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Ricci solitons on almost Kenmotsu 3-manifolds

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Abstract: Let (M^3, g) be an almost Kenmotsu 3-manifold such that the Reeb vector field is an eigenvector field of the Ricci operator. In this paper, we prove that if g represents a Ricci soliton whose potential vector field is orthogonal to the Reeb vector field, then M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure. In particular, when g represents a gradient Ricci soliton whose potential vector field is orthogonal to the Reeb vector field, then M^3 is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$.

Keywords: Almost Kenmotsu 3-manifold, Ricci soliton, Non-unimodular Lie group

MSC: 53D15, 53C25

1 Introduction

Boyer and Galicki [1] proved that a compact Einstein K-contact manifold is Sasakian. Such result can be regarded as an odd-dimensional analog of the well known Goldberg's conjecture which says that a compact Einstein almost Kähler manifold is Kähler. Since then many authors started to study the generalization of Boyer-Galicki's result from various points of view. One of the most attractive methods to extend Boyer-Galicki's result is weakening Einstein condition to a Ricci soliton.

In this paper, by a Ricci soliton we mean a Riemannian manifold (M, g) whose Ricci tensor Ric satisfies

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} + \omega g = 0, \tag{1}$$

where V is a tangent vector field called the potential vector field, ω is a constant called the soliton constant and $\mathcal{L}$ is the Lie differentiation. A Ricci soliton is a self-similar solution of the well known Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t))$, up to diffeomorphisms and scalings. Usually, a Ricci soliton is said to be shrinking, steady and expanding according with ω is negative, zero and positive respectively. In particular, if the potential vector field is the gradient of a smooth function $-f$ on M, then (1) becomes

$$\nabla\nabla f = \mathrm{Ric} + \omega g \tag{2}$$

and in this case a Ricci soliton is said to be a gradient Ricci soliton and f is called a potential function. It is known that a compact Ricci soliton is always a gradient Ricci soliton. In (1) or (2), when the potential vector field vanishes or is Killing, then the soliton becomes an Einstein metric and in this case the soliton is said to be trivial.

Almost contact metric manifolds can be viewed as an odd-dimensional version of almost Hermitian manifolds (see [2]). Almost Kenmotsu manifolds, viewed as a special class of almost contact metric manifolds, have recently been increasing interest in contact geometry. For the studies of Ricci solitons on contact metric manifolds, we refer

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the reader to [3–5]. The study of Ricci solitons on Kenmotsu geometry was initiated by Ghosh in [6] in which the author proved that a Kenmotsu 3-manifold admitting a Ricci soliton is of constant sectional curvature -1. Later, such result was generalized by Ghosh, in [7], to an η-Einstein Kenmotsu manifold of dimension > 3. Cho in [8] proved that a Kenmotsu 3-manifold admitting a Ricci soliton with unit potential vector field orthogonal to ξ is of constant sectional curvature -1. This in fact is a special case of Ghosh's main result in [6].

Recently, the study of Ricci solitons on almost Kenmotsu manifolds was started by the present author and Liu in [9]. In this paper, we extend Ghosh's result (see [6]) to η-Einstein almost Kenmotsu 3-manifolds. Obviously, all Kenmotsu-Ricci solitons mentioned above are trivial. If the metric g of a Riemannian manifold M satisfies (2) for a smooth function ω, then (M, g) is called a gradient Ricci almost soliton. Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds were considered by the present author in [10]. It was proved that a non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold admitting a gradient Ricci almost soliton is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. To my best knowledge, such product is the first nontrivial almost Kenmotsu-Ricci soliton.

In this paper, we continue the study of Ricci solitons on an almost Kenmotsu 3-manifold $(M^3, \phi, \xi, \eta, g)$. Firstly, we show that there exists no Ricci soliton on a Kenmotsu 3-manifold such that the potential vector field is pointwise colinear with the Reeb vector field. Also, we prove that there exists no Ricci soliton on an almost Kenmotsu 3-manifold such that the potential vector field is a constant multiple of ξ. This leads us to consider a potential vector field $\mathcal{V}$ orthogonal to ξ. We prove that when M^3 is an almost Kenmotsu 3-manifold with ξ an eigenvector field of the Ricci operator admitting a gradient Ricci soliton $(g, \mathcal{V})$, then the manifold is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$.

Generalizing the above result, we show that an almost Kenmotsu 3-manifold with ξ an eigenvector field of the Ricci operator admitting a Ricci soliton $(g, \mathcal{V})$ is locally isometric to either $\mathbb{H}^3(-1)$ or a non-unimodular Lie group (whose Lie algebra is given in Theorem 3.6). Our main results mentioned above are natural generalizations of those in [6] and [8, 9].

2 Three-dimensional almost Kenmotsu manifolds

An *almost contact structure* on a smooth differentiable manifold M^{2n+1} of dimension $2n+1$ means a triple (ϕ, ξ, η) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{3}$$

where ϕ is a $(1, 1)$-type tensor field, ξ is a vector field called the Reeb vector field and η is a 1-form called the almost contact 1-form. If there exists a Riemannian metric g on an almost contact manifold M^{2n+1} such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

for any vector fields X, Y, then M^{2n+1} is said to be an *almost contact metric manifold* and g is said to be a *compatible metric* with respect to the almost contact structure (see [2]).

From Janssens and Vanhecke [11], in this paper by an *almost Kenmotsu manifold* we mean an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$, where the *fundamental 2-form* Φ of the almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M^{2n+1}.

We consider the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$ and define on it an almost complex structure J by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ. If

$$[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$$

holds, then the almost contact metric structure is said to be *normal*.

A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (cf. [11, 12]). It is well known that an almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y.

Let M^{2n+1} be an almost Kenmotsu manifold. We consider three tensor fields $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. From Dileo and Pastore [13, 14], we know that the three $(1,1)$-type tensor fields l, h' and h are symmetric and satisfy $h\xi = 0$, $l\xi = 0$, $\mathrm{tr}h = 0$, $\mathrm{tr}(h') = 0$, $h\phi + \phi h = 0$ and

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{5}$$

for any vector field X.

In this paper, we denote by Ric and Q the Ricci tensor and the Ricci operator with respect to the metric g respectively, that is, $\mathrm{Ric}(X, Y) = g(QX, Y)$.

The following proposition was proved in [13].

Proposition 2.1. *A 3-dimensional almost Kenmotsu manifold is Kenmotsu if and only if h vanishes.*

Let $\mathcal{U}_1$ be the open subset of a 3-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where λ is a positive function on $\mathcal{U}_1$. Note that the eigenvalue function λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 2.2 ([15, Lemma 6])**.** *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_\xi e = a\phi e,\ \nabla_\xi \phi e = -ae,\\
&\nabla_e \xi = e - \lambda \phi e,\ \nabla_e e = -\xi - b\phi e,\ \nabla_e \phi e = \lambda \xi + be,\\
&\nabla_{\phi e} \xi = -\lambda e + \phi e,\ \nabla_{\phi e} e = \lambda \xi + c\phi e,\ \nabla_{\phi e} \phi e = -\xi - ce,
\end{aligned} \tag{6}$$

where a, b, c are smooth functions.

Moreover, applying Lemma 2.2 we have (see also [15]) the following

Lemma 2.3. *On $\mathcal{U}_1$, the Ricci operator can be written as*

$$\begin{aligned}
&Q\xi = -2(\lambda^2 + 1)\xi - \sigma(e)e - \sigma(\phi e)\phi e,\\
&Qe = -\sigma(e)\xi - (A + 2\lambda a)e + (\xi(\lambda) + 2\lambda)\phi e,\\
&Q\phi e = -\sigma(\phi e)\xi + (\xi(\lambda) + 2\lambda)e - (A - 2\lambda a)\phi e,
\end{aligned} \tag{7}$$

with respect to the local basis $\{\xi, e, \phi e\}$, where we set $A = e(c) + \phi e(b) + b^2 + c^2 + 2$, $\sigma(e) := -g(Q\xi, e) = \phi e(\lambda) + 2\lambda b$ and $\sigma(\phi e) := -g(Q\xi, \phi e) = e(\lambda) + 2\lambda c$.

Throughout the paper, we denote by $\mathcal{D}$ the distribution $\{\xi\}^\perp = \ker \eta$.

3 Ricci solitons on almost Kenmotsu 3-manifolds

Firstly, we present the following propositions which explain explicitly why we study a potential vector field orthogonal to the Reeb vector field ξ on an almost Kenmotsu 3-manifold.

Proposition 3.1. *On a Kenmotsu 3-manifold there exists no non-trivial Ricci soliton with potential vector field pointwise colinear with the Reeb vector field.*

Proof. Let M^3 be a Kenmotsu 3-manifold admitting a Ricci soliton whose potential vector field V is pointwise colinear with ξ. We may set $V = \alpha\xi$, where α is a non-zero smooth function on M^3. In this case, applying Proposition 2.1 and (5), we have $\nabla_X\xi = X - \eta(X)\xi$ and hence $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$ for any vector fields X, Y. It follows directly that $Q\xi = -2\xi$. Putting $V = \alpha\xi$ into (1) gives

$$QX = -(\alpha+\omega)X + \alpha\eta(X)\xi - \frac{1}{2}X(\alpha)\xi - \frac{1}{2}\eta(X)D\alpha, \tag{8}$$

where D denotes the gradient operator.

Let X in the above relation be orthogonal to ξ; then we obtain $QX = -(\alpha+\omega)X - \frac{1}{2}X(\alpha)\xi$. Taking the inner product of this relation with ξ and using $Q\xi = -2\xi$, we obtain $D\alpha = \xi(\alpha)\xi$. Using this again in (8), because of $Q\xi = -2\xi$, we get $\xi(\alpha) = 2-\omega$. Thus, we have $QX = -(\alpha+\omega)X$ for any $X \in \mathcal{D}$ and hence the scalar curvature $r = -2(\alpha+\omega+1)$. Let us recall the well known formula $\sum_{i=1}^{3} g((\nabla_{e_i}Q)e_i, X) = \frac{1}{2}X(r)$ for any vector field X and any local orthonormal basis $\{e_i, i = 1,2,3\}$. In this formula, considering $X = \xi$ we have $\alpha = 1 - \frac{1}{2}\omega$, a constant. In view of $\xi(\alpha) = 2-\omega$, we obtain $\omega = 2$ and hence $\alpha = 0$, a contradiction. This completes the proof. □

Proposition 3.2. *Let (M^3, g) be a non-Kenmotsu almost Kenmotsu 3-manifold. If g is a non-trivial Ricci soliton with potential vector field pointwise colinear with the Reeb vector field, then we have*

$$\begin{aligned} &\frac{1}{2}\xi(\alpha) = 2\lambda^2 + 2 - \omega,\ e(\alpha) = 2\sigma(e),\ \phi e(\alpha) = 2\sigma(\phi e),\\ &A = \alpha + \omega,\ a = 0,\ \lambda\alpha = \xi(\lambda) + 2\lambda. \end{aligned} \tag{9}$$

Proof. Suppose that the potential vector field V is given by $V = \alpha\xi$, where α is a non-zero smooth function on M^3. As M^3 is assumed to be non-Kenmotsu, $\mathcal{U}_1$ is nonempty and Lemmas 2.2 and 2.3 are true. Using (5) in (1) we have

$$QX = -(\alpha+\omega)X + \alpha\eta(X)\xi - \alpha h'X - \frac{1}{2}X(\alpha)\xi - \frac{1}{2}\eta(X)D\alpha. \tag{10}$$

Replacing X in (10) by ξ we have $Q\xi = -(\omega + \frac{1}{2}\xi(\alpha))\xi - \frac{1}{2}D\alpha$. Comparing this with the first term of (7) we get the first term of (9). Similarly, replacing X in (10) by e we have $Qe = -(\alpha+\omega)e + \lambda\alpha\phi e - \frac{1}{2}e(\alpha)\xi$. Comparing this with the second term of (7) we get $\alpha + \omega = A + 2\lambda a$ and $\lambda\alpha = \xi(\lambda) + 2\lambda$. Also, replacing X in (10) by ϕe we have $Q\phi e = -(\alpha+\omega)e + \lambda\alpha e - \frac{1}{2}\phi e(\alpha)\xi$. Comparing this with the third term of (7) we obtain $\alpha + \omega = A - 2\lambda a$ and $\lambda\alpha = \xi(\lambda) + 2\lambda$. This implies the second term of (9). □

Applying the above two propositions, we have

Corollary 3.3. *On an almost Kenmotsu 3-manifold there exists no Ricci soliton such that the potential vector field is given by $\alpha\xi$, where α is a function invariant along the Reeb vector field.*

Proof. Suppose that there exists a Ricci soliton on an almost Kenmotsu 3-manifold such that the potential vector field $V = \alpha\xi$ satisfies $\xi(\alpha) = 0$, where α is a smooth function. Because of Proposition 3.1, next we need only to consider non-Kenmotsu case. From the first term of (9) we have $\lambda^2 + 1 = \frac{1}{2}\omega$ and hence λ is a positive constant. From the second term of (9) we have $\alpha = 2$. Applying this again in the first term of (9) we have $\sigma(e) = 0$ and $\sigma(\phi e) = 0$ and hence $b = c = 0$. By Lemma 2.3 we have $A = 2$. Because of $\alpha = 2$, from the second term of (9) we have $\omega = A - \alpha = 0$. This is impossible since it implies that $\lambda^2 + 1 = 0$. This completes the proof. □

Corollary 3.4. *On an almost Kenmotsu 3-manifold there exists no Ricci soliton whose potential vector field is constant multiple of the Reeb vector field.*

Corollary 3.4 follows directly from Corollary 3.3.

Corollary 3.5. *On an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$, there exists no Ricci soliton whose potential vector field is pointwise colinear with the Reeb vector field.*

Proof. By (6) we know that on a non-Kenmotsu almost Kenmotsu 3-manifold M^3 the condition $\nabla_\xi h = 0$ is equivalent to $a = 0$ and $\xi(\lambda) = 0$. Thus, if there exists a Ricci soliton whose potential vector field is pointwise

colinear with the Reeb vector field, i.e., $V = \alpha\xi$, α a function, we have from the second term of (9) that $\alpha = 2$. The remaining proof is similar with that of Corollary 3.3. □

The conclusion in Corollary 3.5 is still true even when $\nabla_\xi h = 0$ is replaced by a weaker condition $\xi(\mathrm{tr}h^2) = 0$.

In view of the above results, it is reasonable to consider a potential vector orthogonal to the Reeb vector field on an almost Kenmotsu 3-manifold under a condition that ξ is an eigenvector field of the Ricci operator. Notice that such condition holds naturally on any Kenmotsu 3-manifold (see proof of Proposition 3.1). Next we construct some examples of non-Kenmotsu almost Kenmotsu 3-manifolds such that the Reeb vector field is an eigenvector field of the Ricci operator.

A 3-dimensional almost Kenmotsu manifold is called a (k, μ, ν)-almost Kenmotsu manifold if the Reeb vector field satisfies the (k, μ, ν)-nullity condition, that is,

$$\begin{aligned} R(X,Y)\xi =& k(\eta(Y)X - \eta(X)Y) \\ &+ \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y) \end{aligned}$$

for any vector fields X, Y and some smooth functions k, μ, ν. It was shown in [17] that a (k, μ, ν)-almost Kenmotsu manifold with $k < -1$ is non-Kenmotsu and ξ is an eigenvector field of the Ricci operator, i.e, $Q\xi = 2k\xi$. For some concrete examples of (k, μ, ν)-almost Kenmotsu manifolds we refer the reader to [18]. Moreover, from Theorem 3.2 of [17] one can check that the Reeb vector field of the almost Kenmotsu structure defined on any non-unimodular Lie group of dimension three is an eigenvector field of the Ricci operator.

Now we are ready to give our main results.

Theorem 3.6. *Let (M^3, g) be an almost Kenmotsu 3-manifold with ξ an eigenvector field of the Ricci operator. If g is a gradient Ricci soliton with potential vector field orthogonal to the Reeb vector field, then M^3 is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. Ghosh in [6] proved that a Kenmotsu 3-manifold admitting a Ricci soliton is of constant sectional curvature -1 with soliton constant $\omega = 2$. Thus, next we need only to consider M^3 a non-Kenmotsu 3-manifold and in this case Lemmas 2.2 and 2.3 are applicable. Since the metric g of M^3 is assumed to be a gradient Ricci soliton, (2) can be written as the following form

$$QX = \nabla_X Df - \omega X \tag{11}$$

for any vector field X. Suppose that the potential vector field is orthogonal to the Reeb vector field ξ. We set $Df = f_2 e + f_3 \phi e$, where f_2 and f_3 are smooth functions. Replacing X in (11) by ξ and using (6) we have

$$Q\xi = -\omega\xi + (\xi(f_2) - af_3)e + (\xi(f_3) + af_2)\phi e.$$

Since ξ is an eigenvector field of the Ricci operator, we have $\sigma(e) = \sigma(\phi e) = 0$. Thus, comparing the previous relation with the first term of (7) gives

$$\omega = 2(\lambda^2 + 1),\ \xi(f_2) = af_3,\ \xi(f_3) = -af_2. \tag{12}$$

The first term of (12) means that λ is a positive constant. Therefore, because of $\sigma(e) = \sigma(\phi e) = 0$, we obtain $b = c = 0$ and hence $A = 2$.

Similarly, replacing X in (11) by e and using (6) we have

$$Qe = (\lambda f_3 - f_2)\xi + (e(f_2) - \omega)e + e(f_3)\phi e.$$

Comparing this relation with the second term of (7) gives

$$f_2 = \lambda f_3,\ e(f_2) = 2\lambda(\lambda - a),\ e(f_3) = 2\lambda. \tag{13}$$

Similarly, replacing X in (11) by ϕe and using (6) we have

$$Q\phi e = (\lambda f_2 - f_3)\xi + \phi e(f_2)e + (\phi e(f_3) - \omega)\phi e.$$

Comparing this relation with the second term of (7) gives

$$f_3 = \lambda f_2,\ \phi e(f_2) = 2\lambda,\ \phi e(f_3) = 2\lambda(\lambda + a). \tag{14}$$

Since λ is a positive constant, putting the first term of (14) into that of (13) we obtain $(\lambda^2 - 1) f_2 = 0$. Assuming that $\lambda^2 \neq 1$ holds, it follows that $f_2 = 0$ and using this in the second term of (14) we have $\lambda = 0$, a contradiction. Consequently, we conclude that $\lambda = 1$ and hence $\omega = 2$. By the first term of (14), it is easy to see $f_2 = f_3$. Thus, from the last two terms of (14) we have $a = 0$. In this context, applying Lemma 2.3, by a direct calculation we see that the Ricci operator is parallel. This is equivalent to the fact that the manifold is locally symmetric. The present author in [16] and Cho in [19] proved that any non-Kenmotsu almost Kenmotsu 3-manifold is locally symmetric if and only if it is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. This completes the proof. □

Note that the product $\mathbb{H}^2(-4) \times \mathbb{R}$ is a rigid gradient Ricci soliton (see [20, 21]). For other results on existence of quasi-Einstein metrics on this product we refer the reader to [22].

Theorem 3.7. *Let (M^3, g) be an almost Kenmotsu 3-manifold with ξ an eigenvector field of the Ricci operator. If g is a Ricci soliton with potential vector field orthogonal to the Reeb vector field, then M^3 is locally isometric to either $\mathbb{H}^3(-1)$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure whose Lie algebra is given by* (25)*.*

Proof. As seen in the proof of Theorem 3.6, the proof for the Kenmotsu case has been considered in [6]. Now let M^3 be a non-Kenmotsu almost Kenmotsu 3-manifold. Since g is a Ricci soliton, from (1) we have

$$\frac{1}{2} g(\nabla_X V, Y) + \frac{1}{2} g(\nabla_Y V, X) + g(QX, Y) + \omega g(X, Y) = 0 \tag{15}$$

for any vector fields X, Y. Next we assume that $V = f_2 e + f_3 \phi e$, where f_2 and f_3 are smooth functions. Also, by the hypothesis we have $\sigma(e) = \sigma(\phi e) = 0$.

Replacing $X = Y$ in (1) by ξ and using (6) we have

$$\omega = 2(\lambda^2 + 1). \tag{16}$$

This means that λ is a positive constant. Thus, in view of $\sigma(e) = \sigma(\phi e) = 0$, we have $b = c = 0$ and hence $A = 2$. Replacing $X = Y$ in (1) by e and using (6) we have

$$e(f_2) + 2\lambda(\lambda - a) = 0. \tag{17}$$

Replacing $X = Y$ in (1) by ϕe and using (6) we have

$$\phi e(f_3) + 2\lambda(\lambda + a) = 0. \tag{18}$$

Putting $X = \xi$ and $Y = e$ into (15) and using (6) we have

$$(\lambda - a) f_3 + \xi(f_2) - f_2 = 0. \tag{19}$$

Putting $X = e$ and $Y = \phi e$ into (15) and using (6) we have

$$e(f_3) + \phi e(f_2) + 4\lambda = 0. \tag{20}$$

Putting $X = \xi$ and $Y = \phi e$ into (15) and using (6) we have

$$(\lambda + a) f_2 + \xi(f_3) - f_3 = 0. \tag{21}$$

Applying Lemma 2.2 we have the following Lie brackets.

$$[e, \xi] = e - (\lambda + a)\phi e,\ [e, \phi e] = 0,\ [\phi e, \xi] = (a - \lambda)e + \phi e. \tag{22}$$

Applying (22) in the well known Jacobi identity

$$[[e, \xi], \phi e] + [[\xi, \phi e], e] + [[\phi e, e], \xi] = 0$$

we have $e(a) = \phi e(a) = 0$. Therefore, by a direct calculation, using (21) and (17) we have

$$e(\xi(f_3)) - \xi(e(f_3)) = e(f_3) + 2\lambda(\lambda - a)(\lambda + a) - \xi(e(f_3)).$$

However, by (18) and the first term of (22) we obtain $e(\xi(f_3)) - \xi(e(f_3)) = e(f_3) - (\lambda + a)\phi e(f_3) = e(f_3) + 2\lambda(\lambda + a)^2$. Comparing this with the previous relation and using (18) we have

$$\xi(e(f_3)) + 4\lambda a(\lambda + a) = 0. \tag{23}$$

Similarly, by a direct calculation, using (19) and (18) we have

$$\phi e(\xi(f_2)) - \xi(\phi e(f_2)) = \phi e(f_2) + 2\lambda(\lambda - a)(\lambda + a) - \xi(\phi e(f_2)).$$

However, by (17) and the third term of (22) we obtain $\phi e(\xi(f_2)) - \xi(\phi e(f_2)) = \phi e(f_2) + (a - \lambda)e(f_2) = \phi e(f_2) + 2\lambda(\lambda - a)^2$. Comparing this with the previous relation and using (17) we have

$$\xi(\phi e(f_2)) - 4\lambda a(\lambda - a) = 0. \tag{24}$$

From (20) we know that $e(f_3) + \phi e(f_2) = -4\lambda$ is a negative constant. Therefore, differentiating this relation with respect to ξ we have $\xi(e(f_3)) = -\xi(\phi e(f_2))$. Finally, since λ is a positive constant, using the previous relation, (23) and (24) we have $a = 0$. In this context, (22) becomes

$$[e, \xi] = e - \lambda\phi e,\ [e, \phi e] = 0,\ [\phi e, \xi] = -\lambda e + \phi e. \tag{25}$$

According to Milnor's classification (see [23]), we see that M^3 is locally isometric to a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure. For more details regarding the construction of such structure we refer the reader to [14, Theorem 5.2]. This completes the proof. □

Note that if $\lambda = 1$ for the non-unimodular Lie group whose Lie algebra is given by (25), then M^3 is in fact locally isometric to the product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Since on a Kenmotsu 3-manifold we have $Q\xi = -2\xi$, Theorem 3.6 generalizes naturally Cho's result (see [8, Theorem 3]).

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