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Chain conditions on composite Hurwitz series rings

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Abstract: In this paper, we study chain conditions on composite Hurwitz series rings and composite Hurwitz polynomial rings. More precisely, we characterize when composite Hurwitz series rings and composite Hurwitz polynomial rings are Noetherian, S -Noetherian or satisfy the ascending chain condition on principal ideals.

Keywords: Composite Hurwitz series ring $H(R, D)$, Composite Hurwitz polynomial ring $h(R, D)$, Noetherian ring, S -Noetherian ring, Présimplifiable ring, Ascending chain condition on principal ideals

MSC: 13A15, 13E05, 13E99, 13F15

1 Introduction

1.1 Hurwitz series rings

The formal power series rings and polynomial rings have been of interest and have had important applications in many areas, one of which has been differential algebra. In [1], Keigher introduced a variant of the ring of formal power series and studied some of its properties. In [2], Keigher called such a ring the ring of Hurwitz series and examined its ring theoretic properties. Since then, many works on the ring of Hurwitz series have been done ([3–5]).

Let R be a commutative ring with identity, $R[[X]]$ (resp., $R[X]$) the formal power series ring (resp., polynomial ring) over R , and $H(R)$ the set of formal expressions of the form $\sum_{n=0}^{\infty} a_n X^n$, where $a_n \in R$. Define addition and $*$ -product on $H(R)$ as follows: for $f = \sum_{n=0}^{\infty} a_n X^n$, $g = \sum_{n=0}^{\infty} b_n X^n \in H(R)$,

$$f + g = \sum_{n=0}^{\infty} (a_n + b_n) X^n \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$ and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ for nonnegative integers $n \geq k$. Then $H(R)$ becomes a commutative ring with identity containing R under these two operations, i.e., $H(R) = (R[[X]], +, *)$. The ring $H(R)$ is called the *Hurwitz series ring* over R . The *Hurwitz polynomial ring* $h(R)$ over R is the subring of $H(R)$ consisting of formal expressions of the form $\sum_{k=0}^n a_k X^k$, i.e., $h(R) = (R[X], +, *)$.

Let $R \subseteq D$ be an extension of commutative rings with identity, and let $H(R, D) = \{f \in H(D) \mid \text{the constant term of } f \text{ belongs to } R\}$ (resp., $h(R, D) = \{f \in h(D) \mid \text{the constant term of } f \text{ belongs to } R\}$). Then $H(R, D)$ (resp., $h(R, D)$) is a commutative ring with identity. We call $H(R, D)$ (resp., $h(R, D)$) a *composite Hurwitz series ring* (resp., *composite Hurwitz polynomial ring*). More precisely, $H(R, D)$ (resp., $h(R, D)$) is a subring of $H(D)$ (resp., $h(D)$) containing $H(R)$ (resp., $h(R)$), i.e., $H(R, D) = (R + XD[[X]], +, *)$ (resp., $h(R, D) = (R +$

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$XD[X], +, *)$), where $R + XD[[X]] = \{f \in D[[X]] \mid \text{the constant term of } f \text{ belongs to } R\}$ (resp., $R + XD[X] = \{f \in D[X] \mid \text{the constant term of } f \text{ belongs to } R\}$). Hence if $R \subsetneq D$, then $H(R, D)$ (resp., $h(R, D)$) gives algebraic properties of Hurwitz series (resp., Hurwitz polynomial) type rings strictly between two Hurwitz series rings (resp., Hurwitz polynomial rings). Also, it is easy to see that $H(R, D)$ (resp., $h(R, D)$) is a pullback of R and $H(D)$ (resp., $h(D)$).

1.2 Noetherian rings and related rings

Chain conditions have for many years been important tools in commutative algebra and algebraic geometry because of their use in producing many theorems and applications. For example, a relation between the ascending chain conditions on ideals and finitely generatedness of ideals in rings permits an interesting measure of the size and behavior of such rings, and the Noetherian condition plays a significant role to prove many results on varieties, homology and cohomology. Recently, Anderson and Dumitrescu [6] introduced the notion of S -Noetherian rings and gave a number of S -variants of well-known results for Noetherian rings. After them, S -Noetherian rings have been studied by some mathematicians (see [7–11]).

In [10, 12–14], the authors characterized when composite rings $R + XD[[X]]$ and $R + XD[X]$ are Noetherian rings, S -Noetherian rings, or satisfy the ascending chain condition on principal ideals. It was shown that $R + XD[[X]]$ (resp., $R + XD[X]$) is a Noetherian ring if and only if R is a Noetherian ring and D is a finitely generated R -module [13, Theorem 4] (or [12, Proposition 2.1]) (resp., [12, Proposition 2.1] (or [14, Corollary 2.2])); and that $R + XD[X]$ (resp., $R + XD[[X]]$) is an S -Noetherian ring if and only if R is an S -Noetherian ring and D is an S -finite R -module [10, Theorems 3.6 and 4.4]. Also, it was shown that if D is a présimplifiable ring, then $R + XD[[X]]$ satisfies the ascending chain condition on principal ideals if and only if $U(D) \cap R = U(R)$ and for each sequence $(d_n)_{n \geq 1}$ of D with the property that for each $n \geq 1$, $d_n = d_{n+1}r_n$ for some $r_n \in R$, $d_1D \subseteq d_2D \subseteq \cdots$ is stationary [12, Proposition 4.21].

In [4], Benhissi and Kojia studied when the Hurwitz rings $H(R)$ and $h(R)$ are Noetherian rings, S -Noetherian rings or satisfy the ascending chain condition on principal ideals. They showed that if $\text{char}(R) = 0$, then $H(R)$ is a Noetherian ring if and only if $h(R)$ is a Noetherian ring, if and only if R a Noetherian ring containing $\mathbb{Q}$ [4, Corollary 7.7]. Also, they proved that for an anti-Archimedean subset S of R with zero characteristic containing an element $s_0 \in S$ divisible in R by all the nonzero positive integers, if R is an S -Noetherian ring, then $h(R)$ is an S -Noetherian ring [4, Theorem 9.4]; and if R is an S -Noetherian ring and S consists of nonzerodivisors, then $H(R)$ is an S -Noetherian ring [4, Theorem 9.6].

In this paper, we study chain conditions on composite Hurwitz series rings $H(R, D)$ and composite Hurwitz polynomial rings $h(R, D)$, where $R \subseteq D$ is an extension of commutative rings with identity. In Section 2, we give necessary and sufficient conditions for the rings $H(R, D)$ and $h(R, D)$ to be Noetherian rings. We show that if $\text{char}(R) = 0$, then $H(R, D)$ is a Noetherian ring if and only if $h(R, D)$ is a Noetherian ring, if and only if R is a Noetherian ring and D is a finitely generated R -module containing $\mathbb{Q}$. In Section 3, we give equivalent conditions for the rings $H(R, D)$ and $h(R, D)$ to be S -Noetherian rings, where S is an anti-Archimedean subset of R . We show that if $\text{char}(R) = 0$ and S is an anti-Archimedean subset of R consisting of nonzerodivisors of D which contains an element divisible in D by all the positive integers, then $H(R, D)$ is an S -Noetherian ring if and only if R is an S -Noetherian ring and D is an S -finite R -module; and if $\text{char}(R) = 0$ and S is an anti-Archimedean subset of R which contains an element divisible in D by all the positive integers, then $h(R, D)$ is an S -Noetherian ring if and only if R is an S -Noetherian ring and D is an S -finite R -module. In Section 4, we study when the rings $H(R, D)$ and $h(R, D)$ are présimplifiable. We prove that $H(R, D)$ is présimplifiable if and only if $Z(D) \cap R \subseteq 1 + U(R)$, where $Z(D)$ is the set of zero-divisors of D and $U(R)$ is the set of units in R . We also prove that if D is a torsion-free $\mathbb{Z}$ -module, then $h(R, D)$ is présimplifiable if and only if D is a domainlike ring. Finally, in Section 5, we characterize when the rings $H(R, D)$ and $h(R, D)$ satisfy the ascending chain condition on principal ideals. We show that if D is a présimplifiable ring, then $H(R, D)$ satisfies the ascending chain condition on principal ideals if and only if $U(D) \cap R = U(R)$ and for each sequence $(d_n)_{n \geq 1}$ of D with the property that for each $n \geq 1$, there exists an element $r_n \in R$ such that $d_n = d_{n+1}r_n$, $d_1D \subseteq d_2D \subseteq \cdots$ is stationary; and if D is a présimplifiable ring with $\text{char}(D) > 0$, then $h(R, D)$ satisfies the ascending chain condition on principal ideals if and only if

$U(D) \cap R = U(R)$ and for each sequence $(d_n)_{n \geq 1}$ of D with the property that for each $n \geq 1$, there exists an element $r_n \in R$ such that $d_n = d_{n+1}r_n$, $d_1 D \subseteq d_2 D \subseteq \dots$ is stationary.

2 Noetherian rings

Let R be a commutative ring with identity. Then the mapping $\psi : R[[X]] \rightarrow H(R)$ (resp., $\phi : R[X] \rightarrow h(R)$) defined by

$$\psi\left(\sum_{n=0}^{\infty} a_n X^n\right) = \sum_{n=0}^{\infty} n! a_n X^n \quad (\text{resp., } \phi\left(\sum_{k=0}^n a_k X^k\right) = \sum_{k=0}^n k! a_k X^k)$$

is a ring homomorphism [2, Proposition 2.3]; and ψ is an isomorphism if and only if ϕ is an isomorphism, if and only if the $\mathbb{Z}$ -module R is divisible and torsion-free, if and only if R contains $\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers ([2, Proposition 2.4] and [4, Theorem 1.4 and Corollary 1.5]).

We start this section with the following simple observation without proof.

Lemma 2.1. *Let $R \subseteq D$ be an extension of commutative rings with identity. Then the following conditions are equivalent.*

- (1) D contains $\mathbb{Q}$.
- (2) The $\mathbb{Z}$ -module D is divisible and torsion-free.
- (3) The mapping $\psi : R + XD[[X]] \rightarrow H(R, D)$ defined by $\psi\left(\sum_{n=0}^{\infty} a_n X^n\right) = \sum_{n=0}^{\infty} n! a_n X^n$ is a ring isomorphism.
- (4) The mapping $\phi : R + XD[X] \rightarrow h(R, D)$ defined by $\phi\left(\sum_{k=0}^n a_k X^k\right) = \sum_{k=0}^n k! a_k X^k$ is a ring isomorphism.

Let $R \subseteq D$ be an extension of commutative rings with identity, and set $XD[[X]] = \{f \in H(R, D) \mid \text{the constant term of } f \text{ is zero}\}$ (resp., $XD[X] = \{f \in h(R, D) \mid \text{the constant term of } f \text{ is zero}\}$). Then it is easy to see that $XD[[X]]$ (resp., $XD[X]$) is an $H(R, D)$ -module (resp., $h(R, D)$ -module).

We are now ready to study when composite Hurwitz rings $H(R, D)$ and $h(R, D)$ are Noetherian rings.

Theorem 2.2. *Let $R \subseteq D$ be an extension of commutative rings with identity. If $\text{char}(R) = 0$, then the following statements are equivalent.*

- (1) $H(R, D)$ (resp., $h(R, D)$) is a Noetherian ring.
- (2) R is a Noetherian ring and D is a finitely generated R -module containing $\mathbb{Q}$.
- (3) R is a Noetherian ring and $XD[[X]]$ (resp., $XD[X]$) is a Noetherian $H(R, D)$ -module (resp., $h(R, D)$ -module).

Proof. (1) $\Rightarrow$ (2) Suppose that $H(R, D)$ (resp., $h(R, D)$) is a Noetherian ring, and let p be any prime number. Since $(X, X^2, \dots)$ is finitely generated, there exists a positive integer n such that $X^{p^n} \in (X, X^2, \dots, X^{p^n-1})$; so we can find suitable elements $g_1, \dots, g_{p^n-1} \in H(R, D)$ (resp., $g_1, \dots, g_{p^n-1} \in h(R, D)$) such that $X^{p^n} = X * g_1 + \dots + X^{p^n-1} * g_{p^n-1}$. Comparing the coefficients of X^{p^n} in both sides, we get

$$1 = \binom{p^n}{1} b_1 + \dots + \binom{p^n}{p^n-1} b_{p^n-1}$$

for some $b_1, \dots, b_{p^n-1} \in D$. Note that p divides $\binom{p^n}{k}$ for all $k = 1, \dots, p^n - 1$ [4, Lemma 7.3]; so p is a unit in D . Since all the prime numbers are units in D , all the nonzero integers are also units in D . Therefore D contains $\mathbb{Q}$, and hence by Lemma 2.1, $R + XD[[X]]$ (resp., $R + XD[X]$) is a Noetherian ring. Thus R is a Noetherian ring and D is a finitely generated R -module [13, Theorem 4] (or [12, Proposition 2.1]) (resp., [12, Proposition 2.1] (or [14, Corollary 2.2])).

(2) $\Rightarrow$ (1) Assume that R is a Noetherian ring and D is a finitely generated R -module. Then $R + XD[[X]]$ (resp., $R + XD[X]$) is Noetherian [13, Theorem 4] (or [12, Proposition 2.1]) (resp., [12, Proposition 2.1] (or [14, Corollary 2.2])). Since D contains $\mathbb{Q}$, Lemma 2.1 forces $H(R, D)$ (resp., $h(R, D)$) to be a Noetherian ring.

(1) $\Leftrightarrow$ (3) We first show the composite Hurwitz series ring case. Let $u : R \rightarrow D$ be the natural injection and $v : H(D) \rightarrow D$ the canonical projection. Consider the following commutative diagram

$$\begin{array}{ccc} H(R, D) = R \times_D H(D) & \longrightarrow & R \\ \downarrow & & \downarrow u \\ H(D) & \xrightarrow{v} & D \simeq H(D)/XD[[X]]. \end{array}$$

Then $H(R, D)$ is the pullback of u and v . Thus the equivalence follows from [15, Proposition 4.10].

The proof for the composite Hurwitz polynomial ring case is the same as that for the composite Hurwitz series ring case. $\square$

When $R = D$ in Theorem 2.2, we obtain

Corollary 2.3 ([4, Corollary 7.7]). *Let R be a commutative ring with identity. If $\text{char}(R) = 0$, then the following assertions are equivalent.*

- (1) R is a Noetherian ring containing $\mathbb{Q}$.
- (2) $H(R)$ is a Noetherian ring.
- (3) $h(R)$ is a Noetherian ring.

We next show that in Theorem 2.2, the condition that $\text{char}(R) = 0$ is essential.

Theorem 2.4. *Let $R \subseteq D$ be an extension of commutative rings with identity and let E be either $H(R, D)$ or $h(R, D)$. If $\text{char}(R) > 0$, then E is never a Noetherian ring.*

Proof. Suppose on the contrary that E is a Noetherian ring. Then $(X, X^2, \dots)$ is a finitely generated ideal of E ; so there exists a positive integer q such that $(X, X^2, \dots) = (X, X^2, \dots, X^q)$. Let $\text{char}(R) = p_1^{k_1} \cdots p_m^{k_m}$, where $p_1, \dots, p_m$ are distinct prime numbers. Then we can take a positive integer n such that $p_i^n > q$ for all $i = 1, \dots, m$; so $X^{p_i^n} \in (X, X^2, \dots, X^{p_i^n-1})$ for all $i = 1, \dots, m$. Therefore for each $i = 1, \dots, m$, there exist suitable elements $g_{i1}, \dots, g_{i(p_i^n-1)} \in E$ such that $X^{p_i^n} = X * g_{i1} + \cdots + X^{p_i^n-1} * g_{i(p_i^n-1)}$. By comparing the coefficients of $X^{p_i^n}$ in both sides, we get

$$1 = \binom{p_i^n}{1} b_{i1} + \cdots + \binom{p_i^n}{p_i^n-1} b_{i(p_i^n-1)}$$

for some $b_{i1}, \dots, b_{i(p_i^n-1)} \in D$. Hence we obtain

$$1 = \prod_{i=1}^m \left(\binom{p_i^n}{1} b_{i1} + \cdots + \binom{p_i^n}{p_i^n-1} b_{i(p_i^n-1)} \right)^{k_i}.$$

Note that p_i divides $\binom{p_i^n}{j}$ for all $i = 1, \dots, m$ and all $j = 1, \dots, p_i^n - 1$ [4, Lemma 7.3]; so $\text{char}(R)$ divides 1 in D . Hence $1 = 0$ in D , which is a contradiction. Thus E is not a Noetherian ring. $\square$

3 S -Noetherian rings

Let R be a commutative ring with identity, S a (not necessarily saturated) multiplicative subset of R , and M a unitary R -module. Recall from [6, Definition 1] that an ideal I of R is S -finite if there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$; and R is an S -Noetherian ring if each ideal of R is S -finite. Also, we say that the R -module M is S -finite if $sM \subseteq F \subseteq M$ for some $s \in S$ and some finitely generated R -module F ; and M is S -Noetherian if each R -submodule of M is S -finite.

Our first result in this section gives a necessary condition for composite Hurwitz rings $H(R, D)$ and $h(R, D)$ to be S -Noetherian rings, where $R \subseteq D$ is an extension of commutative rings with identity and S is a multiplicative subset of R .

Proposition 3.1. Let $R \subseteq D$ be an extension of commutative rings with identity, S be a (not necessarily saturated) multiplicative subset of R , and E be either $H(R, D)$ or $h(R, D)$. If E is an S -Noetherian ring, then the following assertions hold.

- (1) S contains an element s divisible in D by all the prime numbers.
- (2) $\text{char}(R) = 0$.

Proof. (1) Suppose that E is an S -Noetherian ring. Then $(X, X^2, \dots)$ is S -finite; so there exist $s \in S$ and $f_1, \dots, f_m \in (X, X^2, \dots)$ such that $s * (X, X^2, \dots) \subseteq (f_1, \dots, f_m)$. Let p be any prime number. Since $f_1, \dots, f_m \in (X, X^2, \dots)$, we can find a positive integer n such that $s * (X, X^2, \dots) \subseteq (X, X^2, \dots, X^{p^n-1})$. Therefore $sX^{p^n} \in (X, X^2, \dots, X^{p^n-1})$, and hence we can write $sX^{p^n} = X * g_1 + \dots + X^{p^n-1} * g_{p^n-1}$ for some $g_1, \dots, g_{p^n-1} \in E$. Comparing the coefficients of X^{p^n} in both sides, we obtain

$$s = \binom{p^n}{1} d_1 + \dots + \binom{p^n}{p^n-1} d_{p^n-1}$$

for some $d_1, \dots, d_{p^n-1} \in D$. Note that p divides $\binom{p^n}{k}$ for all $k = 1, \dots, p^n - 1$ [4, Lemma 7.3]; so p divides s in D . Thus s is divisible in D by all the prime numbers.

(2) Suppose on the contrary that $\text{char}(R) \neq 0$, and let $\text{char}(R) = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $p_1, \dots, p_m$ are distinct prime numbers. Fix an $i \in \{1, \dots, m\}$. Since E is an S -Noetherian ring, by (1), there exist elements $s_i \in S$ and $d_i \in D$ such that $s_i = p_i d_i$. Since $\text{char}(R) = \text{char}(D)$, we get

$$s_1^{\alpha_1} \cdots s_m^{\alpha_m} = \text{char}(R) d_1^{\alpha_1} \cdots d_m^{\alpha_m} = 0,$$

which indicates that $0 \in S$. However this is absurd. Thus $\text{char}(R) = 0$. □

Let R be a commutative ring with identity and S a (not necessarily saturated) multiplicative subset of R . We say that S is *anti-Archimedean* if $(\bigcap_{n \geq 1} s^n R) \cap S \neq \emptyset$ for every $s \in S$. We also say that an integral domain R is an *anti-Archimedean domain* if $\bigcap_{n \geq 1} a^n R \neq 0$ for each $0 \neq a \in R$ (see [16]). Thus R is an anti-Archimedean domain if and only if $R \setminus \{0\}$ is an anti-Archimedean subset of R . Clearly, every multiplicative subset consisting of units is anti-Archimedean. Also, if V is a valuation domain with no height-one prime ideal (or equivalently, every nonzero prime ideal of V has infinite height), then $V \setminus \{0\}$ is an anti-Archimedean subset of V [16, Proposition 2.1].

We next characterize when a composite Hurwitz series ring $H(R, D)$ is an S -Noetherian ring under the assumption that S is an anti-Archimedean subset of R .

Theorem 3.2. Let $R \subseteq D$ be an extension of commutative rings with identity and $\text{char}(R) = 0$, and let S be an anti-Archimedean subset of R consisting of nonzerodivisors of D . If S contains an element divisible in D by all the positive integers, then the following statements are equivalent.

- (1) $H(R, D)$ is an S -Noetherian ring.
- (2) R is an S -Noetherian ring and $XD[[X]]$ is an S -Noetherian $H(R, D)$ -module.
- (3) R is an S -Noetherian ring and D is an S -finite R -module.

Proof. (1) $\Leftrightarrow$ (2) Consider the following commutative diagram

$$\begin{array}{ccc} H(R, D) = R \times_D H(D) & \longrightarrow & R \\ \downarrow & & \downarrow u \\ H(D) & \xrightarrow{v} & D \simeq H(D)/XD[[X]], \end{array}$$

where u is the natural injection and v is the canonical projection. Then $H(R, D)$ is the pullback of u and v . Thus the equivalence follows from [9, Proposition 2.3].

(1) $\Rightarrow$ (3) Suppose that $H(R, D)$ is an S -Noetherian ring. Since R is a homomorphic image of $H(R, D)$, R is an S -Noetherian ring [11, Lemma 2.2]. Note that $XD[[X]]$ is an ideal of $H(R, D)$; so there exist $s \in S$ and $f_1, \dots, f_n \in D[[X]]$ such that $s * XD[[X]] \subseteq (Xf_1, \dots, Xf_n)$. Therefore for any $d \in D$, we have

$$s * dX = Xf_1 * g_1 + \dots + Xf_n * g_n$$

for some $g_1, \dots, g_n \in H(R, D)$. Comparing the coefficients of X in both sides, we get

$$sd = f_1(0)g_1(0) + \dots + f_n(0)g_n(0),$$

where for each $i = 1, \dots, n$, $f_i(0)$ and $g_i(0)$ denote the constant terms of f_i and g_i , respectively. Note that $f_i(0) \in D$ and $g_i(0) \in R$ for all $i = 1, \dots, n$. Hence $sD \subseteq f_1(0)R + \dots + f_n(0)R$. Thus D is an S -finite R -module.

(3) $\Rightarrow$ (1) Suppose that R is an S -Noetherian ring and D is an S -finite R -module. Then D is an S -Noetherian R -module [10, Lemma 3.5(2)]; so D is an S -Noetherian ring. Since D is an S -finite R -module, there exist $s \in S$ and $d_1, \dots, d_m \in D$ such that $sD \subseteq d_1R + \dots + d_mR$; so we have

$$\begin{aligned} s * H(D) &\subseteq d_1 * H(R) + \dots + d_m * H(R) \\ &\subseteq d_1 * H(R, D) + \dots + d_m * H(R, D). \end{aligned}$$

Hence $H(D)$ is an S -finite $H(R, D)$ -module. Clearly, S is an anti-Archimedean subset of D . Note that $\text{char}(D) = \text{char}(R) = 0$; so by our assumption, $H(D)$ is an S -Noetherian ring [4, Theorem 9.6]. Since $H(D)$ is an S -finite $H(R, D)$ -module, $H(R, D)$ is an S -Noetherian ring [6, Corollary 7] (or [10, Lemma 3.5(3)]). $\square$

Recall that if R is an S -Noetherian ring with $\text{char}(R) = 0$ and S is an anti-Archimedean subset of R which contains an element divisible in R by all the positive integers, then $h(R)$ is also an S -Noetherian ring [4, Theorem 9.4]. By combining this result with a similar argument as in the proof of Theorem 3.2, we obtain equivalent conditions for a composite Hurwitz polynomial ring $h(R, D)$ to be an S -Noetherian ring.

Theorem 3.3. *Let $R \subseteq D$ be an extension of commutative rings with identity and $\text{char}(R) = 0$, and let S be an anti-Archimedean subset of R . If S contains an element divisible in D by all the positive integers, then the following statements are equivalent.*

- (1) $h(R, D)$ is an S -Noetherian ring.
- (2) R is an S -Noetherian ring and $XD[X]$ is an S -Noetherian $h(R, D)$ -module.
- (3) R is an S -Noetherian ring and D is an S -finite R -module.

Remark 3.4. *Let $R \subseteq D$ be an extension of commutative rings with identity and S a multiplicative subset of R . If R contains $\mathbb{Q}$, then all the integers are units in D ; so every element in S is divisible in D by all the positive integers. Hence if R contains $\mathbb{Q}$, then it follows from Lemma 2.1 that Theorems 3.2 and 3.3 are nothing but parts of [10, Theorems 4.4 and 3.6].*

By Proposition 3.1 and Theorems 3.2 and 3.3, we may ask the following question.

Question 3.5. *Do Theorems 3.2 and 3.3 still hold if S contains an element divisible in D by all the prime numbers?*

We end this section with an example satisfying the conditions in Theorems 3.2 and 3.3. More precisely, we construct an integral domain R , not containing $\mathbb{Q}$, with $\text{char}(R) = 0$ such that there exists an anti-Archimedean subset S of R containing an element divisible in R by all the positive integers.

Example 3.6. *Let $\mathbb{Z}$ be the ring of integers and G the weak direct sum of $\{Z_i\}_{i=1}^{\infty}$ which has the reverse lexicographic order, where $Z_i = \mathbb{Z}$ for all positive integers i . Let $\{X_i\}_{i=1}^{\infty} \cup \{Y_i\}_{i=1}^{\infty}$ be a set of indeterminates over $\mathbb{Q}$ and v be the valuation on $\mathbb{Q}(\{X_i\}_{i=1}^{\infty}, \{Y_i\}_{i=1}^{\infty})$ induced by the mapping $X_i \mapsto 0$ and $Y_i \mapsto e_i$, of $\{X_i\}_{i=1}^{\infty} \cup \{Y_i\}_{i=1}^{\infty}$ into G , where e_i is an element of G whose i -th component is 1 and j -th component is 0 for $j \neq i$.*

- (1) *Let V be the valuation ring of v and set $V^* = V \setminus \{0\}$. Then V is a valuation domain containing $\mathbb{Q}$ with no height-one prime ideals [17, page 254, Exercise 20] and V^* is an anti-Archimedean subset of V [16, Proposition 2.1]. Clearly, V is a V^* -Noetherian ring; so $V[[X]]$ (resp., $V[X]$) is a V^* -Noetherian ring [6, Proposition 10] (resp., [6, Proposition 9]). Since V contains $\mathbb{Q}$, $H(V)$ (resp., $h(V)$) is isomorphic to $V[[X]]$ (resp., $V[X]$) [4, Theorem 1.4] (resp., [4, Corollary 1.5]). Hence $H(V)$ (resp., $h(V)$) is a V^* -Noetherian ring.*

- (2) Let $R = \mathbb{Z} + M$, where M is the maximal ideal of V . Then R is an anti-Archimedean ring [16, Proposition 2.6] and each ideal of R is comparable with M under the set theoretic inclusion [17, page 202, Exercise 12]. For any integer n , $nR = n\mathbb{Z} + nM = n\mathbb{Z} + M$ contains M (since $\mathbb{Q}M = M$, $nM = M$). Hence every element in M is divisible in R by all the positive integers. Note that $H(R)$ (resp., $h(R)$) is not isomorphic to $R[[X]]$ (resp., $R[X]$) because R does not contain $\mathbb{Q}$. Since $R^* = R \setminus \{0\}$ is anti-Archimedean subset of R and R is R^* -Noetherian, it follows from [4, Theorem 9.6] (resp., [4, Theorem 9.4]) that $H(R)$ (resp., $h(R)$) is an R^* -Noetherian ring.
- (3) Let $R \subseteq D$ be an extension of integral domains, where R is defined in (2). If D is a finitely generated R -module, then D is an R^* -finite R -module. Hence $H(R, D)$ (resp., $h(R, D)$) is an R^* -Noetherian ring.

4 Présimplifiable rings

Let R be a commutative ring with identity, $U(R)$ the set of units of R , and $Z(R)$ the set of zero-divisors of R . Recall that R is *présimplifiable* if whenever $a, b \in R$ satisfy $ab = a$, either $a = 0$ or $b \in U(R)$. It was shown in [18] that R is *présimplifiable* if and only if $Z(R) \subseteq 1 + U(R)$. In [4], the authors studied when Hurwitz rings $H(R)$ and $h(R)$ are *présimplifiable*. In this section, we modify some properties of elements (units and nilpotent) of $H(R)$ and $h(R)$ in [4] to give equivalent conditions for composite Hurwitz series rings and composite Hurwitz polynomial rings to be *présimplifiable*.

Our first result in this section is a necessary and sufficient condition for a composite Hurwitz series ring $H(R, D)$ to be *présimplifiable*, where $R \subseteq D$ is an extension of commutative rings with identity.

Proposition 4.1. *Let $R \subseteq D$ be an extension of commutative rings with identity. Then $H(R, D)$ is *présimplifiable* if and only if $Z(D) \cap R \subseteq 1 + U(R)$.*

Proof. ($\Rightarrow$) Let $r \in Z(D) \cap R$. Since $H(R, D)$ is *présimplifiable*, we obtain

$$r \in Z(H(R, D)) \subseteq 1 + U(H(R, D)).$$

Thus $r \in 1 + U(R)$ [19, Lemma 2.2(1)].

($\Leftarrow$) Let $f = \sum_{i=0}^{\infty} a_i X^i \in Z(H(R, D))$. Then by the assumption, we obtain

$$a_0 \in Z(R) \subseteq Z(D) \cap R \subseteq 1 + U(R).$$

Therefore $f - 1 \in U(H(R, D))$ [19, Lemma 2.2(1)], and hence $f \in 1 + U(H(R, D))$. Thus $H(R, D)$ is *présimplifiable*. $\square$

Remark 4.2. *For an extension $R \subseteq D$ of commutative rings with identity, $Z(R) \subseteq Z(D) \cap R$. Hence if $H(R, D)$ is *présimplifiable*, then so is $H(R)$. However, the converse does not hold in general. Consider $R = \mathbb{Z} \subseteq D = \mathbb{Z}[Y]/(3Y)$. Then clearly $H(R)$ is *présimplifiable* but $H(R, D)$ is not *présimplifiable* because $3 \in Z(H(R, D))$ and $3 \notin 1 + U(H(R, D))$.*

We next study when a composite Hurwitz polynomial ring $h(R, D)$ is *présimplifiable*, where $R \subseteq D$ is an extension of commutative rings with identity. To do this, we need two lemmas.

Lemma 4.3. *Let $R \subseteq D$ be an extension of commutative rings with identity and $f = \sum_{i=0}^n a_i X^i \in h(R, D)$. Then the following assertions hold.*

- (1) *f is nilpotent if and only if a_0 is nilpotent and for each $i = 1, \dots, n$, a_i is nilpotent or some power of a_i is with torsion.*
- (2) *f is a unit if and only if a_0 is a unit in R and for each $i = 1, \dots, n$, a_i is nilpotent or some power of a_i is with torsion.*

Proof. (1) The proof is identical to that of [4, Theorem 2.4].

(2) $(\Rightarrow)$ Assume that f is a unit in $h(R, D)$. Then we can find an element $g = \sum_{i=0}^m b_i X^i \in h(R, D)$ such that $f * g = 1$; so $a_0 b_0 = 1$. Hence a_0 is a unit in R . Since $h(R, D) \subseteq h(D)$, f is a unit in $h(D)$; so for each $i = 1, \dots, n$, a_i is nilpotent or some power of a_i is with torsion [4, Theorem 3.1].

($\Leftarrow$) Assume that for each $i = 1, \dots, n$, a_i is nilpotent or some power of a_i is with torsion. Then by (1), $\sum_{i=1}^n a_i X^i$ is nilpotent in $h(R, D)$. Since a_0 is a unit in $h(R, D)$, f is a unit in $h(R, D)$. $\square$

Lemma 4.4. *Let $R \subseteq D$ be an extension of commutative rings with identity. Then the following assertions are equivalent.*

(1) *For each $f \in Z(h(R, D))$, there exists an element $d \in D \setminus \{0\}$ such that $d * f = 0$.*

(2) *D is a torsion-free $\mathbb{Z}$ -module.*

Proof. (1) $\Rightarrow$ (2) Let a be any nonzero element in D . If there exists a positive integer n such that $na = 0$, then $0 = naX^n = aX * X^{n-1}$; so X^{n-1} is a zero-divisor of $h(R, D)$. By the assumption, we can find an element $d \in D \setminus \{0\}$ such that $dX^{n-1} = 0$, which is absurd. Thus D is a torsion-free $\mathbb{Z}$ -module.

(2) $\Rightarrow$ (1) Since $h(R, D) \subseteq h(D)$, the implication follows directly from [4, Theorem 4.1]. $\square$

Let R be a commutative ring with identity. Recall that R is a *domainlike ring* if every zero-divisor of R is nilpotent. It is easy to see that R is domainlike if and only if (0) is primary.

Proposition 4.5. *Let $R \subseteq D$ be an extension of commutative rings with identity. If D is a torsion-free $\mathbb{Z}$ -module, then $h(R, D)$ is présimplifiable if and only if D is a domainlike ring.*

Proof. ($\Rightarrow$) Let $d \in Z(D)$. Since $h(R, D)$ is présimplifiable, we have

$$dX \in Z(h(R, D)) \subseteq 1 + U(h(R, D));$$

so $-1 + dX \in U(h(R, D))$. Since D is a torsion-free $\mathbb{Z}$ -module, d is nilpotent by Lemma 4.3(2). Thus D is a domainlike ring.

($\Leftarrow$) Let $f = \sum_{i=0}^n a_i X^i \in Z(h(R, D))$. Since D is a torsion-free $\mathbb{Z}$ -module, by Lemma 4.4, there exists an element $d \in D \setminus \{0\}$ such that $d * f = 0$. Hence for all $i = 0, \dots, n$, a_i is a zero-divisor of D . By the assumption, a_i is nilpotent for all $i = 0, \dots, n$; so by Lemma 4.3(1), f is nilpotent. Therefore $f - 1$ is a unit in $h(R, D)$, and hence $f \in 1 + U(h(R, D))$. Thus $h(R, D)$ is présimplifiable. $\square$

We next show that in Proposition 4.5, the condition that D is a torsion-free $\mathbb{Z}$ -module is essential.

Proposition 4.6. *Let $R \subseteq D$ be an extension of commutative rings with identity and $\text{char}(D) = 0$. If D is not a torsion-free $\mathbb{Z}$ -module, then $h(R, D)$ is never présimplifiable.*

Proof. Assume that D is not a torsion-free $\mathbb{Z}$ -module. Then we can find an element $d \in D \setminus \{0\}$ and a positive integer n such that $nd = 0$; so $X * dX^{n-1} = ndX^n = 0$. Hence $X \in Z(h(R, D))$. If $h(R, D)$ is présimplifiable, then $Z(h(R, D)) \subseteq 1 + U(h(R, D))$; so $-1 + X$ is a unit in $h(R, D)$. Hence by Lemma 4.3(2), 1 is with torsion. However, this is impossible because $\text{char}(D) = 0$. Thus $h(R, D)$ is not présimplifiable. $\square$

Let $R \subseteq D$ be an extension of commutative rings with identity. Then it follows directly from Lemma 4.3(2) that if $\text{char}(R) > 0$, then $\sum_{i=0}^n a_i X^i \in h(R, D)$ is a unit if and only if a_0 is a unit in R . Hence a similar argument as in the proof of Proposition 4.1 shows the following result.

Proposition 4.7. *Let $R \subseteq D$ be an extension of commutative rings with identity. If $\text{char}(D) > 0$, then $h(R, D)$ is présimplifiable if and only if $Z(D) \cap R \subseteq 1 + U(R)$.*

5 Rings satisfying ascending chain condition on principal ideals

Let R be a commutative ring with identity. We say that R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist a strict ascending chain of principal ideals of R . It was shown in [19, Theorem 2.4] that if $R \subseteq D$ is an extension of integral domains with $\text{char}(D) = 0$, then $H(R, D)$ satisfies ACCP if and only if $h(R, D)$ satisfies ACCP, if and only if $\bigcap_{n \geq 1} r_1 \cdots r_n D = (0)$ for each infinite sequence $(r_n)_{n \geq 1}$ consisting of nonzero nonunits of R . In this section, we study an equivalent condition for $H(R, D)$ and $h(R, D)$ to satisfy ACCP, where $R \subseteq D$ is an extension of présimplifiable rings with identity.

Theorem 5.1. *Let $R \subseteq D$ be an extension of commutative rings with identity. If D is a présimplifiable ring, then the following statements are equivalent.*

- (1) $H(R, D)$ satisfies ACCP.
- (2) $U(D) \cap R = U(R)$ and for each sequence $(d_n)_{n \geq 1}$ of D with the property that for each $n \geq 1$, there exists an element $r_n \in R$ such that $d_n = d_{n+1}r_n$, $d_1 D \subseteq d_2 D \subseteq \cdots$ is stationary.

Proof. (1) $\Rightarrow$ (2) Let $u \in U(D) \cap R$. Then $\frac{1}{u}X * H(R, D) \subseteq \frac{1}{u^2}X * H(R, D) \subseteq \cdots$ is an ascending chain of principal ideals of $H(R, D)$. Since $H(R, D)$ satisfies ACCP, there exists a positive integer m such that $\frac{1}{u^{m+1}}X * H(R, D) = \frac{1}{u^m}X * H(R, D)$. Hence $u \in U(R)$. Clearly, $U(R) \subseteq U(D) \cap R$, which shows that $U(D) \cap R = U(R)$.

Let $(d_n)_{n \geq 1}$ be a sequence of D with the property that for each $n \geq 1$, there exists an element $r_n \in R$ such that $d_n = d_{n+1}r_n$. Then $d_1X * H(R, D) \subseteq d_2X * H(R, D) \subseteq \cdots$ is an ascending chain of principal ideals of $H(R, D)$. Since $H(R, D)$ satisfies ACCP, the chain $d_1X * H(R, D) \subseteq d_2X * H(R, D) \subseteq \cdots$ is stationary; so we can find a positive integer m such that $d_nX * H(R, D) = d_mX * H(R, D)$ for all $n \geq m$. Hence $d_n D = d_m D$ for all $n \geq m$. Thus the chain $d_1 D \subseteq d_2 D \subseteq \cdots$ stops.

(2) $\Rightarrow$ (1) Let $f_1 * H(R, D) \subseteq f_2 * H(R, D) \subseteq \cdots$ be an ascending chain of nonzero principal ideals of $H(R, D)$. Then for each $n \geq 1$, $f_n = f_{n+1} * g_n$ for some $g_n \in H(R, D)$. If f_n is a unit for some $n \geq 1$, then there is nothing to prove; so we assume that f_n is a nonunit for all $n \geq 1$. For each $n \geq 1$, write $f_n = \sum_{m=k_n}^{\infty} a_{nm}X^m$ and $g_n = \sum_{m=0}^{\infty} b_{nm}X^m$, where $a_{nk_n} \neq 0$. Since f_n is a multiple of f_{n+1} , $k_1 \geq k_2 \geq \cdots \geq 0$; so there exists a positive integer q such that $k_n = k_q$ for all $n \geq q$. Hence $a_{nk_n} = a_{n+1k_{n+1}}b_{n0}$ for all $n \geq q$. By the assumption, the chain $a_qk_q D \subseteq a_{q+1k_{q+1}} D \subseteq \cdots$ is stationary; so we can find an integer $p \geq q$ such that $a_mk_m D = a_pk_p D$ for all $m \geq p$. Therefore for each $n \geq p$, there exists an element $d_n \in D$ such that $a_{n+1k_{n+1}} = a_{nk_n}d_n$. Hence $a_{nk_n} = a_{nk_n}d_nb_{n0}$ for all $n \geq p$. Since D is présimplifiable and $a_{nk_n} \neq 0$, d_nb_{n0} is a unit in D , which indicates that $b_{n0} \in U(D) \cap R = U(R)$. Hence g_n is a unit in $H(R, D)$ [19, Lemma 2.2(1)], which shows that $f_n * H(R, D) = f_p * H(R, D)$ for all $n \geq p$. Thus $H(R, D)$ satisfies ACCP. $\square$

Let $R \subseteq D$ be an extension of commutative rings with identity. Note that by Lemma 4.3(2), if $\text{char}(R) > 0$, then $\sum_{i=0}^n a_i X^i \in h(R, D)$ is a unit if and only if a_0 is a unit in R . Hence a similar argument as in the proof of Theorem 5.1 shows the following result.

Theorem 5.2. *Let $R \subseteq D$ be an extension of commutative rings with identity. If D is a présimplifiable ring with $\text{char}(D) > 0$, then the following statements are equivalent.*

- (1) $h(R, D)$ satisfies ACCP.
- (2) $U(D) \cap R = U(R)$ and for each sequence $(d_n)_{n \geq 1}$ of D with the property that for each $n \geq 1$, there exists an element $r_n \in R$ such that $d_n = d_{n+1}r_n$, $d_1 D \subseteq d_2 D \subseteq \cdots$ is stationary.

When $R = D$ in Theorems 5.1 and 5.2, we obtain

Corollary 5.3. *Let R be a présimplifiable ring with identity. Then the following assertions hold.*

- (1) R satisfies ACCP if and only if $H(R)$ satisfies ACCP.
- (2) If $\text{char}(R) > 0$, then R satisfies ACCP if and only if $h(R)$ satisfies ACCP.

We are closing this paper with an example which shows that if a ring has characteristic zero, then ACCP property does not ascend into the Hurwitz polynomial ring extension.

Example 5.4. Let K be a field with $\text{char}(K) = 0$, $\{A_n\}_{n=1}^\infty$ a set of indeterminates over K , and set $S = K[\{A_n\}_{n=1}^\infty]/(\{A_{n+1}(A_n - A_{n+1})\}_{n=1}^\infty)$. Let a_n be the image of A_n in S and R be the localization of S at the ideal $(a_1, a_2, \dots)S$.

- (1) R is a *présimplifiable* ring which satisfies ACCP ([12, Remark 4.17] and [20, Example]).
- (2) Note that for all $n \geq 1$, $a_n X + 1 = (a_{n+1} X + 1) * ((a_n - a_{n+1})X + 1)$; so $(a_1 X + 1) * h(R) \subseteq (a_2 X + 1) * h(R) \subseteq \dots$ is an ascending chain of principal ideals of $h(R)$. Suppose on the contrary that the chain stops. Then there exists a positive integer m such that $a_{m+1} X + 1 = (a_m X + 1) * f$ for some $f \in h(R)$. Now, an easy calculation shows that $f = \sum_{n=0}^\infty b_n X^n$, where $b_0 = 1$ and $b_n = (-1)^{n+1} n! a_m^{n-1} (a_{m+1} - a_m)$ for all $n \geq 1$. Since $\text{char}(K) = 0$, $b_n \neq 0$ for all nonnegative integers n [20, Example]. Hence $f \notin h(R)$, which is absurd. Thus $h(R)$ does not satisfy ACCP.

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