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Research Article

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Two new eigenvalue localization sets for tensors and theirs applications

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Abstract: A new eigenvalue localization set for tensors is given and proved to be tighter than those presented by Qi (J. Symbolic Comput., 2005, 40, 1302-1324) and Li *et al.* (Numer. Linear Algebra Appl., 2014, 21, 39-50). As an application, a weaker checkable sufficient condition for the positive (semi-)definiteness of an even-order real symmetric tensor is obtained. Meanwhile, an S-type E-eigenvalue localization set for tensors is given and proved to be tighter than that presented by Wang *et al.* (Discrete Cont. Dyn.-B, 2017, 22(1), 187-198). As an application, an S-type upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

Keywords: Nonnegative tensors, Tensor eigenvalue, Localization set, Positive definite, Spectral radius

MSC: 15A18, 15A42, 15A69

1 Introduction

For a positive integer n, $n \geq 2$, N denotes the set $\{1, 2, \cdots, n\}$. $\mathbb{C}$ ($\mathbb{R}$) denotes the set of all complex (real) numbers. We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order m dimension n, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if

$$a_{i_1 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j \in N$ for $j = 1, 2, \cdots, m$. A tensor of order m dimension n is called the unit tensor, denoted by $\mathcal{I}$, if its entries are $\delta_{i_1 \cdots i_m}$ for $i_1, \cdots, i_m \in N$, where

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{A}$ is called nonnegative if $a_{i_1 \cdots i_m} \geq 0$. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called symmetric [1] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called weakly symmetric [2] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1, \cdots, i_m \in N} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$. It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

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To an n-vector $x = (x_1, x_2 \cdots, x_n)^T$, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2,\cdots,i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}$$

and

$$x^{[m-1]} = (x_i^{m-1})_{1 \le i \le n}.$$

Definition 1.1 ([1, 3]). *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

(λ, x) is called an H-eigenpair if both of them are real.

Definition 1.2 ([1, 3]). *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is called an E-eigenpair of $\mathcal{A}$ if*

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^T x = 1.$$

(λ, x) is called an Z-eigenpair if both of them are real.

We define the Z-spectrum of $\mathcal{A}$, denoted $\mathcal{Z}(\mathcal{A})$ to be the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\mathcal{Z}(\mathcal{A}) \neq 0$, then the Z-spectral radius [2] of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \mathcal{Z}(\mathcal{A})\}.$$

It is shown in [1] that a real even-order symmetric tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is positive definite if and only if all of its H-eigenvalues (Z-eigenvalues) are positive. However, when m and n are very large, it is not easy to compute all H-eigenvalues (Z-eigenvalues) of $\mathcal{A}$. Then we can try to give a set in the complex which includes all H-eigenvalues (Z-eigenvalues) of $\mathcal{A}$. If this set is in the right-half complex plane, then we can conclude that all H-eigenvalues (Z-eigenvalues) are positive, consequently, $\mathcal{A}$ is positive definite; for details, see [1, 4–7].

There are other applications of (E-)eigenvalue inclusion sets, for example we can use them to obtain the lower and upper bounds for the H-eigenvalues (Z-spectral radius) of (nonnegative) tensors and the minimum eigenvalue of $\mathcal{M}$-tensors; for details, see [8–19].

In 2005, Qi [1] presented the following Geršgorin-type eigenvalue localization set for real symmetric tensors, which can be easily extended to general tensors [4, 20].

Theorem 1.3 ([1, Theorem 6]). *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ is the set of all eigenvalues of $\mathcal{A}$ and

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \dots i}| \le r_i(\mathcal{A})\}, \; r_i(\mathcal{A}) = \sum_{\delta_{i i_2 \cdots i_m} = 0} |a_{i i_2 \cdots i_m}|.$$

To get a tighter eigenvalue localization set than $\Gamma(\mathcal{A})$, Li *et al.* [4] proposed the following Brauer-type eigenvalue localization set for tensors.

Theorem 1.4 ([4, Theorem 2.1]). *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i,j \in N, j \neq i} \mathcal{K}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^j(\mathcal{A}))|z - a_{j \dots j}| \le |a_{i j \dots j}| r_j(\mathcal{A})\},$$

$$r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{i i_2 \cdots i_m} = 0, \\ \delta_{j i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| = r_i(\mathcal{A}) - |a_{i j \dots j}|.$$

To reduce computations, Li *et al.* [4] gave an S-type eigenvalue localization set by breaking N into disjoint subsets S and $\bar{S}$, where $\bar{S}$ is the complement of S in N.

Theorem 1.5 ([4, Theorem 2.2]). *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in \bar{S}} \mathcal{K}_{i,j}(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in \bar{S}, j\in S} \mathcal{K}_{i,j}(\mathcal{A})\right).$$

In 2017, Wang *et al.* established the following Z-eigenvalue localization set for a real tensor $\mathcal{A}$, which is completely different from eigenvalue localization sets and can be generalized to an E-eigenvalue localization set easily.

Theorem 1.6 ([8, Theorem 3.1]). *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$\mathcal{Z}(\mathcal{A}) \subseteq \hat{\Gamma}(\mathcal{A}) = \bigcup_{i\in N} \hat{\Gamma}_i(\mathcal{A}),$$

where

$$\hat{\Gamma}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\},\ R_i(\mathcal{A}) = \sum_{i_2\cdots i_m \in N} |a_{i i_2\cdots i_m}|.$$

The main aim of this paper is to give a new eigenvalue localization set for tensors, which is tighter than those in Theorems 1.3-1.5, and a new E-eigenvalue localization set for tensors, which is tighter than that in Theorem 1.6. As applications, a weaker checkable sufficient condition for the positive (semi-)definiteness of an even-order real symmetric tensor is obtained based on the eigenvalue localization set, and a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained based on the E-eigenvalue localization set.

2 A new eigenvalue localization set for tensors and its applications

In this section, we propose a new eigenvalue localization set for tensors and establish the comparisons between this set with those in Theorems 1.3-1.5. As an application of this set, we give a weaker checkable sufficient condition for the positive (semi-)definiteness of an even-order real symmetric tensor.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^{\cap}(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \mathcal{K}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : \left(|z - a_{i\cdots i}| - r_i^j(\mathcal{A})\right)|z - a_{j\cdots j}| \leq |a_{ij\cdots j}| r_j(\mathcal{A})\right\}.$$

Proof. Let λ be an eigenvalue of $\mathcal{A}$ with corresponding eigenvector $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n\backslash\{0\}$, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

Let $|x_p| = \max\{|x_i| : i \in N\}$. Then, $|x_p| > 0$. From (1), we have

$$(\lambda - a_{p\cdots p})x_p^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_{i_2}\cdots x_{i_m} + a_{pj\cdots j} x_j^{m-1},\ \forall\, j \in N, j \neq p.$$

Taking modulus in the above equation and using the triangle inequality give

$$|\lambda - a_{p\cdots p}||x_p|^{m-1} \leq \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + |a_{pj\cdots j}||x_j|^{m-1}$$

$$\leq \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}||x_p|^{m-1} + |a_{pj\cdots j}||x_j|^{m-1}$$
$$= r_p^j(\mathcal{A})|x_p|^{m-1} + |a_{pj\cdots j}||x_j|^{m-1},$$

equivalently,

$$(|\lambda - a_{p\cdots p}| - r_p^j(\mathcal{A}))|x_p|^{m-1} \leq |a_{pj\cdots j}||x_j|^{m-1}. \quad (2)$$

If $|x_j| = 0$, by $|x_p| > 0$, we have $|\lambda - a_{p\cdots p}| - r_p^j(\mathcal{A}) \leq 0$. Then

$$(|\lambda - a_{p\cdots p}| - r_p^j(\mathcal{A}))|\lambda - a_{j\cdots j}| \leq 0 \leq |a_{pj\cdots j}|r_j(\mathcal{A}),$$

which implies that $\lambda \in \mathcal{K}_{p,j}(\mathcal{A}) \subseteq \mathcal{K}^{\cap}(\mathcal{A})$. Otherwise, $|x_j| > 0$. Similarly, from (1), we can obtain

$$|\lambda - a_{j\cdots j}||x_j|^{m-1} \leq r_j(\mathcal{A})|x_p|^{m-1}. \quad (3)$$

Multiplying (2) with (3) and noting that $|x_p|^{m-1}|x_j|^{m-1} > 0$, we have

$$(|\lambda - a_{p\cdots p}| - r_p^j(\mathcal{A}))|\lambda - a_{j\cdots j}| \leq |a_{pj\cdots j}|r_j(\mathcal{A}),$$

then $\lambda \in \mathcal{K}_{p,j}(\mathcal{A}) \subseteq \mathcal{K}^{\cap}(\mathcal{A})$. From the arbitrariness of j, we have $\lambda \in \bigcap_{j\in N, j\neq p} \mathcal{K}_{p,j}(\mathcal{A})$. Furthermore, $\lambda \in \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \mathcal{K}_{i,j}(\mathcal{A})$. □

Next, a comparison theorem is given for Theorems 1.3-1.5 and Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, S be a nonempty proper subset of N. Then*

$$\mathcal{K}^{\cap}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. Let $\bar{S}$ be the complement of S in N. According to Theorem 2.3 in [4], $\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. Hence, we only prove $\mathcal{K}^{\cap}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. Let $z \in \mathcal{K}^{\cap}(\mathcal{A})$, then there exists $i_0 \in N$, such that $z \in \mathcal{K}_{i_0,j}(\mathcal{A}), \forall\ j \in N, j \neq i_0$. If $i_0 \in S$, then for any $j \in \bar{S}$, we have $z \in \bigcup_{i_0\in S, j\in\bar{S}} \mathcal{K}_{i_0,j}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. If $i_0 \in \bar{S}$, then for any $j \in S$, we have $z \in \bigcup_{i_0\in\bar{S}, j\in S} \mathcal{K}_{i,j}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. The conclusion follows. □

Remark 2.3. *Theorem 2.2 shows that this set in Theorem 2.1 is tighter than those in Theorem 1.3, Theorem 1.4 and Theorem 1.5, that is, $\mathcal{K}^{\cap}(\mathcal{A})$ can capture all eigenvalues of $\mathcal{A}$ more precisely than $\Gamma(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^S(\mathcal{A})$.*

As shown in [1, 4–7], an eigenvalue localization set can provide a checkable sufficient condition for the positive (semi-)definiteness of tensors. As an application of Theorem 2.1, we give a checkable sufficient condition for the positive (semi-)definiteness of tensors.

Theorem 2.4. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{k\cdots k} > 0$ for all $k \in N$. If $\forall\ i \in N, \exists\ j \in N, j \neq i$,*

$$\big(a_{i\cdots i} - r_i^j(\mathcal{A})\big)a_{j\cdots j} > |a_{ij\cdots j}|r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive definite.

Proof. Let λ be an H-eigenvalue of $\mathcal{A}$. By Theorem 2.1, we have $\lambda \in \mathcal{K}^{\cap}(\mathcal{A})$, that is, there is $i_0 \in N$, for any $j \in N, j \neq i_0$,

$$\big(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^j(\mathcal{A})\big)|\lambda - a_{j\cdots j}| \leq |a_{i_0 j\cdots j}|r_j(\mathcal{A}).$$

Suppose that $\lambda \leq 0$. Then for $i_0 \in N$, $\exists\ j_0$, such that $a_{i_0\cdots i_0} > 0, a_{j_0\cdots j_0} > 0$, and

$$\big(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{j_0}(\mathcal{A})\big)|\lambda - a_{j_0\cdots j_0}| \geq \big(a_{i_0\cdots i_0} - r_{i_0}^{j_0}(\mathcal{A})\big)a_{j_0\cdots j_0} > |a_{i_0 j_0\cdots j_0}|r_{j_0}(\mathcal{A}).$$

This is a contradiction. Hence, $\lambda > 0$, and $\mathcal{A}$ is positive definite. The conclusion follows. □

Similar to the proof of Theorem 2.4, the following sufficient condition is easily obtained.

Theorem 2.5. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{k\cdots k} \geq 0$ for all $k \in N$. If $\forall\, i \in N, \exists\, j \in N, j \neq i$,*

$$\big(a_{i\cdots i} - r_i^j(\mathcal{A})\big)a_{j\cdots j} \geq |a_{ij\cdots j}|r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive semi-definite.

Remark 2.6. *When $n = 2$, Theorem 2.4 is the same as Theorem 4.1 and Theorem 4.2 in [4]. When $n \geq 3$, it is easy to see that the conditions of Theorem 2.4 for determining the positive definiteness of tensors are weaker than those in Theorem 4.1 and Theorem 4.2 in [4].*

Next, an example is given to verify the fact in Remark 2.6.

Example 2.7. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a symmetric tensor with elements defined as follows:*

$$a_{1111} = 12.1, a_{2222} = 4.6, a_{3333} = 3.6, a_{1112} = -0.1, a_{1113} = 0.15, a_{1122} = -0.2, a_{1123} = -0.2,$$

$$a_{1133} = 0, a_{1222} = -0.1, a_{1223} = 0.3, a_{1233} = 0.1, a_{1333} = -0.15, a_{2223} = 0.1, a_{2233} = -0.1, a_{2333} = 0.2.$$

By computations, we get that

$$(a_{1111} - r_1^3(\mathcal{A}))a_{3333} = 29.7 > 0.5550 = |a_{1333}|r_3(\mathcal{A}); \tag{4}$$

$$(a_{2222} - r_2^3(\mathcal{A}))a_{3333} = 1.0800 > 0.7400 = |a_{2333}|r_3(\mathcal{A}); \tag{5}$$

$$(a_{3333} - r_3^1(\mathcal{A}))a_{1111} = 0.6050 > 0.6000 = |a_{3111}|r_1(\mathcal{A}); \tag{6}$$

$$(a_{3333} - r_3^2(\mathcal{A}))a_{2222} = 0 < 0.45 = |a_{3222}|r_2(\mathcal{A}). \tag{7}$$

Let $S = \{1, 2\}, \bar{S} = \{3\}$. Because (7) holds, we can not use Theorem 4.1 and Theorem 4.2 in [4] to determine the positiveness of $\mathcal{A}$ under this division. But from (4)-(6) and Theorem 2.4, we can determine that $\mathcal{A}$ is positive definite. In fact, all the H-eigenvalues of $\mathcal{A}$ are 2.9074, 3.1633, 3.7705, 4.6282 and 12.4216. By Theorem 5 in [1], $\mathcal{A}$ is positive definite.

3 A new E-eigenvalue localization set for tensors and its applications

In this section, we give an S-type E-eigenvalue localization set for tensors, and establish the comparison between this set with that in Theorem 1.6. For simplification, we first denote some notations. Given a nonempty proper subset S of N, let

$$\Delta^N = \{(i_2, i_3, \cdots, i_m) : each\ i_j \in N\ for\ j = 2, \cdots, m\},$$

$$\Delta^S = \{(i_2, i_3, \cdots, i_m) : each\ i_j \in S\ for\ j = 2, \cdots, m\},$$

and then

$$\overline{\Delta^S} = \Delta^N \backslash \Delta^S.$$

This implies that for a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$, we have that for $i \in S$,

$$R_i(\mathcal{A}) = \sum_{i_2,\cdots,i_m \in N} |a_{ii_2\cdots i_m}| = R_i^{\Delta^S}(\mathcal{A}) + R_i^{\overline{\Delta^S}}(\mathcal{A}),$$

where

$$R_i^{\Delta^S}(\mathcal{A}) = \sum_{(i_2,\cdots,i_m)\in\Delta^S} |a_{ii_2\cdots i_m}|, \quad R_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2,\cdots,i_m)\in\overline{\Delta^S}} |a_{ii_2\cdots i_m}|.$$

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\sigma_E(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in\bar{S}} \Omega^S_{i,j}(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in\bar{S}, j\in S} \Omega^{\bar{S}}_{i,j}(\mathcal{A})\right),$$

where $\sigma_E(\mathcal{A})$ is the set of all E-eigenvalues of $\mathcal{A}$ and

$$\Omega^S_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : |z|(|z| - R_j^{\overline{\Delta^S}}(\mathcal{A})) \leq R_i(\mathcal{A}) R_j^{\Delta^S}(\mathcal{A})\right\},$$
$$\Omega^{\bar{S}}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : |z|(|z| - R_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq R_i(\mathcal{A}) R_j^{\Delta^{\bar{S}}}(\mathcal{A})\right\}.$$

Proof. Let λ be an E-eigenvalue of $\mathcal{A}$ with corresponding eigenvector $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n\backslash\{0\}$, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x, ||x||_2 = 1. \tag{8}$$

Let $|x_p| = \max\{|x_i| : i \in S\}$ and $|x_q| = \max\{|x_j| : j \in \bar{S}\}$. Then, at least one of $|x_p|$ and $|x_q|$ is nonzero. We next distinguish two cases to prove.

Case I. Suppose that $|x_q| \geq |x_p|$, then $|x_q| = \max_{j\in N} |x_j|$ and $0 < |x_q|^{m-1} \leq |x_q| \leq 1$. From (8), we have

$$\lambda x_q = \sum_{(i_2\cdots i_m)\in\Delta^S} a_{qi_2\cdots i_m} x_{i_2}\cdots x_{i_m} + \sum_{(i_2\cdots i_m)\in\overline{\Delta^S}} a_{qi_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}|\lambda||x_q|^{m-1} \leq |\lambda||x_q| &\leq \sum_{(i_2\cdots i_m)\in\Delta^S} |a_{qi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{(i_2\cdots i_m)\in\overline{\Delta^S}} |a_{qi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\leq \sum_{(i_2\cdots i_m)\in\Delta^S} |a_{qi_2\cdots i_m}||x_p|^{m-1} + \sum_{(i_2\cdots i_m)\in\overline{\Delta^S}} |a_{qi_2\cdots i_m}||x_q|^{m-1} \\ &= R_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1} + R_q^{\overline{\Delta^S}}(\mathcal{A})|x_q|^{m-1},\end{aligned}$$

i.e.,

$$(|\lambda| - R_q^{\overline{\Delta^S}}(\mathcal{A}))|x_q|^{m-1} \leq R_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1}. \tag{9}$$

If $|x_p| = 0$, by $|x_q| > 0$, we have $|\lambda| - R_q^{\overline{\Delta^S}}(\mathcal{A}) \leq 0$. Then

$$(|\lambda| - R_q^{\overline{\Delta^S}}(\mathcal{A}))|\lambda| \leq 0 \leq R_q^{\Delta^S}(\mathcal{A}) R_p(\mathcal{A}),$$

which implies that $\lambda \in \Omega^S_{p,q}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. If $|x_p| > 0$, from (8), we can obtain

$$|\lambda||x_p|^{m-1} \leq |\lambda||x_p| \leq \sum_{i_2\cdots i_m\in N} |a_{pi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \leq R_p(\mathcal{A})|x_q|^{m-1}. \tag{10}$$

Multiplying (9) with (10) and noting that $|x_p|^{m-1}|x_q|^{m-1} > 0$, we have

$$(|\lambda| - R_q^{\overline{\Delta^S}}(\mathcal{A}))|\lambda| \leq R_q^{\Delta^S}(\mathcal{A}) R_p(\mathcal{A}),$$

which leads to $\lambda \in \Omega^S_{p,q}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

Case II. Suppose that $|x_p| \geq |x_q|$, then $|x_p| = \max_{i\in N} |x_i|$ and $0 < |x_p|^{m-1} \leq |x_p| \leq 1$. Similar to (9), we can obtain

$$(|\lambda| - R_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_p|^{m-1} \leq R_p^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1}. \tag{11}$$

If $|x_q| = 0$, by $|x_p| > 0$, we have $|\lambda| - R_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \leq 0$. Then

$$(|\lambda| - R_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|\lambda| \leq 0 \leq R_p^{\Delta^{\bar{S}}}(\mathcal{A})R_q(\mathcal{A}),$$

which implies that $\lambda \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^{S}(\mathcal{A})$. If $|x_q| > 0$, similar to (10), we have

$$|\lambda||x_q|^{m-1} \leq R_q(\mathcal{A})|x_p|^{m-1}. \tag{12}$$

Multiplying (11) with (12) and noting that $|x_p|^{m-1}|x_q|^{m-1} > 0$, we have

$$(|\lambda| - R_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|\lambda| \leq R_p^{\Delta^{\bar{S}}}(\mathcal{A})R_q(\mathcal{A}),$$

which leads to $\lambda \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^{\bar{S}}(\mathcal{A})$. The conclusion follows from Cases I and II. □

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\Omega^{S}(\mathcal{A}) \subseteq \hat{\Gamma}(\mathcal{A}).$$

Proof. Let $\lambda \in \Omega^{S}(\mathcal{A})$. Then

$$\lambda \in \bigcup_{i \in S, j \in \bar{S}} \Omega_{i,j}^{S}(\mathcal{A}) \text{ or } \lambda \in \bigcup_{i \in \bar{S}, j \in S} \Omega_{i,j}^{\bar{S}}(\mathcal{A}).$$

Without loss of generality, suppose that $\lambda \in \bigcup_{i \in S, j \in \bar{S}} \Omega_{i,j}^{S}(\mathcal{A})$ (we can prove it similarly if $\lambda \in \bigcup_{i \in \bar{S}, j \in S} \Omega_{i,j}^{\bar{S}}(\mathcal{A})$). Then there are $i \in S$ and $j \in \bar{S}$ such that $\lambda \in \Omega_{i,j}^{S}(\mathcal{A})$, i.e.,

$$|\lambda|(|\lambda| - R_j^{\overline{\Delta^{S}}}(\mathcal{A})) \leq R_i(\mathcal{A})R_j^{\Delta^{S}}(\mathcal{A}). \tag{13}$$

If $R_i(\mathcal{A})R_j^{\Delta^{S}}(\mathcal{A}) = 0$, then $\lambda = 0$ or $|\lambda| \leq R_j^{\overline{\Delta^{S}}}(\mathcal{A}) \leq R_j(\mathcal{A})$. Hence, $\lambda \in R_i(\mathcal{A}) \bigcup R_j(\mathcal{A})$. Otherwise, from (13), we have

$$\frac{|\lambda|}{R_i(\mathcal{A})} \frac{|\lambda| - R_j^{\overline{\Delta^{S}}}(\mathcal{A})}{R_j^{\Delta^{S}}(\mathcal{A})} \leq 1.$$

Furthermore,

$$\frac{|\lambda|}{R_i(\mathcal{A})} \leq 1$$

or

$$\frac{|\lambda| - R_j^{\overline{\Delta^{S}}}(\mathcal{A})}{R_j^{\Delta^{S}}(\mathcal{A})} \leq 1,$$

which implies that $\lambda \in R_i(\mathcal{A}) \bigcup R_j(\mathcal{A})$. □

Remark 3.3. *From Theorem 3.2, we known that the set $\Omega^{S}(\mathcal{A})$ in Theorem 3.1 localizes all E-eigenvalues of a tensor $\mathcal{A}$ more precisely than the set $\hat{\Gamma}(\mathcal{A})$ in Theorem 1.6.*

Next, based on Theorem 3.1, we give an S-type upper bound for the Z-spectral radius of a weakly symmetric nonnegative tensor.

Theorem 3.4. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\varrho(\mathcal{A}) \leq \Psi^{S}(\mathcal{A}) = \max\left\{\max_{i \in S, j \in \bar{S}} \Psi_{ij}^{S}(\mathcal{A}), \max_{i \in \bar{S}, j \in S} \Psi_{ij}^{\bar{S}}(\mathcal{A})\right\},$$

where

$$\Psi_{ij}^{S}(\mathcal{A}) = \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}})^2 + 4R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\},$$
$$\Psi_{ij}^{\bar{S}}(\mathcal{A}) = \frac{1}{2}\left\{R_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^{\bar{S}}}})^2 + 4R_i(\mathcal{A})R_j^{\Delta^{\bar{S}}}(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

Proof. By Lemma 4.4 in [8], $\varrho(\mathcal{A})$ is the largest Z-eigenvalue of $\mathcal{A}$. From Theorem 3.1, we know that $\varrho(\mathcal{A}) \in \Omega^S(\mathcal{A})$. Then

$$\varrho(\mathcal{A}) \in \bigcup_{i\in S, j\in\bar{S}} \Omega_{i,j}^{S}(\mathcal{A}) \text{ or } \varrho(\mathcal{A}) \in \bigcup_{i\in\bar{S}, j\in S} \Omega_{i,j}^{\bar{S}}(\mathcal{A}).$$

We next distinguish two cases to prove.

Case I: If $\varrho(\mathcal{A}) \in \bigcup_{i\in S, j\in\bar{S}} \Omega_{i,j}^{S}(\mathcal{A})$, then there exists $i \in S$, $j \in \bar{S}$, such that

$$\varrho(\mathcal{A})\left(\varrho(\mathcal{A}) - R_j^{\overline{\Delta^S}}(\mathcal{A})\right) \le R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A}).$$

Then

$$\varrho(\mathcal{A}) \le \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

Furthermore,

$$\varrho(\mathcal{A}) \le \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

Case II: If $\varrho(\mathcal{A}) \in \bigcup_{i\in\bar{S}, j\in S} \Omega_{i,j}^{\bar{S}}(\mathcal{A})$, similar to the proof of Case I, we can obtain

$$\varrho(\mathcal{A}) \le \max_{i\in\bar{S}, j\in S} \frac{1}{2}\left\{R_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})R_j^{\Delta^{\bar{S}}}(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

The conclusion follows from Cases I and II. □

Theorem 3.5. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\Psi^S(\mathcal{A}) \le \max_{i\in N} R_i(\mathcal{A}). \tag{14}$$

Proof. Here, we only prove that when $\Psi^S(\mathcal{A}) = \max_{i\in S, j\in\bar{S}} \Psi_{ij}^{S}(\mathcal{A})$, (14) holds. Similarly, we can also prove that (14) holds if $\Psi^S(\mathcal{A}) = \max_{i\in\bar{S}, j\in S} \Psi_{ij}^{\bar{S}}(\mathcal{A})$. Next, we divide two cases to prove.

Case I: For any $i \in S$, $j \in \bar{S}$, if $R_i(\mathcal{A}) \le R_j(\mathcal{A})$, then

$$\begin{aligned}
\Psi^S(\mathcal{A}) &= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\
&\le \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_j(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\
&= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4\big(R_j^{\overline{\Delta^S}}(\mathcal{A}) + R_j^{\Delta^S}(\mathcal{A})\big)R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\
&= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[\big(R_j^{\overline{\Delta^S}}(\mathcal{A}) + 2R_j^{\Delta^S}(\mathcal{A})\big)^2\right]^{\frac{1}{2}}\right\} \\
&= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + R_j^{\overline{\Delta^S}}(\mathcal{A}) + 2R_j^{\Delta^S}(\mathcal{A})\right\} \\
&= \max_{j\in\bar{S}} R_j(\mathcal{A})
\end{aligned}$$

$$\leq \max_{j\in N} R_j(\mathcal{A}).$$

Case II: For any $i \in S$, $j \in \bar{S}$, if $R_j(\mathcal{A}) \geq R_i(\mathcal{A})$, then $0 \leq R_j^{\Delta^S}(\mathcal{A}) \leq R_i(\mathcal{A}) - R_j^{\overline{\Delta^S}}(\mathcal{A})$, and

$$\begin{aligned}
\Psi^S(\mathcal{A}) &= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})R_j^{\Delta^S}(\mathcal{A})\right]^{\frac{1}{2}}\right\}\\
&\leq \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[(R_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4R_i(\mathcal{A})\big(R_i(\mathcal{A}) - R_j^{\overline{\Delta^S}}(\mathcal{A})\big)\right]^{\frac{1}{2}}\right\}\\
&= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + \left[\big(2R_i(\mathcal{A}) - R_j^{\overline{\Delta^S}}(\mathcal{A})\big)^2\right]^{\frac{1}{2}}\right\}\\
&= \max_{i\in S, j\in\bar{S}} \frac{1}{2}\left\{R_j^{\overline{\Delta^S}}(\mathcal{A}) + 2R_i(\mathcal{A}) - R_j^{\overline{\Delta^S}}(\mathcal{A})\right\}\\
&= \max_{i\in S} R_i(\mathcal{A})\\
&\leq \max_{i\in N} R_i(\mathcal{A}).
\end{aligned}$$

The conclusion follows from Cases I and II. □

Remark 3.6. *Theorem 3.5 shows that the upper bound in Theorem 3.4 is better than Corollary 4.5 of [9].*

Now, we show that the upper bound in Theorem 3.4 is sharper than those in [8–13] in some cases by the following example.

Example 3.7. *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ with entries defined as follows:*

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 0 & 2 & 0\\ 1.5 & 1 & 2\\ 0 & 2 & 0\end{pmatrix},\ \mathcal{A}(:,:,2) = \begin{pmatrix} 1 & 0.5 & 2\\ 0 & 0 & 2\\ 2 & 2.5 & 3\end{pmatrix},\ \mathcal{A}(:,:,3) = \begin{pmatrix} 0 & 2 & 0.5\\ 2 & 3 & 2.5\\ 1 & 2 & 1\end{pmatrix}.$$

It is not difficult to verify that $\mathcal{A}$ is a weakly symmetric nonnegative tensor. By computation, we obtain $(\varrho(\mathcal{A}), x) = (7.3450, (0.3908, 0.6421, 0.6596))$. By Corollary 4.5 of [9] and Theorem 3.3 of [10], we both have

$$\varrho(\mathcal{A}) \leq 14.$$

By Theorem 3.5 of [11], we have

$$\varrho(\mathcal{A}) \leq 13.9189.$$

By Theorem 4.6 of [8], we have

$$\varrho(\mathcal{A}) \leq 13.9133.$$

By Theorem 4.7 of [8], we have

$$\varrho(\mathcal{A}) \leq 13.8167.$$

By Theorem 4.5 of [8] and Theorem 6 of [12], we both have

$$\varrho(\mathcal{A}) \leq 13.5000.$$

By Theorem 2.9 of [13], we have

$$\varrho(\mathcal{A}) \leq 12.9790.$$

Let $S = \{1\}$, $\bar{S} = \{2, 3\}$. By Theorem 3.4, we obtain

$$\varrho(\mathcal{A}) \leq 11.5440,$$

which shows that the upper bound in Theorem 3.4 is sharper.

4 Conclusions

In this paper, we give a new eigenvalue localization set $\mathcal{K}^{\cap}(\mathcal{A})$ and prove that $\mathcal{K}^{\cap}(\mathcal{A})$ is tighter than those in [1] and [4]. Based on this set, we obtain a weaker checkable sufficient condition to determine the positive (semi-)definiteness for an even-order real symmetric tensor. Meanwhile, we present an S-type E-eigenvalue localization set $\Omega^S(\mathcal{A})$ and prove that $\Omega^S(\mathcal{A})$ is tighter than that in [8]. As an application, we obtain an S-type upper bound $\Psi^S(\mathcal{A})$ for the Z-spectral radius of weakly symmetric nonnegative tensors, and show that $\Psi^S(\mathcal{A})$ is sharper than those in [8–13] in some cases by a numerical example. Then an interesting problem is how to pick S to make $\Psi^S(\mathcal{A})$ as small as possible. But this is difficult when n is large. In the future, we will focus on this problem.

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