H_\alpha$  and  $T_{\alpha_{\eta+1}}^{g \wedge \xi} \cup T_{\alpha_\eta}^g$  are not  $\kappa$ -directed whenever  $\alpha_\delta < \alpha_\eta < \alpha$ . Since  $|H_{\alpha_{\eta+1}}| \leq 2^\mu$  we have  $|[H_{\alpha_{\eta+1}}]^{<\kappa}| \leq (2^\mu)^{<\kappa} = 2^\mu$ , the function  $f_{\alpha_{\eta+1}, g}$  determines a partition of  $A_{\alpha_\eta}^g \setminus \{\alpha_{\eta+1}\}$  into at most  $2^\mu < \tau$  elements. Since  $\tau$  is regular the family

$$P_{\alpha_{\eta+1}} = \{T_{\alpha_{\eta+1}}^{g \wedge \xi} \in [H_{\alpha_{\eta+1}}]^{<\kappa} : |f_{\alpha_{\eta+1}, g}^{-1}(T_{\alpha_{\eta+1}}^{g \wedge \xi})| = \tau, \text{ for some } g \wedge \xi \in {}^{\eta+1} \tau\}$$

is non-empty and fulfills (ii)-(iii).

Now we assume that  $\eta$  is limit and for all  $\delta < \eta$  the families  $P_{\alpha_\delta} = \{T_{\alpha_\delta}^g \in [H_{\alpha_\delta}]^{<\kappa} : |f_{\alpha_\delta, h}^{-1}(T_{\alpha_\delta}^g)| = \tau, \text{ for some } h \in {}^\lambda \tau, g \in {}^\delta \tau, \lambda < \delta\}$  fulfilling (ii)-(iii) have been defined. We set

$$B_{\alpha_\eta} = \left\{ \bigcap_{\delta < \eta} \bigcup_{g \in {}^\delta \tau} T_{\alpha_\delta}^g : T_{\alpha_\delta}^g \in P_{\alpha_\delta} \right\}$$

and

$$B'_{\alpha_\eta} = \{T \in B_{\alpha_\eta} : T \neq \emptyset\}.$$

Obviously  $|B'_{\alpha_\eta}| \leq 2^\mu$ . By previous steps we have  $|f_{\alpha_\delta, h}^{-1}(T_{\alpha_\delta}^g)| = \tau$  and  $A_{\alpha_\delta}^h = f_{\alpha_\delta, h}^{-1}(T_{\alpha_\delta}^g)$  for all  $T_{\alpha_\delta}^g \in P_{\alpha_\delta}$  and  $h \in {}^\lambda \tau, g \in {}^\delta \tau, \lambda < \delta < \tau$ . Let

$$F_{\alpha_\delta} = \{h \in {}^\delta \tau : |f_{\alpha_\delta, h}^{-1}(T_{\alpha_\delta}^g)| = \tau\}.$$

Take  $A_{\alpha_\eta} = \bigcup_{h \in F_{\alpha_\delta}} A_{\alpha_\delta}^h$ . Consider a function

$$f_{\alpha_\eta}: A_{\alpha_\eta} \setminus \{\alpha_\eta\} \rightarrow [B'_{\alpha_\eta}]^{<\kappa}$$

given by the formula  $f_{\alpha_\eta}(\xi) = T_{\alpha_\eta}^\xi$  such that  $T_{\alpha_\eta}^\xi \cup H_\alpha$  and  $T_{\alpha_\eta}^\xi \cup T_{\alpha_\delta}^h$  are not  $\kappa$ -directed for all  $h \in F_{\alpha_\delta}$ ,  $\alpha_\delta < \alpha_\eta < \alpha$ . Then the family  $P_{\alpha_\eta} = \{T_{\alpha_\eta}^\xi \in [B_{\alpha_\eta}]^{<\kappa}: |f_{\alpha_\eta}^{-1}(T_{\alpha_\eta}^\xi)| = \tau \text{ for some } \xi < \tau\}$  fulfills (ii)-(iii).

Thus we have constructed at least one  $\kappa$ -strong sequence of the form

$$\{T_{\alpha_\gamma}^g \in P_{\alpha_\gamma}: \gamma < \beta, g \in {}^\nu \tau\}.$$

Suppose now that at least one of defined above  $\kappa$ -strong sequences has length  $\zeta > \beta$ . Each set  $T_{\alpha_\gamma}^g$  determines a set  $A_{\alpha_\gamma}^g \subset \tau$  of cardinality  $\tau$ , i. e. there is defined a function  $f_{\alpha_\gamma, g}$  as in (iii). Let  $\nu = \sup\{|P_{\alpha_\gamma}|: \gamma < \zeta\}$ . Then there would exist  $\nu^\zeta > \tau$  pairwise disjoint sets  $A_{\alpha_\gamma}^g$  of cardinality  $\tau$ . A contradiction.  $\square$

**Corollary 2.3.** *Let  $\beta, \kappa, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a set of cardinality  $\tau$ . Then either  $X$  contains a  $\kappa$ -directed subset of cardinality  $\tau$  or there exists a family of cardinality  $\beta$  consisting of subsets of  $[X]^{<\kappa}$  with the property: for each  $A, B \in [X]^{<\kappa}$  the set  $A \cup B$  is not  $\kappa$ -directed.*

*Proof.* Without the loss of generality we can assume that  $X \subseteq \tau$ . Suppose that each  $\kappa$ -directed subset of  $X$  has cardinality less than  $\tau$ . We will construct a  $\kappa$ -strong sequence  $\{H_\alpha: \alpha < \tau\}$  via transfinite recursion.

Assume that for  $\alpha < \tau$  the  $\kappa$ -strong sequence  $\{H_\eta \subset X \setminus \bigcup_{\gamma < \eta} H_\gamma: \eta < \alpha\}$  has been defined. Since  $|H_\eta| < \tau$  then  $|\bigcup_{\eta < \alpha} H_\eta| < \tau$  and  $\tau$  is regular we have  $|X \setminus \bigcup_{\eta < \alpha} H_\eta| = \tau$ . Now we will construct  $H_\alpha$ .

Let  $\alpha$  be successor. Let  $H_\alpha \subset X \setminus \bigcup_{\eta < \alpha} H_\eta$  be a maximal  $\kappa$ -directed set.

Let  $\alpha$  be limit. Let  $x = \min(X \setminus \bigcup_{\eta < \alpha} H_\eta)$ . Let  $H_\alpha = \bigcup_{\eta < \alpha} H_\eta \cup \{x\}$ .

Let  $H_\alpha$  be the next element of the  $\kappa$ -strong sequence. By Theorem 2.2 there exists a  $\kappa$ -strong sequence  $\{T_\alpha: \alpha < \beta\}$  such that  $|T_\alpha| < \kappa$  for all  $\alpha < \beta$ . If  $T_\alpha$  are not pairwise disjoint then we take the family consisting of sets  $T'_\alpha$  such that  $T'_0 = T_0$  and  $T'_\alpha = T_\alpha \setminus \bigcup_{\beta < \alpha} T_\beta$  for  $\alpha$  - successor and  $T'_\alpha = \sup(\bigcup_{\beta < \alpha} T'_\beta)$  for  $\alpha$  - limit. This completes the proof.  $\square$

The next corollary follows immediately from Corollary 2.3.

**Corollary 2.4.** *Let  $\beta, \kappa, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a set of cardinality  $\tau$ . Then either  $X$  contains a  $\kappa$ -directed subset of cardinality  $\tau$  or there exists a subset of  $X$  of cardinality  $\beta$  consisting of pairwise incompatible elements.*

The next result in this paragraph will be the special case of previous ones for  $(P(X), \subseteq)$ .

A family  $\mathcal{A} \subset P(X)$  is *closed under taking  $\kappa$  - intersections* i.e. for all  $\mathcal{A}' \subset \mathcal{A}$  such that  $|\mathcal{A}'| < \kappa$  we have  $\bigcap \mathcal{A}' \in \mathcal{A}$ . Let  $\kappa, \tau$  be cardinals with  $\kappa < \tau$ . A family of sets  $\mathcal{A} \subset P(X)$ , with  $|\mathcal{A}| \geq \tau$ , is called a  $\kappa$ -vaulted family iff for each subfamily  $\mathcal{B} \subset \mathcal{A}$  of cardinality less than  $\kappa$  we have  $\bigcap \mathcal{B} \neq \emptyset$ .

**Theorem 2.5.** *Let  $\beta, \kappa, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a topological space of cardinality  $\tau$ . Let  $\mathcal{A} \subset \mathcal{P}(X)$  be a family of sets with  $|\mathcal{A}| = \tau$  closed under taking  $\kappa$ -intersections. Then  $\mathcal{A}$  contains a  $\kappa$ -vaulted family of cardinality  $\tau$  or  $\mathcal{A}$  contains a subfamily of cardinality  $\beta$  which consists of pairwise disjoint sets.*

*Proof.* Let  $\mathcal{A} = \{A_\gamma: \gamma < \tau\}$  be a family as is required in the theorem. Define a partial ordered set  $\mathcal{P} = \{\gamma < \tau: A_\gamma \in \mathcal{A}\}$  with the following relation.

$$(\gamma, \beta) \in r \Leftrightarrow A_\gamma \subseteq A_\beta.$$

If  $\gamma, \beta$  are incompatible, then  $A_\gamma \cap A_\beta = \emptyset$ . By Corollary 2.4 the proof is complete.  $\square$

### 3 Generalized independent families

In [7] the following definition was introduced

**Definition 3.1** ([7]). Let  $\mathcal{I} = \{I_\alpha^\beta: \beta < \lambda_\alpha, \alpha < \tau\}$  be a family of partitions of an infinite set  $S$  with each  $\lambda_\alpha \geq 2$  and let  $\kappa, \lambda, \theta$  be cardinals. If for any  $J \in [\tau]^{<\theta}$  and for any  $f \in \Pi_{\alpha \in J} \lambda_\alpha$  the intersection  $\bigcap \{I_\alpha^{f(\alpha)}: \alpha \in J\}$  has cardinality at least  $\kappa$ , then  $\mathcal{I}$  is called a  $(\theta, \kappa)$ -generalized independent family on  $S$ . Moreover, if  $\lambda_\alpha = \lambda$  for all  $\alpha < \tau$ , then  $\mathcal{I}$  is called a  $(\theta, \kappa, \lambda)$ -generalized independent family on  $S$ .

Notice that an independent family considered in [6] is  $(\omega, 1, |S|)$ -independent family on a set  $S$ . Moreover in [8] and [7] there are shown some results concerning cardinality of generalized independent families. In Theorem 3.2 we show the relation between the existence of generalized independent families and  $\kappa$ -strong sequences.

Let  $\mathcal{A} \subset P(X)$ . Then

$$c(\mathcal{A}) = \sup\{|\mathcal{B}|: \mathcal{B} \text{ is a subfamily of pairwise disjoint sets of } \mathcal{A}\}.$$

**Theorem 3.2.** Let  $\beta, \kappa, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a topological space of cardinality  $\tau$ . Let  $\mathcal{A} \subset \mathcal{P}(X)$  be a family of cardinality  $\tau$  closed under taking  $\kappa$ -intersections and such that  $c(\mathcal{A}) < \beta$ . Then there exists a  $(\kappa, 1)$ -generalized independent family of cardinality  $\tau$ .

*Proof.* Consider a family  $Part = \{P^\alpha: |P^\alpha| = \lambda_\alpha, \alpha < \xi\}$  of all partitions of  $\tau$ . Since  $\lambda_\alpha \leq \tau$  for all  $\alpha < \xi$ ,  $|Part| \geq \tau$ . By Theorem 2.5 there exists a  $\kappa$ -vaulted family  $\mathcal{A} \subset Part$  of cardinality  $\tau$ . We will construct a  $(\kappa, 1)$ -generalized independent family via transfinite recursion.

Assume that for  $\eta < \tau$  the  $(\kappa, 1)$ -generalized independent family  $\mathcal{I} = \{P^\alpha \in \mathcal{A} \setminus \{P^\gamma: \gamma < \alpha\}: \alpha < \eta\}$  has been defined. Clearly,  $\mathcal{I}$  has the property that for any  $J \in [\eta]^{<\kappa}$  and  $f \in \Pi_{\alpha \in J} \lambda_\alpha$  the intersection  $\bigcap \{I_\alpha^{f(\alpha)}: \alpha \in J\}$  is non-empty, where  $I_\alpha^{f(\alpha)}$  is a  $f(\alpha)$ -element of the partition  $P^\alpha$ . Since  $\eta < \tau$  and  $|\mathcal{A}| = \tau$ ,  $\mathcal{A} \setminus \{P^\alpha: \alpha < \eta\} \neq \emptyset$ . Hence we can continue our construction. Let

$$\mathcal{I} = \{I: I = \bigcap \{I_\alpha^{f(\alpha)}: \alpha \in J\} \text{ for some } J \in [\eta]^{<\kappa} \text{ and some } f \in \Pi_{\alpha \in J} \lambda_\alpha\}.$$

Observe that  $\bigcup \mathcal{I} = \tau$ . If not, then there exists  $\delta \in \tau \setminus \bigcup \mathcal{I}$ . It would mean that there is no  $J \in [\eta]^{<\kappa}$  and  $f \in \Pi_{\alpha \in J} \lambda_\alpha$  such that  $\delta \in \bigcap \{I_\alpha^{f(\alpha)}: \alpha \in J\}$ . Then  $\delta \notin P^\alpha$  for some  $P^\alpha \in \mathcal{I}$ . A contradiction because  $P^\alpha \in Part$ .

Let  $\eta$  be a successor. Let  $P^\eta \in \mathcal{A} \setminus \{P^\alpha: \alpha < \eta\}$  be a partition with the property that for all  $I \in \mathcal{I}$  there exists  $I_\eta \in P^\eta$  such that  $I \subset I_\eta$ . For a set  $J \in [\tau]^{<\kappa}$  and  $f \in \Pi_{\alpha \in J} \lambda_\alpha$  choose  $\{I_\alpha^{f(\alpha)}: \alpha \in J\}$ . Since the family  $\mathcal{A}$  is  $\kappa$ -vaulted,  $\bigcap \{I_\alpha^{f(\alpha)}: \alpha \in J\} \neq \emptyset$ .

If  $\eta$  is the limit, then we take  $P^\eta = \bigcap_{\alpha < \eta} P^\alpha$ .

Thus the  $(\kappa, 1)$ -generalized independent family  $\{P^\alpha = \{I_\alpha^\beta: \beta < \lambda_\alpha\}: \alpha < \tau\}$  has been defined.  $\square$

In [6] it is proved the result related to the density of product of topological spaces (Theorem 8) using the theorem on the existence of independent families (Theorem 3). Following [8], we give the definition of  $\kappa$ -box product.

Let  $\kappa, \mu$  be cardinals with  $\aleph_0 \leq \kappa \leq \mu$  and  $\{X_i\}_{i \in \mu}$  be a family of topological spaces. Then  $\square_{i \in \mu}^\kappa X_i$  denotes the  $\kappa$ -box product which is induced on the full Cartesian product  $\Pi_{i \in \mu} X_i$  by the canonical base

$$\mathcal{B} = \left\{ \bigcap_{i \in I} pr_i^{-1}(U_i): I \in P_{<\kappa}(\mu) \text{ and } U_i \text{ is open in } X_i \right\},$$

where  $P_{<\kappa}(\mu) := \{I \subset \mu: |I| < \kappa\}$ .

The next two corollaries follow from Theorem 2.5 and Corollary 2.3 and Theorem 4.3 in [7].

**Corollary 3.3.** Let  $\beta, \kappa, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\beta, \tau$  be regular. Let  $X$  be a topological space of cardinality  $\tau$ . Let  $\mathcal{A} \subset \mathcal{P}(X)$  be a family of cardinality  $\tau$  closed under taking  $\kappa$ -intersections and such that  $c(\mathcal{A}) < \beta$ . Let  $\{X_\alpha\}_{\alpha < \tau}$  be a family of topological spaces such that  $d(X_\alpha) \leq \lambda_\alpha$  for all  $\alpha < \tau$ . Then  $d(\square_{\alpha \in \tau}^\kappa X_\alpha) \leq |S|$ .

A topological space  $X$  is *irresolvable* if  $X$  does not have disjoint dense subsets.

Let  $\lambda$  be a cardinal. An ideal  $\mathcal{I} \subset P(X)$  is called  $\lambda$ -complete if  $\bigcup_{\alpha < \lambda} A_\alpha \in \mathcal{I}$  for  $\alpha < \lambda$  and  $A_\alpha \in \mathcal{I}$ .

In [7], one can find the following lemma

**Lemma 3.4.** Suppose  $(X, \mathcal{T})$  is an open-hereditarily irresolvable space and  $\mathcal{T}$  is a  $P_\theta$ -topology for some regular cardinal  $\theta$ . Let  $\mathcal{N}$  denote the ideal of nowhere dense subsets, and let  $\lambda$  be the smallest cardinal such that  $\mathcal{N}$  is not  $\lambda$ -complete. Then for any  $\gamma < \gamma^+ < \lambda$  and  $\eta < \theta$ ,  $\mathcal{N}$  is  $(\gamma^\eta)^+$ -complete.

**Corollary 3.5.** Let  $\beta, \kappa, \theta, \tau$  be cardinals such that  $\omega \leq \beta \ll \tau$ ,  $\kappa < \tau$  and  $\theta < \tau$  with  $\beta, \tau, \theta$  are regular. Let  $X$  be a topological space of cardinality  $\tau$ . Let  $\mathcal{A} \subset \mathcal{P}(X)$  be a family of cardinality  $\tau$  closed under taking  $\kappa$ -intersections and such that  $c(\mathcal{A}) < \beta$ . Suppose that  $\mathcal{I}$  is a family of partitions of a set  $S$  with each  $\lambda_\alpha < \theta$ . Let  $\mathcal{N}$  be the ideal of nowhere dense set of the simple topology induced by  $\mathcal{I}$  and let  $\lambda$  be the smallest cardinal such that  $\mathcal{I}$  is not  $\lambda$ -complete. Then

- 1) there is a nonempty open set  $U$  of the simple topology such that  $U$  with the subspace topology satisfies all conditions in Lemma 3.4 and the ideal  $\mathcal{I}_U$  of nowhere dense set of  $U$  is  $\lambda$ -saturated and
- 2)  $2^{<\theta} = \theta$ .

We finish this paper by showing relations between cardinal invariants associated with both considered notions. Let  $\kappa$  be a cardinal and let  $X$  be an infinite set of the cardinality  $\kappa$ . Accept the following notations:

$$\hat{s}_\kappa = \sup\{\alpha: \text{there exists a } \kappa\text{-strong sequence in } X \text{ of length } \alpha\}.$$

$$i(\kappa, 1) = \min\{\alpha: \text{there is no } (\kappa, 1) - \text{generalized independent family on } X \text{ of length } \alpha\}.$$

**Theorem 3.6.** Let  $\kappa, \tau$  be cardinals such that  $\kappa < \tau$  and  $\tau$  - regular and there exists a regular cardinal  $\beta$  such that  $\omega \leq \beta \ll \tau$ . Let  $X$  be a topological space of cardinality  $\tau$ . Then  $\hat{s}_\kappa \leq i(\kappa, 1)$ .

*Proof.* Without the loss of generality we can assume that  $X \subseteq \tau$ . Let  $\mathcal{I}$  be a maximal  $(\kappa, 1)$ -generalized independent family. If  $|\mathcal{I}| = \tau$  and  $\mathcal{I}$  contains a  $\kappa$ -vaulted family of cardinality  $\tau$  then the theorem is complete. Suppose that each  $\kappa$ -vaulted family has cardinality less than  $\tau$ . (Then by Theorem 2.5 there are only  $\beta$  pairwise disjoint sets in  $\mathcal{I}$  for  $\omega \leq \beta \ll \tau$ ,  $\beta, \tau$ -regular). By transfinite recursion we will construct a  $\kappa$ -strong sequence  $\{H_\alpha: \alpha < \tau\}$ . Assume that for  $\gamma < \alpha < \tau$  the  $\kappa$ -strong sequence  $\{H_\gamma: \gamma < \alpha\}$  such that  $H_\gamma \subset X \setminus \bigcup_{\eta < \gamma} H_\eta$  has been defined. Since  $|H_\gamma| < \tau$ ,  $|\bigcup_{\gamma < \alpha} H_\gamma| < \tau$  and  $\tau$  is regular we have  $|X \setminus \bigcup_{\gamma < \alpha} H_\gamma| = \tau$ .

Let  $\alpha$  be a successor. Hence there exists a maximal  $\kappa$ -directed set  $H_\alpha \subset X \setminus \bigcup_{\gamma < \alpha} H_\gamma$  such that  $H_\alpha \cup H_\gamma$  is not  $\kappa$ -directed for all  $\gamma < \alpha$ .

If  $\alpha$  is limit, then  $H_\alpha = \bigcup_{\gamma < \alpha} H_\gamma \cup \{x\}$ , where  $x = \min(X \setminus \bigcup_{\gamma < \alpha} H_\gamma)$ . Obviously,  $H_\alpha \cup H_\gamma$  is not  $\kappa$ -directed for all  $\gamma < \alpha$ . Let  $H_\alpha$  be the next element of the  $\kappa$ -strong sequence. The proof is complete.  $\square$

The easy consequence of Theorem 3.6 and Theorem 3.2. in [7] is

**Corollary 3.7.** Let  $\kappa, \tau$  be cardinals such that  $\kappa < \tau$  and  $\tau$  - regular and there exists a regular cardinal  $\beta$  such that  $\omega \leq \beta \ll \tau$ . Let  $X$  be a topological space of cardinality  $\tau$ . Then the following statements are equivalent:

- 1)  $\hat{s}_\kappa \leq i(\kappa, 1)$
- 2)  $d(\square_\kappa^{\hat{s}_\kappa}(X_\alpha)) \leq |X|$  holds for any family of topological spaces  $\{X_\alpha\}_{\alpha < \hat{s}_\kappa}$ , with each  $d(X_\alpha) \leq \lambda$  for some  $\lambda < \tau$ .

## References

- [1] Efimov b. A., Diaditcheskie bikompakty, (in Russian), Trudy Mosk. Matem. O-va 14 (1965), 211-247.
- [2] Jureczko J. On inequalities among some cardinal invariants, Math.Aeterna, 6(1), (2016), 87 - 98.
- [3] Jureczko J. Strong sequences and independent sets, Math. Aeterna, 6(2), (2016), 141-152.
- [4] Fichtenholz G. M. , Kantorovitch L.V. Sur le opérations linéaires dans l'espace de fonctions bornées, Studia Math. 5 (1935), 69-98.

- [5] Hausdorff F. Über zwei Sätze von G Fichtenholz and L. Kantorovitch, *Studia Math* 6 (1936), 18-19.
- [6] Engelking R. , Karłowicz M. Some theorems of set theory and their topological consequences, *Fund. Math.* LVII (1965).
- [7] Hu W., Generalized independent families and dense sets of Box-Product spaces, *App. Gen. Top.* 7(2), (2006), 203-209.
- [8] Elser S. O., Density of  $\kappa$ -Box Products and the existencxe of generalized independent families, *App. Gen. Top.*, 12(2) (2011), 221-225.
- [9] Engelking R., *General topology*, PWN 1977.
- [10] Jech T. , *Set Theory*, Academic Press New York San Francisco London, 1978.