Open Math. 2017; 15: 1090–1098 DE GRUYTER OPEN

Open Mathematics

Research Article

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OD-characterization of alternating groups A_{p+d}

https://doi.org/10.1515/math-2017-0092
Received November 2, 2016; accepted June 20, 2017.

Abstract: Let A_n be an alternating group of degree n. Some authors have proved that A_{10}, A_{147} and A_{189} cannot be OD-characterizable. On the other hand, others have shown that A_{16}, A_{23+4}, and A_{23+5} are OD-characterizable. We will prove that the alternating groups A_{p+d} except A_{10}, are OD-characterizable, where p is a prime and d is a prime or equals to 4. This result generalizes other results.

Keywords: Order component, Element order, Alternating group, Degree pattern, Prime graph, Simple group

MSC: 20D05, 20D06, 20D60

1 Introduction

All groups under consideration are finite and for a simple group we mean a non-abelian simple group. For a given group G, the socle of G is defined as the subgroups generated by the minimal normal subgroups of G, denoted by $\mathrm{Soc}(G)$. $\pi(G)$ is the set of prime divisors of $|G|$. $\mathrm{Syl}_p(G)$ is the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, $\mathrm{Syl}_r(G)$ is the set of the Sylow r-subgroups of G for $r \in \pi(G)$. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ denote the automorphism and outer-automorphism of G, respectively. S_n and A_n denote the symmetric and alternating groups of degree n, respectively. $\omega(G)$ is the set of orders of its elements of G. Associated to $\omega(G)$ a graph is named prime graph of G, which is written by $GK(G)$. The vertex set of $GK(G)$ is $\pi(G)$, and two distinct vertices p, q are joined by an edge if $p \cdot q \in \omega(G)$ which is denoted by $p \sim q$. We use $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. If $n = p^a m$ with $(p, m) = 1$, then $n_p = p^a$. The other symbols are standard (see [1], for instance).

Definition 1.1 ([2])**.** *Let G be a finite group and* $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$*, where* p_i*s are different primes and* α_i*s are positive integers. For* $p \in \pi(G)$*, let* $\deg(p) := |\{q \in \pi(G) | p \sim q\}|$*, which we call the degree of* p*. We also define* $D(G) := (\deg(p_1), \deg(p_2), \cdots, \deg(p_k))$*, where* $p_1 < p_2 < \cdots < p_k$*. We call* $D(G)$ *the degree pattern of* G*.*

Given a finite group M, denote by $h_{OD}(M)$ the number of isomorphism classes of finite groups G such that (1) $|G| = |M|$ and (2) $D(G) = D(M)$.

Definition 1.2 ([2])**.** *A finite group* M *is called* k*-fold OD-characterizable if* $h_{OD}(M) = k$*. Moreover, a 1-fold* OD*-characterizable group is simply called an* OD*-characterizable group.*

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For the groups which are k-fold OD-characterizable, please see Tables 1 and 2 and corresponding references of [3]. Many finite groups are k-fold OD-characterizable.

Some authors have proved that some alternating groups with $s(G) \geq 2$ are OD-characterizable.

Proposition 1.3. *The alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime, are OD-characterizable [4].*

But if $s(G) = 1$, then what is happening to the influence of OD on the structure of groups? Recently the following results have been shown.

Proposition 1.4. *Let A_n be alternating group of degree n. Then the following statements hold.*
(1) A_{p+3}, where p is a prime and $7 \neq p$, are OD-characterizable [5-7].
(2) A_{10} is 2-fold OD-characterizable [8].
(3) A_n with that either $5 \leq n \leq 100$ and $n \neq 10$ or $n = 106, 112$, is OD-characterizable [5,7,9-12].
(4) A_{p+5} with $p \leq 1000$ a prime is OD-characterizable [13].
(5) A_{147} and A_{189} are 7-fold and 14-fold OD-characterizable, respectively [14].
(6) Assume that p is a prime satisfying the following three conditions:
(6a) $p \neq 139$ and $p \neq 181$,
(6b) $\pi((p+8)!) = \pi(p!)$,
(6c) $p \leq 997$.
Then A_{p+8} is OD-characterizable [14].

Form Proposition 1.4(2)(3), some alternating groups cannot be OD-characterizable. So naturally, we will ask this question: which of the alternating groups is OD-characterizable? Thus in this paper, we give a new characterization of alternating group A_{p+d} by OD. In fact, we will prove the following result.

Main theorem 1.5. *Let p be a prime and d positive. If d is equal to 2 or 4 or d is a prime, then alternating groups A_{p+d} except A_{10} are OD-characterizable.*

This result confirms the conjecture [15, Conjecture 5.1] put forward by of Yan et al.

2 Some lemmas

In this section, we will give some results which will be used.

Lemma 2.1. *Let A_n (or S_n) be an alternating (or a symmetric group) of degree n. Then the following hold.*
(1) Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.
(2) Let $p \in \pi(A_n)$ be odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.
(3) Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.

Proof. It is easy to get from [16]. □

By [1], we have that $|A_n| = n!/2$ and $|S_n| = n!$.

Let $\exp(n, r) = a$ or $r^a \| n$, where a is a positive integer satisfying $r^a \big| n$ but $r^{a+1} \nmid n$.

Lemma 2.2. *Let A_{p+k} be an alternating group of degree $p + k$ such that $p > k \geq 3$ and $p + i$ is composite, $i = 3, \cdots, k$, where p is a prime. Let $|\pi(A_{p+k})| = d$. Then the following hold.*
(1) $\deg(2) = \deg(3) = d - 1$ if $k \geq 4$; $\deg(2) = d - 2$ and $\deg(3) = d - 1$ if $k = 3$.
(2) $\deg(p) = |\pi(k!)|$ if $k \geq 4$; $\deg(p) = |\pi(k!)| - 1$ if $k = 3$.
(3) $\exp(|A_{p+k}|, 2) = \sum\limits_{i=1}^{\infty} [\frac{p+k}{2^i}] - 1$. In particular, $\exp(|A_{p+k}|, 2) \leq p + k - 1$.

(4) $\exp(|A_{p+k}|, r) = \sum_{i=1}^{\infty}[\frac{p+k}{r^i}]$ *for each* $r \in \pi(A_{p+k})\backslash\{2\}$*. Furthermore,* $\exp(|A_{p+k}|, r) < \frac{p+k}{2}$*, where* $3 \leq r \in \pi(A_{p+k})$*. In particular, if* $r > [\frac{p+k}{2}]$*, then* $\exp(|A_{p+k}|, r) = 1$.

Proof. (1) By Lemma 2.1, we have that $2 \sim p$ and $3 \sim p$. Hence $\deg(2) = d - 1 = \deg(3)$.

(2) For $r \in \pi(A_{p+k})\backslash\{2, p\}$, by Lemma 2.1, it is easy to see that $r \sim p$ if and only if $p+r \leq p+k$. It follows that $r \leq k$ and $r \in \pi(k!)$. Hence $\deg(p) = |\pi(k!)|$.

(3) By the definition of Gaussian integer function, we have that

$$\begin{aligned}\exp(|A_{p+k}|, 2) &= \sum_{i=1}^{\infty}[\frac{p+k}{2^i}] - 1 \\ &= ([\frac{p+k}{2}] + [\frac{p+k}{2^2}] + [\frac{p+k}{2^3}] + \cdots) - 1 \\ &\leq (\frac{p+k}{2} + \frac{p+k}{2^2} + \frac{p+k}{2^3} + \cdots) - 1 \\ &= (p+k)(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots) - 1 \\ &= p + k - 1.\end{aligned}$$

(4) By the same as in proof of (3), we have that

$$\begin{aligned}\exp(|A_{p+k}|, r) &\leq (p+k)(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots) \\ &= \frac{p+k}{r-1} \\ &\leq \frac{p+k}{2}\end{aligned}$$

for an odd prime $r \in \pi(A_{p+k})$. If $r > [\frac{p+k}{2}]$, $\exp(|A_{p+k}|, r) = 1$.

The proof is completed. □

Lemma 2.3. *Let L be a nonabelian simple group. Then the order and its outer-automorphism of L are as listed in Tables 1–3.*

Proof. See [17]. □

Lemma 2.4. *If* $n \geq 6$ *is a natural number, then there are at least* $s(n)$ *prime numbers* p_i *such that* $\frac{n+1}{2} < p_i < n$*. Here*

- $s(n) = 6$ *for* $n \geq 48$*;*
- $s(n) = 5$ *for* $42 \leq n \leq 47$*;*
- $s(n) = 4$ *for* $38 \leq n \leq 41$*;*
- $s(n) = 3$ *for* $18 \leq n \leq 37$*;*
- $s(n) = 2$ *for* $14 \leq n \leq 17$*;*
- $s(n) = 1$ *for* $6 \leq n \leq 13$*.*

In particular, for every natural number $n > 6$*, there exists a prime* p *such that* $\frac{n+1}{2} < p < n-1$*, and for every natural number* $n > 3$*, there exists an odd prime number* p *such that* $n - p < p < n$*.*

Proof. See Lemma 1 of [18]. □

Lemma 2.5. *Let G be a finite non-abelian simple group and p is the largest prime divisor of* $|G|$ *with* $p \| |G|$*. Then* $p \big| |\mathrm{Out}(G)|$*.*

Proof. We know that if a prime p divides the order of outer-automorphism of a group G, then the Sylow p-subgroups of G are noncyclic (see [19, Corollary 11.21]). □

Lemma 2.6. *Let G be a simple group of Lie type in characteristic p. Assume that $p^k \mid |G|$ but $p^{k+1} \nmid |G|$ where k is a positive integer, then $|G| < p^{3k}$, and if $G \ncong L_2(q)$ with $q = p^k$, then $|G| < p^{\frac{8}{3}k}$.*

Proof. It is easy to get from Tables 1 and 2 of Lemma 2.3. □

3 Proof of the main theorem

In this section, we give the proof of main theorem.

For easy reading, we rewrite the main theorem.

Theorem 3.1. *Let p be a prime. Assume that d is a positive integer such that either d is a prime if d is odd or d is equal to at most 4 if d is even. Then alternating groups A_{p+d} except A_{10} are OD-characterizable.*

Proof. By Proposition 1.4(1), we can assume that $d \geq 4$. Propositions 1.3 and 1.4(3)(5) implies that if $p + d$ is a prime, then the alternating group is OD-characterizable, hence we always consider the case: $\pi((p+d)!) = \pi(p!)$ and $p \geq 113$. We have divided the proof of theorem into a series of lemmas. Note that p is always the largest prime divisor of $|G|$.

Step 1: G is insoluble.

Proof. Since $n \geq 118$ and $|G| = \frac{n!}{2}$, then by Lemmas 2.2 and 2.4 there exists a prime p_1 such that $|G|_{p_1} = p_1$, and $p_1 \nmid (p-1)$. Assume that G is soluble, then there is a subgroup H of order $p_1 p$. Since $p_1 \nmid (p-1)$, then H is cyclic and so G has an element of order $p_1 p$. It follows that $\deg(p) > |\pi(d!)|$, a contradiction to Lemma 2.2. Therefore G is insoluble. □

We use the notation of [20]. Let $|A_n| = \frac{n!}{2} = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$ where $\alpha_1 = \sum_{s=1}^{\infty}[\frac{n}{2^s}] - 1$ and $\alpha_i = \sum_{s=1}^{\infty}[\frac{n}{q_i^s}]$, $2 \leq i \leq k$ and the q_i are prime numbers. If $\frac{n}{2} < p$ and $n \geq 113$, then by Lemma 2.4, $\alpha_i = 1$ for $i \in \{k-5, k-4, k-3, k-2, k-1, k\}$. For convenience, we set $t_n(1) = \prod_{\frac{n}{2} < p_1 \leq n} p_1$, $t_n(2) = \prod_{\frac{n}{3} < p_2 < \frac{n}{2}} p_2^2 t_n(1), \cdots$, and $t_n(k) = \prod_{i=1}^{k}(\prod_{\frac{n}{i+1} < p_i < \frac{n}{i}} p_i)^i$ where $p_j = 1$ if there is no prime between $\frac{n}{j+1}$ and $\frac{n}{j}$.

Step 2: There exists a normal series $1 < K < H < G$ such that $S \lesssim G/K \lesssim \mathrm{Aut}(S)$ where $S \cong H/K$.

Let $\overline{G} = G/K$ and $S = \mathrm{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where the B_i are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. In the following, we will prove that $m = 1$.

Assume that $m \geq 2$. Then there is a prime $p \mid t_n(1)$ such that $p \nmid |S|$. Otherwise, by Lemma 2.4 there is a prime $p' \mid t_n(1)$ with that $pp' \in \omega(G)$, in contradiction to Lemma 2.2 (since $p' \nmid (p-1)$ and $p' \notin \pi(K)$, then $p' \in \pi(C_{G/K}(SK/K))$). Thus for every i, $B_i \in \mathfrak{F}_{p'}$, where $p' < p$ is the second maximal prime of $\pi(G)$ and $\mathfrak{F}_{p'}$ is the set of non-abelian finite simple group S such that $p' \in \pi(S) \subseteqq \{2, 3, 5, \cdots, p'\}$.

We shall prove that $p \nmid |K|$. Otherwise, by NC theorem, $N_G(K_p)/C_G(K_p)$ is isomorphic to a subgroup of $\mathrm{Aut}(K_p)$ where K_p is a Sylow p-subgroup of K and also a Sylow p-subgroup of G. Then there is a prime p' that $|G|_{p'} = p'$ divides $|K|$. It follows that there exists an element of order pp' contradicting to $\deg(p) = |\pi(d!)|$ by Lemma 2.2. Hence $p \big| |\mathrm{Out}(S)|$. We know that

$$\mathrm{Out}(S) = \mathrm{Out}(S_1) \times \mathrm{Out}(S_2) \times \cdots \times \mathrm{Out}(S_r),$$

where the groups $S_j\,(j = 1, 2, \cdots, r)$ are direct products of all isomorphic B_i's such that

$$S = S_1 \times S_2 \times \cdots \times S_r.$$

Therefore for certain j, p divides the order of an outer-automorphism of a direct product S_j of t isomorphic simple groups B_i for some $1 \leq j \leq m$. Since $B_i \in \mathfrak{F}_p$, it follows from Lemma 2.5, that $p \nmid |\mathrm{Out}(B_i)|$. But by [21], $|\mathrm{Aut}(S)| = |\mathrm{Aut}(S_j)|^t \cdot t!$. Thus $t \geq p$. Since B_j is non-abelian simple group, then $4^p \big| |\mathrm{Aut}(B_i)|^t$, and hence, $2^{2p} \big| |G|$ which contradicts Lemma 2.2. Hence $m = 1$ and $S = B_i$.

Let $|\pi(n)|$ be the number of primes $\leq n$.

Step 3: $t_n(1)\big||S|$. In particular, if $n \geq 118$, then $t_n(9)\big||S|$.

First we show that $t_n(1)$ divides the order $|H/K|$ of H/K. We know that $|G_p| = p$, and so we can assume that p divides H/K and $p \nmid |K|$. Then H/K is nonabelian. If not, H/K is a group of order p. By NC theorem, $\frac{G/K}{C_{G/K}(H/K)} \cong$ a subgroup of $Z_{p-1}$where Z_n is a cyclic group of order n. As $n \geq 118$, there exists another prime $p'(\neq p)$ dividing $t_n(1)$. Since $p' \nmid p-1$ and $p' \notin \pi(K)$, $p' \in \pi(C_{G/K}(H/K))$ and so $pp' \in \omega(G)$ contradicting to $\deg(p) = |\pi(q)|$. Thus H/K is non-abelian. If there is a prime $p' \notin \pi(H/K)$ dividing $t_n(1)$, then $p' \in \pi(G/H)$. By Frattini's argument, $G = N_G(P)H$ where $P \in \mathrm{Syl}_p(H)$. Now we have $\frac{N_G(P)}{C_G(P)} \cong$ a subgroup of Z_{p-1} and so $p' \in \pi(C_G(P))$, whence $pp' \in \omega(G)$ a contradiction since $\deg(p) = |\pi(d!)|$. Therefore, $t_n(1)$ divides the order $|H/K|$ of H/K.

Secondly, we prove $t_n(9)\big||S|$. If n is ≥ 118, then there are two primes q_1, q_2 such that $\frac{9}{10}n < q_2 < q_1 \leq n$. Indeed, using Theorem 1 of [22],

$$\frac{x}{\log x}(1+\frac{1}{2\log x}) < |\pi(x)| < \frac{x}{\log x}(1+\frac{3}{2\log x}).$$

Hence we have

$$|\pi(n)| - |\pi(\frac{9}{10}n)| > \frac{n}{\log n}(1+\frac{1}{2\log n}) - \frac{\frac{9}{10}n}{\log\frac{9}{10}n}(1+\frac{3}{2\log\frac{9}{10}n}).$$

By direct computation, $|\pi(n)| - |\pi(\frac{9}{10}n)| > 1$ when $n \geq 18000$. While $118 \leq n < 18000$, we can check it directly by [23].

Let $\frac{n}{10} < p_1 \leq \frac{n}{2}$ where p_1 is a prime. If $p_1\big||S|$, then $p_1\big||K|$ or $p_1\big||G/H|$. If $p_1\big||G/H|$, then G has an element of order $p_1 p$, a contradiction according to the above arguments. If $p_1\big||K|$, then $p_1^j\big||K|$ for $j \in \{2, 3, 4, 5, 6, 7, 8, 9\}$. We conclude that there is a subgroup M of order pp_i for $i \leq 9$. Obviously, G has no element of order pp_i as $\deg(p) = |\pi(d!)|$ and by Frattini's argument. By Theorem 9.3.1 of [24], $p_i \mid (p-1)$ for $i \leq 9$. Then $p_1 p_2 \cdots p_9 \mid (p-1)$. But $\frac{n}{2} \geq q_1 > q_2 > \cdots > q_9 > \frac{n}{10} > \frac{p}{10}$ and hence, $\prod_{i=1}^{9} q_i > p$ by direct check when $p \geq 113$, a contradiction. Therefore $t_n(9)$ divides the order $|S|$ of S.

Step 4: If $n \geq 118$, then $t_n(9) > 7.82^n$.

If $n \geq 118$, then by Theorem 1 of [22], $|\pi(\frac{n}{i})| > \frac{\frac{n}{i}}{\log\frac{n}{i}}(1+\frac{1}{2\log\frac{n}{i}})$ for $i \in \{1, 2, \cdots, 9\}$ and by Corollary 2 of [22], $|\pi(n)| < \frac{5n}{4\log n}$. Thus for $n \geq 15$

$$\begin{aligned}
\log t_n(9) &= \sum_{\frac{n}{2}<p_2\leq n} \log p_1 + 2\sum_{\frac{n}{3}<p_9\leq\frac{n}{2}} \log p_2 + \cdots + 9\sum_{\frac{n}{10}<p_1\leq\frac{n}{9}} \log p_9 \\
&> (|\pi(n)| - |\pi(\frac{n}{2})|)\log\frac{n}{2} + 2(|\pi(\frac{n}{2}|) - |\pi(\frac{n}{3})|)\log\frac{n}{3} + \cdots + 9(|\pi(\frac{n}{9})| - |\pi(\frac{n}{10})|)\log\frac{n}{10} \\
&= |\pi(n)|\log\frac{n}{2} + |\pi(\frac{n}{2})|(2\log\frac{n}{3} - \log\frac{n}{2}) + \cdots + |\pi(\frac{n}{9})|(6\log\frac{n}{7} - 5\log\frac{n}{6}) - 6|\pi(\frac{n}{7})|\log(\frac{n}{7}) \\
&> \frac{n}{\log n}(1+\frac{1}{2\log n})\log(\frac{n}{2}) + \frac{\frac{n}{2}}{\log\frac{n}{2}}(1+\frac{1}{2\log\frac{n}{2}})\log(\frac{2n}{3^2}) \\
&\quad + \frac{\frac{n}{9}}{\log\frac{n}{9}}(1+\frac{1}{2\log\frac{n}{9}})\log\frac{9^8 n}{10^9} - 9\cdot 1.25\cdot\frac{n}{10} \\
&= \sum_{i=1}^{9} \frac{\frac{n}{i}}{\log\frac{n}{i}}(1+\frac{1}{2\log\frac{n}{i}})\log\frac{i^{i-1}n}{(i+1)^i} - 1.25\cdot\frac{9n}{10} \\
&= 18.93167097 - 16.875 > 2.05667n
\end{aligned}$$

So we conclude that $t_n(9) > e^{2.05667n} > (7.82)^n$.

Step 5: S is isomorphic to A_n with $n = p, p+1, p+2, p+3, \cdots, p+d$.

According to Classification Theorem of Finite Simple Groups and Step 2, S is isomorphic to a sporadic simple group, a simple group of Lie type or an alternating group. If S is isomorphic to a sporadic simple group and $p \geq 113$, then there is no group satisfying $t_n(9)\big||S|$ by Table 3. If S is a simple group of Lie type in characteristic p, then $p \leq \frac{n}{10}$. If not and we assume that $\frac{n}{2} < p \leq n$, then $t_n(1) > (|\pi(n)| - |\pi(\frac{n}{2})|)(\frac{n}{2}) > (\frac{n}{2})^8$ (By corollary 3 of

[22], $|\pi(n)| - |\pi(\frac{n}{2})| > \frac{3\cdot\frac{n}{2}}{5\cdot\log\frac{n}{2}} > 8$ for $n \geq 118$). So $t_n(1) > n^7 > p^7$, a contradiction to Lemma 2.6($p\big||S|$ by Step 3). If $\frac{n}{k+1} < p \leq \frac{n}{k}$ with $k = 2, 3, 4$, then $p^k\big||S|$ by Step 3 and so $|S| < p^{3k}$ by Lemma 2.6. Thus $\frac{|S|}{p^k} < p^{2k} < p^{10} < (\frac{n}{2})^{10} < t_n(1)$ and hence $|S| < p^k t_n(1) < t_n(9)$, a contradiction since $t_n(9)$ divides the order $|S|$ of S by Step 3. If $k = 5, 6, 7, 8, 9$, then similarly we get $\frac{|S|}{p^k} < p^{2k} < p^{18} < (\frac{n}{5})^{18} < t_n(5)$ and so $|S| < p^k t_n(5) < t_n(9)$, a contradiction (when $n \geq 118$, $|\pi(\frac{n}{k})| - |\pi(\frac{n}{k+1})| > 1$ for $k \in \{5, 6, 7, 8, 9\}$).

Let $p^l\big||S|$. If $p \geq 3$, then by Lemmas 2.6 and Step 4,

$$7.82^n < t_n(9) \leq \frac{|S|}{p^l} < p^{2l} \leq (p^{\sum_{i=1}^{\infty}[\frac{n}{p^i}]})^2 = (p^{\frac{n}{p-1}})^2 < (3^{\frac{2}{3-1}})^n = 3^n$$

(if $x \geq 3$ and $f(x) = x^{\frac{1}{x-1}}$, then the differentiate $f'(x)$ of $f(x)$ is < 0), a contradiction since $7.82 \nleq 3$. If $p = 2$ and $S \ncong L_2(2^l)$, then $7.82^n < t_n(9) < \frac{|S|}{2^l} < 2^{\frac{5l}{3}} \leq (2^{\sum_{i=1}^{\infty}[\frac{n}{2^i}]})^{\frac{5}{3}} = (2^{\frac{5}{3}})^n$ and so $7.82 < 2^{5/3} < 2^2 = 4$, a contradiction. If $S \cong L_2(2^l)$, then S contains maximal cyclic subgroups of orders $p^l + 1$ and $p^l - 1$ and so S contains at most two prime factors which are larger than $n/2$, a contradiction to the fact that $|\pi(n)| - |\pi(\frac{n}{2})| > 8$ for $n \geq 118$.

Since p is the largest prime divisor of $|G|$ and by Step 3, p divides $|S|$, then S is isomorphic to A_n for $n = p$, $p+1, \cdots, p+d$.

Step 6: G is isomorphic to A_{p+d}.

The proof of Step 5 implies that S is isomorphic to A_n with $n = p, p+1, \cdots, p+d$. By Steps 2 and 5, we have that $A_n \leq G/K \leq S_n$.

If $H/K \cong A_p$, then $A_p \leq G/K \leq S_p$. If $G/K \cong A_p$, then $|K| = (p+1)(p+2)\cdots(p+d)$. It is easy to see that there is a prime divisor p' of $|K|$ such that $p' > d$ and $p' \nmid p-1$. Note that p is the largest prime of $\pi(G)$ and $d \geq 4$. Then there is cyclic subgroup of order pp' and so p' is adjacent to p. It follows that $pp' \in \omega(G)$ and so $\deg(p) > |\pi(d!)|$, a contradiction. If $G/K \cong S_p$, then $|K| = (p+1)(p+2)\cdots(p+d)/2$. In this case, we also get that there exists a prime p' of $|K|$ with the properties: $p' > d$ and $p' \nmid p-1$. We similarly get that $pp' \in \omega(G)$ and so we have a contradiction as $\deg(p) = |\pi(d!)|$.

Similarly to the above arguments, we also can rule out these cases: $H/K \cong A_n$ with $n \in \{p+1, p+2, \cdots, p+d-1\}$.

If $H/K \cong A_{p+d}$, then $A_{p+d} \leq G/K \leq S_{p+d}$. If $G/K \cong A_{p+d}$, then $K = 1$, the desired result. If $G/K \cong S_{p+d}$, then $|G|_2 < |S_{p+d}|_2$, a contradiction.

Steps 1-6 complete the proof of the Maintheorem 1.5. □

By Propositions 1.3 and 1.4, some alternating groups are OD-characterizable and by our main theorem 1.5 we have the following.

Theorem 3.2. *Let p be a prime. Then the alternating group A_{p+d} except A_{10} where d is a prime or equals to 4, is OD-characterizable.*

The conjecture in [25] was proved true by some joint works of many mathematicians and the last important part of the proof was given by V.D.Mazurov, Chen, SHI, etc. (see [26] and related references). Therefore we can get the following theorem which is also a conjecture.

Theorem 3.3. *Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.*

Corollary 3.4. *Let G be a group and $p \geq 5$ a prime. Assume that d is either a prime or 4. Then $G \cong A_{p+d}$ if and only if $\omega(G) = \omega(A_{p+d})$ and $|G| = |A_{p+d}|$.*

The conflict of interest disclosure

The authors declare that there is no conflict of interest regarding the publication of this paper.

Tables

Table 1. The simple classical groups

L	Lie;rankL	d	O	$\|L\|$
$L_n(q)$	$A_{n-1}(q)$ $n-1$	$(n,q-1)$	$2df$, if $n \geq 3$; df, if $n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1)$
$U_n(q)$	${}^2A_{n-1}(q)$ $[n/2]$	$(n,q+1)$	$2df$, if $n \geq 3$ df, if $n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)$
$PSp_{2m}(q)$	$C_m(q)$ m	$(2,q-1)$	df, $m \geq 3$; $2f$, if $m=2$	$\frac{1}{d}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$\Omega_{2m+1}(q)$ q odd	$B_m(q)$ m	2	$2f$	$\frac{1}{2}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$P\Omega^{+}_{2m}(q)$ $m \geq 3$	$D_m(q)$ m	$(4,q^m-1)$	$2df$, if $m \neq 4$ $6df$, if $m=4$	$\frac{1}{d}q^{m(m-1)(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1)}$
$P\Omega^{-}_{2m}(q)$ $m \geq 2$	${}^2D_m(q)$ $m-1$	$(4,q^m+1)$	$2df$	$\frac{1}{d}q^{m(m-1)(q^m+1)\prod_{i=1}^{m-1}}(q^{2i}-1)$

Table 2. The simple exceptional groups

L	L	d	O	$\|L\|$
$G_2(q)$	2	1	f, if $p \neq 3$ $2f$, if $p=3$	$q^6(q^2-1)(q^6-1)$
$F_4(q)$	4	1	$(2,p)f$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$E_6(q)$	6	$(3,q-1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-1)$
$E_7(q)$	7	$(2,q-1)$	df	$\frac{1}{d}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$
$E_8(q)$	8	1	f	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$
${}^2B_2(q), q=2^{2m+1}$	1	1	f	$q^2(q^2+1)(q-1)$
${}^2G_2(q), q=3^{2m+1}$	1	1	f	$q^3(q^3+1)(q-1)$
${}^2F_4(q), q=2^{2m+1}$	2	1	f	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$
${}^3D_4(q)$	2	1	$3f$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
${}^2E_6(q)$	4	$(3,q+1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-(-1)^i)$

Acknowledgement: The corresponding author was supported by the Department of Education of Sichuan Province (Grant No: 15ZA0235) and by the Opening Project of Sichuan Province University Key Laborstory of Bridge Non-destruction Detecting and Engineering Computing (Grant Nos: 2016QYJ01 and 2017QZJ01). The authors are very grateful for the helpful suggestions of the referee.

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Table 3. The simple sporadic groups

L	d	O	$\|L\|$
M_{11}	1	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	2	2	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	12	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}	1	1	$2^{10} \cdot 3^3 \cdot 5.7 \cdot 11 \cdot 23$
J_1	1	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	2	2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	3	2	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	1	1	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	2	2	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Suz	6	2	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
McL	3	2	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Ru	2	1	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$He\,(F_7)$	1	2	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ly	1	1	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
ON	3	2	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	2	1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	1	1	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	1	1	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	6	2	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	1	1	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	3	2	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$HN(F_5)$	1	2	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
$Th(F_3)$	1	1	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
$BM(F_2)$	2	1	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$M(F_1)$	1	1	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

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