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Research Article

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On generalized Ehresmann semigroups

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Abstract: As a generalization of the class of inverse semigroups, the class of Ehresmann semigroups is introduced by Lawson and investigated by many authors extensively in the literature. In particular, Gomes and Gould construct a fundamental Ehresmann semigroup C_E from a semilattice E which plays for Ehresmann semigroups the role that T_E plays for inverse semigroups, where T_E is the Munn semigroup of a semilattice E. From a varietal perspective, Ehresmann semigroups are derived from reduction of inverse semigroups. In this paper, from varietal perspective Ehresmann semigroups are extended to *generalized Ehresmann semigroups* derived instead from normal orthodox semigroups (i.e. regular semigroups whose idempotents form normal bands) with an inverse transversal. We present here a semigroup $C_{(I,\Lambda,E^\circ)}$ from an *admissible triple* (I, Λ, E°) that plays for generalized Ehresmann semigroups the role that C_E from a semilattice E plays for Ehresmann semigroups. More precisely, we show that a semigroup is a fundamental generalized Ehresmann semigroup whose admissible triple is isomorphic to (I, Λ, E°) if and only if it is (2,1,1,1)-isomorphic to a *quasi-full* (2,1,1,1)-subalgebra of $C_{(I,\Lambda,E^\circ)}$. Our results generalize and enrich some results of Fountain, Gomes and Gould on weakly E-hedges semigroups and Ehresmann semigroups.

Keywords: Generalized Ehresmann semigroup, Fundamental semigroup, Fundamental representation

MSC: 20M10

1 Introduction

Let S be a semigroup. We denote the set of all idempotents of S by $E(S)$ and the set of all inverses of $x \in S$ by $V(x)$. Recall that

$$V(x) = \{a \in S | xax = x, axa = a\}$$

for all $x \in S$. A semigroup S is called *regular* if $V(x) \neq \emptyset$ for any $x \in S$, and a regular semigroup S is called *inverse* if $E(S)$ is a commutative subsemigroup (i.e. a subsemilattice) of S, or equivalently, the cardinal of $V(x)$ is equal to 1 for all $x \in S$.

Recall that a regular semigroup S is *fundamental* if the largest congruence contained in $\mathcal{H}$ on S is trivial. Structure theorems for certain important subclasses of the class of fundamental regular semigroups are already known. Munn [1] initiated the work in this direction. He proved that given a semilattice E, the *Munn semigroup* T_E of all isomorphisms of principal ideals of E is “maximal" in the class of all fundamental inverse semigroups whose semilattices of idempotents are E, that is, every semigroup belonging to this class is isomorphic to a full inverse subsemigroup of T_E. Further from Munn [1] if S is an inverse semigroup such that $E(S)$ is isomorphic to a given semilattice E, then there exists a homomorphism $f : S \to T_E$ and the kernel of f is the largest congruence contained in $\mathcal{H}$ on S.

The pioneer work of Munn was generalized firstly by Hall in 1971 to orthodox semigroups (i.e. regular semigroups whose idempotents form subsemigroups) in [2] in which the Hall semigroup W_B of a band B was constructed. Recall that *a band* is a semigroup in which every element is idempotent. The Hall semigroup W_B

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has properties analogous to those described above for T_E (see Hall [2] for details). As another direction, Fountain [3] generalized Munn's result to a class of non-regular semigroup, namely *ample semigroups*. The next step in this direction was made by Fountain, Gomes and Gould in [4] where by using of a completely fresh technology they considered the class of *weakly E-hedged semigroups* which is a special class of Ehresmann semigroups first defined by Lawson in [5]. The natural next step in this direction is to look at the whole class of Ehresmann semigroups. This is done by Gomes and Gould in [6] in which they removed the "weakly E-hedged" condition from [4] and considered the whole calss of Ehresmann semigroups. For a given semilattice E, in [6] the authors have constructed a semigroup C_E which plays the T_E role for Ehresmann semigroups and generalized the main results in [4]. Furthermore, El-Qallali-Fountain-Gould [7], Gomes-Gould [8] and Wang [9] went a step further to extend Hall's approach for orthodox semigroups to some classes of non-regular semigroups having a band of idempotents. It is worth remarking that the class of Ehresmann semigroups and its subclasses are investigated extensively in literature by many famous semigroup researchers (see [10-13] for example). In particular, Jones [12] provided a common framework for Ehresmann semigroups and regular *-semigroups from varietal perspective. More recent developments in this area can be found in good survey articles by Gould [11, 14] and Hollings [15, 16].

On the other hand, Blyth-McFadden [17] introduced the concept of inverse transversals for regular semigroups. A subsemigroup S° of a regular semigroup S is called an *inverse transversal* of S if $V(x) \cap S^\circ$ contains exactly one element for all $x \in S$. Clearly, in this case, S° is an inverse subsemigroup of S. Since an inverse semigroup can be regarded as an inverse transversal of itself, the class of regular semigroups with inverse transversals contains the class of inverse semigroups as a proper subclass. Regular semigroups with inverse transversals are investigated extensively by many authors (see [18-21] and their references) and some generalizations of inverse transversals are proposed, see [22, 23] for example.

Inspired by the approach used in Jones [12], in this paper a common framework, termed *generalized Ehresmann semigroups*, for Ehresmann semigroups and normal orthodox semigroups with an inverse transversal is introduced from varietal perspective, where a *normal orthodox semigroup* means a regular semigroup whose idempotents form a normal band. We construct a semigroup $C_{(I,\Lambda,E^\circ)}$ from the so-called *admissible triple* (I, Λ, E°) that plays for generalized Ehresmann semigroups the role that C_E from a semilattice E plays for Ehresmann semigroups. More precisely, we show that a semigroup is a fundamental generalized Ehresmann semigroup whose admissible triple is isomorphic to (I, Λ, E°) if and only if it is (2,1,1,1)-isomorphic to a so-called *quasi-full* (2,1,1,1)-subalgebra of $C_{(I,\Lambda,E^\circ)}$. This generalizes and enriches some results of Fountain, Gomes and Gould on weakly E-hedges semigroups and Ehresmann semigroups obtained in texts [4] and [6].

2 Generalized Ehresmann semigroups

In this section, after giving some preliminary results on Ehresmann semigroups and inverse transversals, we introduce generalized Ehresmann semigroups and consider some basic properties of this class of semigroups. Firstly, we consider Ehresmann semigroups. Let S be a semigroup and let $E \subseteq E(S)$. The relation $\widetilde{\mathcal{R}}_E$ is defined on S by the rule that for any $x, y \in S$, we have $x \widetilde{\mathcal{R}}_E y$ if

$$ex = x \text{ if and only if } ey = y \text{ for all } e \in E.$$

Dually, we have the relation $\widetilde{\mathcal{L}}_E$ on S. Observe that both $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalences on S but $\widetilde{\mathcal{R}}_E$ (resp. $\widetilde{\mathcal{L}}_E$) may not be a left congruence (resp. a right congruence). From Lawson [5], a semigroup S is an *Ehresmann semigroup with respect to* E (or (S, E) *is an Ehresmann semigroup*) if

(1) E is a subsemilattice of S,
(2) every $\widetilde{\mathcal{R}}_E$-class contains a unique element of E and $\widetilde{\mathcal{R}}_E$ is a left congruence,
(3) every $\widetilde{\mathcal{L}}_E$-class contains a unique element of E and $\widetilde{\mathcal{L}}_E$ is a right congruence.

If this is the case, E is called *the distinguished semilattice* of S. We say that a semigroup S is *Ehresmann* in the sequel if (S, E) is an Ehresmann semigroup for some $E \subseteq E(S)$. From Lemma 2.2 and its dual in Gould [14], we have the following characterization of Ehresmann semigroups from a varietal perspective.

Lemma 2.1. *A semigroup $(S,\cdot)$ is Ehresmann if and only if there are two unary operations "$+$" and "$*$" on S such that the following identities hold:*

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y^+)^+ = x^+y^+, (xy)^+ = (xy^+)^+;$$

$$xx^* = x, x^*y^* = y^*x^*, (x^*y^*)^* = x^*y^*, (xy)^* = (x^*y)^*; (x^+)^* = x^+, (x^*)^+ = x^*.$$

In this case, S is an Ehresmann semigroup with distinguished semilattice $\{x^+|x \in S\}(= \{x^*|x \in S\})$, and we shall call $(S\cdot,\ ^+,\ ^*)$ an Ehresmann semigroup.

Now we consider regular semigroups with an inverse transversal. Let S be a regular semigroup and S° an inverse transversal of S. For any $x \in S$, we use x° to denote the unique inverse of x in S° and let $x^{\circ\circ} = (x^\circ)^\circ$. We can consider the induced tri-unary semigroup $(S,\ \cdot,\ ^+,\ ^*,\ ^-)$, where

$$x^+ = xx^\circ, x^* = x^\circ x, \overline{x} = x^{\circ\circ}.$$

Observe that $x^{\circ\circ\circ} = x^\circ$ and $x^{\circ\circ}x^\circ, x^\circ x^{\circ\circ} \in E(S^\circ)$. Here we are only interested in normal orthodox semigroups with an inverse transversal. Recall that a *normal orthodox semigroup* means a regular semigroup whose idempotents form a normal band, and a band B is called *left normal* (resp. *right normal, normal*) if $efg = egf$ (resp. $efg = feg$, $efge = egfe$) for all $e, f, g \in B$. Observe that a band B is normal if and only if $efgh = egfh$ for all $e, f, g, h \in B$. The following proposition can be deduced from the text [20]. However, we give its direct proof for the sake of completeness.

Proposition 2.2. *Let S be a normal orthodox semigroup with an inverse transversal S°. The tri-unary semigroup $(S,\ \cdot,\ ^+,\ ^*,\ ^-)$ satisfies the following identities:*

Table 1. Generalized Ehresmann conditions

(1)	$x^+x = x$	(1)′	$xx^* = x$
(2)	$x^+y^+x^+ = x^+y^+$	(2)′	$x^*y^*x^* = y^*x^*$
(3)	$(x^+y^+)^+ = x^+y^+$	(3)′	$(x^*y^*)^* = x^*y^*$
(4)	$(xy)^+ = (xy^+)^+$	(4)′	$(xy)^* = (x^*y)^*$
(5)	$(x^+)^* = \overline{x}^+$	(5)′	$(x^*)^+ = \overline{x}^*$
(6)	$\overline{x}^+ = \overline{x^+}$	(6)′	$\overline{x}^* = \overline{x^*}$
(7)	$x = x^+\overline{x}x^*$	(8)	$x^*y^+ = \overline{x}^*\overline{y}^+$

Proof. By symmetry, we only need to show (1)–(8).

(1) This is equivalent to the equality $(xx^\circ)x = x$.

(2) As $E(S)$ is a normal band, we have

$$x^+y^+ = (xx^\circ)[(yy^\circ)(y^{\circ\circ}y^\circ)](y^{\circ\circ}y^\circ) = (xx^\circ)[(y^{\circ\circ}y^\circ)(yy^\circ)](y^{\circ\circ}y^\circ) = xx^\circ y^{\circ\circ}y^\circ. \quad (1)$$

This implies that

$$x^+y^+x^+ = x^+(y^+x^+) = xx^\circ(yy^\circ x^{\circ\circ}x^\circ) = (xx^\circ yy^\circ)(x^{\circ\circ}x^\circ)$$
$$= xx^\circ(y^{\circ\circ}y^\circ)(x^{\circ\circ}x^\circ) = xx^\circ(x^{\circ\circ}x^\circ)(y^{\circ\circ}y^\circ) = xx^\circ y^{\circ\circ}y^\circ = x^+y^+.$$

(3) We first observe that

$$(xy)^\circ = y^\circ x^\circ \text{ for all } x, y \in S. \quad (2)$$

In fact, since S is orthodox, $y^\circ x^\circ$ is an inverse of xy. But $y^\circ x^\circ \in S^\circ$, so $(xy)^\circ = y^\circ x^\circ$. By (1) and (2), we have

$$(x^+y^+)^+ = (xx^\circ y^{\circ\circ}y^\circ)(xx^\circ y^{\circ\circ}y^\circ)^\circ = xx^\circ y^{\circ\circ}y^\circ y^{\circ\circ}y^{\circ\circ\circ}x^{\circ\circ}x^\circ$$
$$= xx^\circ y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ = xx^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ = xx^\circ y^{\circ\circ}y^\circ = x^+y^+.$$

(4) By (2), we have

$$(xy^+)^+ = (xyy^\circ)(xyy^\circ)^\circ = xyy^\circ y^{\circ\circ} y^\circ x^\circ = xyy^\circ x^\circ = xy(xy)^\circ = (xy)^+.$$

(5) From (2),

$$(x^+)^* = (xx^\circ)^\circ(xx^\circ) = x^{\circ\circ}x^\circ xx^\circ = x^{\circ\circ}x^\circ = x^{\circ\circ}x^{\circ\circ\circ} = \overline{x}^+.$$

(6) It follows that $\overline{x^+} = (xx^\circ)^{\circ\circ} = x^{\circ\circ}x^\circ = x^{\circ\circ}(x^{\circ\circ})^\circ = \overline{x}^+$ from (2).
(7) This is equivalent to the statement $x = (xx^\circ)x^{\circ\circ}(x^\circ x)$.
(8) Since $E(S)$ is a normal band, using (1) and its dual, we have

$$x^*y^+ = x^\circ xyy^\circ = x^\circ x^{\circ\circ}x^\circ xyy^\circ y^{\circ\circ}y^\circ = (x^\circ x^{\circ\circ}yy^\circ)(x^\circ xy^{\circ\circ}y^\circ)$$

$$= (x^\circ x^{\circ\circ}y^{\circ\circ}y^\circ)(x^\circ x^{\circ\circ}y^{\circ\circ}y^\circ) = x^\circ x^{\circ\circ}y^{\circ\circ}y^\circ = ((x^{\circ\circ})^\circ x^{\circ\circ})(y^{\circ\circ}(y^{\circ\circ})^\circ) = \overline{x}^*\overline{y}^+,$$

as required. □

We shall term any tri-unary semigroup $(S, \cdot, {}^+, {}^*, {}^-)$ that satisfies the identities in Table 1 a *generalized Ehresmann semigroup*. By Proposition 2.2, any normal orthodox semigroup S with an inverse transversal S° induces the generalized Ehresmann semigroup $(S, \cdot, {}^+, {}^*, {}^-)$ by setting $x^+ = xx^\circ, x^* = x^\circ x$ and $\overline{x} = x^{\circ\circ}$. The following example gives a very special case of this kind of generalized Ehresmann semigroups.

Example 2.3. *Let S be a rectangular band. Fix an element u in S. Consider the tri-unary semigroup $(S, \cdot, {}^+, {}^*, {}^-)$ where $x^+ = xu, x^* = ux, \overline{x} = u$. Then it is routine to check that the identities in Table 1 are satisfied and so $(S, \cdot, {}^+, {}^*, {}^-)$ is a generalized Ehresmann semigroup. In fact, S is indeed a normal band and $\{u\}$ is an inverse transversal of S.*

Example 2.4. *Any Ehresmann semigroup $(S, \cdot, {}^+, {}^*)$ also induces a generalized Ehresmann semigroup, which justifies our term* "generalized Ehresmann semigroups"*. In fact, for an Ehresmann semigroup $(S, \cdot, {}^+, {}^*)$, we define the third unary operation "${}^-$" on S by $\overline{x} = x$. Then we have the tri-unary semigroup $(S, \cdot, {}^+, {}^*, {}^-)$ and it is easy to see that the identities in Table 1 are all satisfied by Lemma 2.1.*

Since a rectangular band having more than one element must not be an Ehresmann semigroup, the class of generalized Ehresmann semigroups contains the class of Ehresmann semigroups and the class of rectangular bands as proper subclasses by the above two examples. We also observe that a generalized Ehresmann semigroup which is also regular may not contain any inverse transversal. In fact, any monoid S with the identity 1 is always a (generalized) Ehresmann semigroup by setting $x^+ = x^* = 1$ and $\overline{x} = x$ for all $x \in S$. Obviously, a regular monoid may not contain any inverse transversal. Here is an example.

Example 2.5. *Let $M = \{1, b, c, x\}$(taken from Exercise 10 in Chapter VI of [24]) with the multiplication*

M	1	b	c	x
1	1	b	c	x
b	b	b	b	b.
c	c	c	c	c
x	x	c	b	1

Then M is a monoid and

$$E(M) = \{1, b, c\}, V(1) = \{1\}, V(b) = \{b, c\} = V(c), V(x) = \{x\}.$$

It is easy to check that M contains no inverse transversal.

In the remainder of this section, we consider some properties associated to generalized Ehresmann semigroups which will be used in the next sections. Let $(S, \cdot, {}^+, {}^*, {}^-)$ be a generalized Ehresmann semigroup. Denote

$$I_S = \{x^+ | x \in S\}, \Lambda_S = \{x^* | x \in S\}, E_S^\circ = \{\overline{x}^+ | x \in S\}.$$

Lemma 2.6. *Let* $(S, \cdot, {}^+, {}^*, {}^-)$ *be a generalized Ehresmann semigroup.*
(a) $i^+ = i, i^* = \overline{i}$ *and* $\lambda^* = \lambda, \lambda^+ = \overline{\lambda}$ *for all* $i \in I_S$ *and* $\lambda \in \Lambda_S$.
(b) $E_S^\circ = \{\overline{x}^* | x \in S\} = I_S \cap \Lambda_S$,
(c) $x^+ \mathcal{L} \overline{x}^+$ *and* $x^* \mathcal{R} \overline{x}^*$.
(d) I_S *is a left normal band,* Λ_S *is a right normal band and* E_S° *is a subsemilattice of* S*, respectively.*

Proof. (a) Using identities (1),(4) and (3) in Table 1, we have

$$x^+ = (x^+ x)^+ = (x^+ x^+)^+ = x^+ x^+$$

and so $x^+ \in E(S)$. Take $i \in I_S$. Then $i = x^+$ for some $x \in S$. This implies that

$$i^+ = (x^+)^+ = (x^+ x^+)^+ = x^+ x^+ = x^+$$

by $x^+ \in E(S)$ and the identity (3). Moreover,

$$i^* = (x^+)^* = \overline{x}^+ = \overline{x^+} = \overline{i}$$

by the identities (5) and (6). By symmetry, $\lambda^* = \lambda$ and $\lambda^+ = \overline{\lambda}$ for all $\lambda \in \Lambda_S$.

(b) By identity (5), $\overline{x}^+ = (x^+)^* \in I_S \cap \Lambda_S$ for all $x \in S$. Now let $u = x^+ \in I_S \cap \Lambda_S$ for some $x \in S$. Using item (a) and the identity (5), we have

$$u = u^* = (x^+)^* = \overline{x}^+ \in E_S^\circ.$$

Thus $E_S^\circ = I_S \cap \Lambda_S$. Dually, $\{\overline{x}^* | x \in S\} = I_S \cap \Lambda_S$.

(c) Using identities (7), (4), (3), (4), (5)′ and (1)′ in Table 1 in that order, we have

$$x^+ = (x^+ \overline{x} x^*)^+ = (x^+ (\overline{x} x^*)^+)^+ = x^+ (\overline{x} x^*)^+ = x^+ (\overline{x} (x^*)^+)^+ = x^+ (\overline{x}\, \overline{x}^*)^+ = x^+ \overline{x}^+.$$

Since $\overline{x}^+ \in \Lambda_S$ by (b), we have $(\overline{x}^+)^* = \overline{x}^+$ by (a). Using the identities (5), (8) and (6), we have

$$\overline{x}^+ x^+ = (x^+)^* x^+ = (\overline{x^+})^* \overline{x}^+ = (\overline{x}^+)^* \overline{x}^+ = \overline{x}^+ \overline{x}^+ = \overline{x}^+.$$

This shows that $x^+ \mathcal{L} \overline{x}^+$. Dually, $x^* \mathcal{R} \overline{x}^*$.

(d) In view of the proof of item (a), every element in I_S is idempotent. By the identity (3), $x^+ y^+ = (x^+ y^+)^+ \in I_S$ for all $x^+, y^+ \in I_S$. So I_S is a subband of S. By the identity (2) and (2)′ and item (b), E_S° is a subsemilattice of S. Moreover, for $x, y, z \in S$, by identities (5), (8) and (6) and items (b) and (a), we have

$$\overline{x}^+ y^+ = (x^+)^* y^+ = \overline{x^+}^* \overline{y}^+ = (\overline{x}^+)^* \overline{y}^+ = \overline{x}^+ \overline{y}^+.$$

Similarly, $\overline{y}^+ z^+ = \overline{y}^+ \overline{z}^+$. In view of item (c) and the fact that E_S° is a subsemilattice,

$$x^+ y^+ z^+ = x^+ \overline{x}^+ y^+ z^+ = x^+ \overline{x}^+ \overline{y}^+ z^+ = x^+ \overline{x}^+ \overline{y}^+ \overline{z}^+$$

$$= x^+ \overline{x}^+ \overline{y}^+ \overline{z}^+ = x^+ \overline{y}^+ \overline{z}^+ = x^+ \overline{z}^+ \overline{y}^+.$$

Similarly, we have $x^+ z^+ y^+ = x^+ \overline{z}^+ \overline{y}^+$. This yields that I_S is a left normal band. Dually, Λ_S is a right normal band. □

Let S be a semigroup. Recall that the natural partial order " $\leq$ " on $E(S)$ is defined as follows:

$$e \leq f \text{ if and only if } ef = fe = e \text{ for all } e, f \in E(S). \tag{3}$$

Lemma 2.7. *Let* $(S, \cdot, {}^+, {}^*, {}^-)$ *be a generalized Ehresmann semigroup and* $x, y \in S$.
(a) $(xy)^+ = x^+ (\overline{x}\, \overline{y})^+, (xy)^* = (\overline{x}\, \overline{y})^* y^*$.
(b) $(xy)^+ \leq x^+, (xy)^* \leq y^*$.

(c) $\overline{xy} = \overline{x}\,\overline{y}$.

Proof. (a) Using the identities (4), (7), (8), (1)′, (4) and (3) in that order, we have

$$(xy)^+ = (xy^+)^+ = (x^+\overline{x}x^*y^+)^+ = (x^+\overline{x}\,\overline{x}^*\overline{y}^+)^+$$
$$= (x^+\overline{x}\,\overline{y}^+)^+ = (x^+\overline{x}\,\overline{y})^+ = (x^+(\overline{x}\,\overline{y})^+)^+ = x^+(\overline{x}\,\overline{y})^+.$$

Dually, $(xy)^* = (\overline{x}\,\overline{y})^*y^*$.

(b) Since $(xy)^+ = x^+(\overline{x}\,\overline{y})^+$ by item (a), we have $x^+(xy)^+ = (xy)^+$. Moreover, by the identity (2),

$$(xy)^+x^+ = x^+(\overline{x}\,\overline{y})^+x^+ = x^+(\overline{x}\,\overline{y})^+ = (xy)^+.$$

So $(xy)^+ \le x^+$. Dually, $(xy)^* \le y^*$.

(c) In view of the identity (8) and items (b), (d) in Lemma 2.6, we have $(x^+)^*(\overline{x}\,\overline{y})^+ = \overline{x^+}^{\,*}\,\overline{\overline{x}\,\overline{y}}^{\,+} \in E_S^\circ$ so that $((x^+)^*(\overline{x}\,\overline{y})^+)^* = (x^+)^*(\overline{x}\,\overline{y})^+$ by Lemma 2.6 (a). Using the identity (5), item (a) of this lemma, the identities (4)′, (5), (3), (4), (1) in that order, we obtain that

$$\overline{xy}^+ = ((xy)^+)^* = (x^+(\overline{x}\,\overline{y})^+)^* = ((x^+)^*(\overline{x}\,\overline{y})^+)^*$$
$$= (x^+)^*(\overline{x}\,\overline{y})^+ = \overline{x}^+(\overline{x}\,\overline{y})^+ = (\overline{x}^+(\overline{x}\,\overline{y})^+)^+ = (\overline{x}^+\overline{x}\,\overline{y})^+ = (\overline{x}\,\overline{y})^+.$$

Dually, $\overline{xy}^* = (\overline{x}\,\overline{y})^*$. Using the identities (1) and (1)′, Lemma 2.6 (c), the identities (7), (8), (1), (1)′, item (b) of this lemma, Lemma 2.6 (c), the identities (1) and (1)′ in that order, we get

$$\overline{xy} = \overline{xy}^+\overline{xy}\,\overline{xy}^* = \overline{xy}^+(xy)^+\overline{xy}\,(xy)^*\overline{xy}^* = \overline{xy}^+xy\overline{xy}^* = \overline{xy}^+x^+\overline{x}x^*y^+\overline{y}y^*\overline{xy}^*$$
$$= \overline{xy}^+x^+\overline{x}\,\overline{x}^*\overline{y}^+\overline{y}y^*\overline{xy}^* = \overline{xy}^+x^+\overline{x}\,\overline{y}y^*\overline{xy}^* = (\overline{x}\,\overline{y})^+x^+\overline{x}\,\overline{y}y^*(\overline{x}\,\overline{y})^*$$
$$= (\overline{x}\,\overline{y})^+\overline{x}^+x^+\overline{x}\,\overline{y}y^*\overline{y}^*(\overline{x}\,\overline{y})^* = (\overline{x}\,\overline{y})^+\overline{x}^+\overline{x}\,\overline{y}\,\overline{y}^*(\overline{x}\,\overline{y})^* = (\overline{x}\,\overline{y})^+\overline{x}\,\overline{y}(\overline{x}\,\overline{y})^* = \overline{x}\,\overline{y},$$

as required. □

3 The semigroup $C_{(I,\Lambda,E^\circ)}$

We call a generalized Ehresmann semigroup $(S, \cdot, ^+, ^*, ^-)$ *fundamental* if the largest semigroup congruence μ_S contained in the equivalence

$$\{(a,b) \in S \times S \,|\, a^+ = b^+, a^* = b^*\}$$

is the identity relation on S. In this section, we shall construct a fundamental generalized Ehresmann semigroup which plays the similar role in the class of generalized Ehresmann semigroups as the Munn semigroup of a semilattice in the class of inverse semigroups. To do this, we need to introduce the notion of admissible triples, which is motivated by Lemma 2.6.

Definition 3.1. *Let I (resp. Λ) be a left normal band (resp. a right normal band), $E^\circ = I \cap \Lambda$ a subsemilattice of I and Λ. The triple (I, Λ, E°) is called* admissible *if for all $g \in I$ and $f \in \Lambda$, there exist $g^\circ, f^\circ \in E^\circ$ such that $g\mathcal{L}g^\circ$ and $f\mathcal{R}f^\circ$.*

Remark 3.2. *Let (I, Λ, E°) be an admissible triple. Since E° is a subsemilattice, the elements g° and f° in Definition 3.1 are uniquely determined by g and f, respectively. In particular, $i \in E^\circ$ if and only if $i^\circ = i$.*

Remark 3.3. *Let $(S, \cdot, ^+, ^*, ^-)$ be a generalized Ehresmann semigroup. By Lemma 2.6 (b), (c) and (d), $(I_S, \Lambda_S, E_S^\circ)$ is an admissible triple which will be called the* admissible triple *of S. In this case, for all $i \in I_S$ and $\lambda \in \Lambda_S$, we have $i^\circ = i^*$ and $\lambda^\circ = \lambda^+$. In fact, if $x^+ \in I_S$, then by the identity (5) in Table 1 and Lemma 2.6 (c), we have $x^+\mathcal{L}\overline{x}^+ = (x^+)^*$. The case for $\lambda \in \Lambda_S$ can be showed dually.*

To construct the semigroup $C_{(I,\Lambda,E^\circ)}$ of an admissible triple (I, Λ, E°), we need some preliminaries. First, we have the following basic facts on admissible triples.

Lemma 3.4. *Let (I, Λ, E°) be an admissible triple and $e, g \in I$, $f, h \in \Lambda$. Then*

$$eg = eg^\circ, (eg)^\circ = e^\circ g^\circ, fh = f^\circ h, (fh)^\circ = f^\circ h^\circ.$$

Moreover, we have $eE^\circ e = eE^\circ$ and $fE^\circ f = E^\circ f$, which are subsemilattices of I and Λ, respectively.

Proof. Since $g\mathcal{L}g^\circ$ and I is a left normal band, we have $eg = egg^\circ = eg^\circ g = eg^\circ$. This implies that $e^\circ g = e^\circ g^\circ$, and so $eg\mathcal{L}e^\circ g = e^\circ g^\circ \in E^\circ$ by the fact that $e\mathcal{L}e^\circ$. This yields that $(eg)^\circ = e^\circ g^\circ$. Finally, it follows that $eE^\circ e = eE^\circ$ by the fact that I is a left normal band. Moreover, for $i, j \in E^\circ$, we have

$$(ei)(ej) = (eie)j = eij = eji = (eje)i = (ej)(ei),$$

whence eE° is a subsemilattice of I. The remaining facts of this lemma can be proved by symmetry. □

Let (I, Λ, E°) be an admissible triple. We use I^1 (resp. Λ^1) to denote I (resp. Λ) with *the identity adjoined*. Furthermore, *we always assume that* $1^\circ = 1$ *for the adjoined identity 1 on both I and Λ*. A function η from I^1 to Λ is called *order-preserving* if $x\eta \leq y\eta$ in Λ for all $x, y \in I^1$ with $x \leq y$ in I^1, where $\leq$ is defined in (3). Denote the set of order-preserving functions from I^1 to Λ by $\mathcal{O}(I^1 \to \Lambda)$. Dually, we also have $\mathcal{O}(\Lambda^1 \to I)$. Similarly, we have $\mathcal{O}(I^1 \to I)$ and $\mathcal{O}(\Lambda^1 \to \Lambda)$. Moreover, we denote the set of the morphisms from I^1 to Λ by $\mathrm{End}(I^1 \to \Lambda)$. Dually, we have $\mathrm{End}(\Lambda^1 \to I)$. Obviously,

$$\mathrm{End}(I^1 \to \Lambda) \subseteq \mathcal{O}(I^1 \to \Lambda), \mathrm{End}(\Lambda^1 \to I) \subseteq \mathcal{O}(\Lambda^1 \to I).$$

For every $e \in I$ (resp. $f \in \Lambda$), define

$$\rho_e : I^1 \to I, x \mapsto ex \ \ (\text{resp. } \sigma_f : \Lambda^1 \to \Lambda, x \mapsto xf).$$

Then it is easy to see that $\rho_e \in \mathcal{O}(I^1 \to I)$ and $\sigma_f \in \mathcal{O}(\Lambda^1 \to \Lambda)$ for all $e \in I$ and $f \in \Lambda$ as I is a left normal band and Λ is a right normal band. Moreover, *we use ρ_1 and σ_1 to denote the identity maps on I and Λ, respectively*.

Lemma 3.5. *Let (I, Λ, E°) be an admissible triple. Define a multiplication " $\diamond$ " on $\mathcal{O}(I^1 \to \Lambda)$as follows: for all $\alpha, \beta \in \mathcal{O}(I^1 \to \Lambda)$,*

$$\alpha \diamond \beta : I^1 \to \Lambda, x \to (x\alpha)^\circ \beta.$$

Then $\mathcal{O}(I^1 \to \Lambda)$ is a semigroup with respect to " $\diamond$ ". Dually, for $\alpha, \beta \in \mathcal{O}(\Lambda^1 \to I)$, define

$$\beta \star \alpha : \Lambda^1 \to I, x \mapsto (x\beta)^\circ \alpha,$$

then $\mathcal{O}(\Lambda^1 \to I)$ forms a semigroup with respect to " $\star$ ".

Proof. Since $\emptyset \neq \mathrm{End}(I^1 \to \Lambda) \subseteq \mathcal{O}(I^1 \to \Lambda)$, it follows that $\mathcal{O}(I^1 \to \Lambda) \neq \emptyset$. Observe that

$$x^\circ \leq y^\circ \text{ for all } x, y \in I^1 \text{ (or } x, y \in \Lambda^1) \text{ with } x \leq y \tag{4}$$

by Lemma 3.4. Let $x, y \in I^1$ and $x \leq y$. Then $x\alpha, y\alpha \in \Lambda$ and $x\alpha \leq y\alpha$ as α is order-preserving. This implies that $(x\alpha)^\circ \leq (y\alpha)^\circ$. Observe that β is also order-preserving, it follows that $((x\alpha)^\circ)\beta \leq ((y\alpha)^\circ)\beta$. Thus $\alpha \diamond \beta \in \mathcal{O}(I^1 \to \Lambda)$. Now let $\alpha, \beta, \gamma \in \mathcal{O}(I^1 \to \Lambda)$ and $x, y \in I^1$. Then

$$x[(\alpha \diamond \beta) \diamond \gamma] = [(x(\alpha \diamond \beta))^\circ]\gamma = [(x\alpha)^\circ \beta]^\circ \gamma = (x\alpha)^\circ (\beta \diamond \gamma) = x[\alpha \diamond (\beta \diamond \gamma)].$$

This implies that $\mathcal{O}(I^1 \to \Lambda)$ is a semigroup with respect to " $\diamond$ ". Dually, $\mathcal{O}(\Lambda^1 \to I)$ forms a semigroup with respect to " $\star$ ". □

Corollary 3.6. *Let (I, Λ, E°) be an admissible triple, and " $\diamond$ " and " $\star$ " be defined as in Lemma 3.5. Then $\mathcal{O}(I^1 \to \Lambda) \times \mathcal{O}(\Lambda^1 \to I)$ forms a semigroup by defining*

$$(\alpha, \beta)(\gamma, \delta) = (\alpha \diamond \gamma, \delta \star \beta)$$

for all $(\alpha, \beta), (\gamma, \delta) \in \mathcal{O}(I^1 \to \Lambda) \times \mathcal{O}(\Lambda^1 \to I)$.

The semigroup $\mathcal{O}(I^1 \to \Lambda)$ is partially ordered by " $\leq$ " where for all $\alpha, \beta \in \mathcal{O}(I^1 \to \Lambda)$,

$$\alpha \leq \beta \text{ if and only if } x\alpha \leq x\beta \text{ in } \Lambda \text{ for all } x \in I^1.$$

Similarly, the semigroups $\mathcal{O}(\Lambda^1 \to I)$, $\mathcal{O}(I^1 \to I)$ and $\mathcal{O}(\Lambda^1 \to \Lambda)$ can be partially ordered, respectively.

Consider the subset

$$C_{(I,\Lambda,E^\circ)} = \{(\alpha, \beta) | (\forall x \in I^1)(\forall y \in \Lambda^1) x^\circ\alpha = x\alpha, y^\circ\beta = y\beta, \sigma_{x\alpha} \leq \beta\rho_x\alpha, \rho_{y\beta} \leq \alpha\sigma_y\beta\}$$

of the product semigroup $\mathcal{O}(I^1 \to \Lambda) \times \mathcal{O}(\Lambda^1 \to I)$.

Lemma 3.7. *Let (I, Λ, E°) be an admissible triple. For $e \in I$, define θ_e and τ_e as follows:*

$$\theta_e : I^1 \to \Lambda, x \mapsto e^\circ x, \quad \tau_e : \Lambda^1 \to I, x \mapsto ex^\circ.$$

Then $(\theta_e, \tau_e) \in C_{(I,\Lambda,E^\circ)}$. Dually, for $f \in \Lambda$, define η_f and ξ_f as follows:

$$\eta_f : I^1 \to \Lambda, x \mapsto x^\circ f, \quad \xi_f : \Lambda^1 \to I, x \mapsto xf^\circ.$$

Then $(\eta_f, \xi_f) \in C_{(I,\Lambda,E^\circ)}$. Moreover, if $e \in I \cap \Lambda = E^\circ$, then $(\theta_e, \tau_e) = (\eta_e, \xi_e)$.

Proof. Firstly, by Lemma 3.4, we have $e^\circ x = e^\circ x^\circ \in E^\circ \subseteq \Lambda$ for all $x \in I$, and $x^\circ \in E^\circ, ex^\circ \in I$ for all $x \in \Lambda$. Therefore θ_e and τ_e are well-defined. Secondly, since I is a left normal band, we have

$$(x\theta_e)(y\theta_e) = (e^\circ x)(e^\circ y) = e^\circ xe^\circ y = e^\circ xy = (xy)\theta_e$$

for all $x, y \in I^1$. On the other hand, by Lemma 3.4, we have

$$(x\tau_e)(y\tau_e) = ex^\circ ey^\circ = ex^\circ y^\circ = e(xy)^\circ = (xy)\tau_e$$

for all $x, y \in \Lambda^1$. This shows that θ_e and τ_e are morphisms and so order-preserving. Thirdly, if $x \in I$, then $x\theta_e = e^\circ x = e^\circ x^\circ = x^\circ\theta_e$. Similarly, $y\tau_e = ey^\circ = e(y^\circ)^\circ = y^\circ\tau_e$ for all $y \in \Lambda$. Finally, let $x \in I^1$ and $u \in \Lambda^1$. Then by Lemma 3.4,

$$u\sigma_{x\theta_e} = u\sigma_{e^\circ x} = u(e^\circ x) = u^\circ(e^\circ x^\circ) = u^\circ e^\circ x^\circ$$

and

$$u(\tau_e\rho_x\theta_e) = e^\circ(x(eu^\circ)) = e^\circ(x(eu^\circ)^\circ) = e^\circ(xe^\circ u^\circ) = e^\circ xu^\circ = e^\circ x^\circ u^\circ = u^\circ e^\circ x^\circ.$$

This implies that $\sigma_{x\theta_e} = \tau_e\rho_x\theta_e$ for all $x \in I^1$. Similarly, $\rho_{y\tau_e} = \theta_e\sigma_y\tau_e$ for all $y \in \Lambda^1$. Thus $(\theta_e, \tau_e) \in C_{(I,\Lambda,E^\circ)}$. Dually, $(\eta_f, \xi_f) \in C_{(I,\Lambda,E^\circ)}$. If $e \in E^\circ$, then $e^\circ = e$ and so

$$x\theta_e = e^\circ x = e^\circ x^\circ = ex^\circ = x^\circ e = x\eta_e$$

for all $x \in I^1$. This shows that $\theta_e = \eta_e$. Dually, $\tau_e = \xi_e$. □

Lemma 3.8. *$C_{(I,\Lambda,E^\circ)}$ is a subsemigroup of $\mathcal{O}(I^1 \to \Lambda) \times \mathcal{O}(\Lambda^1 \to I)$.*

Proof. Denote $C = C_{(I,\Lambda,E^\circ)}$. We have seen that C is non-empty by Lemma 3.7. Let $(\alpha, \beta), (\gamma, \delta) \in C$. Then

$$(\alpha, \beta)(\gamma, \delta) = (\alpha \diamond \gamma, \delta \star \beta).$$

We first show that $\sigma_{x(\alpha\diamond\gamma)} \leq (\delta \star \beta)\rho_x(\alpha \diamond \gamma)$ for all $x \in I^1$. Let $u \in \Lambda^1$ and $x \in I^1$. Then $u\delta \in I$ and $(u\delta)^\circ \in I \cap \Lambda$. Since $\sigma_{x\alpha} \leq \beta\rho_x\alpha$ by the fact that $(\alpha, \beta) \in C$, we have $(u\delta)^\circ\sigma_{x\alpha} \leq (u\delta)^\circ\beta\rho_x\alpha$ whence $(u\delta)^\circ(x\alpha) \leq (x((u\delta)^\circ\beta))\alpha$. In view of (4) and Lemma 3.4, we can obtain that

$$(x\alpha)^\circ(u\delta) = (x\alpha)^\circ(u\delta)^\circ = ((u\delta)^\circ(x\alpha))^\circ \leq ((x((u\delta)^\circ\beta))\alpha)^\circ.$$

Since γ is order-preserving, it follows that

$$u(\delta\rho_{(x\alpha)^\circ}\gamma) = ((x\alpha)^\circ(u\delta))\gamma \leq (((x((u\delta)^\circ\beta))\alpha)^\circ)\gamma = u(\delta \star \beta)\rho_x(\alpha \diamond \gamma).$$

This shows that $\delta\rho_{(x\alpha)^\circ}\gamma \leq (\delta \star \beta)\rho_x(\alpha \diamond \gamma)$. Since $\sigma_{v\gamma} \leq \delta\rho_v\gamma$ for all $v \in I^1$ by the fact $(\gamma, \delta) \in C$ and $(x\alpha)^\circ \in I$, it follows that

$$\sigma_{x(\alpha\diamond\gamma)} = \sigma_{(x\alpha)^\circ\gamma} \leq \delta\rho_{(x\alpha)^\circ}\gamma \leq (\delta \star \beta)\rho_x(\alpha \diamond \gamma).$$

Finally, let $x \in I$. Since $x^\circ\alpha = x\alpha$ by the fact that $(\alpha, \beta) \in C$, we get

$$x(\alpha \diamond \gamma) = (x\alpha)^\circ\gamma = (x^\circ\alpha)^\circ\gamma = x^\circ(\alpha \diamond \gamma).$$

By symmetry, we can obtain $\rho_{y(\delta\star\beta)} \leq (\alpha \diamond \gamma)\sigma_y(\delta \star \beta)$ and $y^\circ(\delta \star \beta) = y(\delta \star \beta)$ for all $y \in \Lambda^1$. Thus $(\alpha, \beta)(\gamma, \delta) \in C$. So C is a subsemigroup. □

Now, we are in a position to state our main result of this section.

Theorem 3.9. *Define three unary operations on the semigroup $C = C_{(I,\Lambda,E^\circ)}$ as follows:*

$$(\alpha, \beta)^+ = (\alpha^+, \beta^+), (\alpha, \beta)^* = (\alpha^*, \beta^*), \overline{(\alpha, \beta)} = (\overline{\alpha}, \overline{\beta}),$$

where

$$\overline{\alpha} : I^1 \to \Lambda, x \mapsto (x\alpha)^\circ, \quad \overline{\beta} : \Lambda^1 \to I, x \mapsto (x\beta)^\circ,$$
$$\alpha^+ = \theta_{1\beta} : I^1 \to \Lambda, x \mapsto (1\beta)^\circ x, \quad \beta^+ = \tau_{1\beta} : \Lambda^1 \to I, x \mapsto (1\beta)x^\circ$$
$$\alpha^* = \eta_{1\alpha} : I^1 \to \Lambda, x \mapsto x^\circ(1\alpha) \quad \beta^* = \xi_{1\alpha} : \Lambda^1 \to I, x \mapsto x(1\alpha)^\circ.$$

Then $(C, \cdot, {}^+, {}^*, {}^-)$ is a generalized Ehresmann semigroup.

Proof. Let $(\alpha, \beta) \in C$. By Lemma 3.7 and the fact that $1\beta \in I$, $1\alpha \in \Lambda$, it follows that

$$(\alpha^+, \beta^+) = (\theta_{1\beta}, \tau_{1\beta}) \in C, (\alpha^*, \beta^*) = (\eta_{1\alpha}, \xi_{1\alpha}) \in C. \tag{5}$$

This shows that " $+$ " and " $*$ " are well-defined.

Now, let $(\alpha, \beta) \in C$. Then

$$x^\circ\alpha = x\alpha, y^\circ\beta = y\beta, \sigma_{x\alpha} \leq \beta\rho_x\alpha, \rho_{y\beta} \leq \alpha\sigma_y\beta$$

for all $x \in I^1$ and $y \in \Lambda^1$. Let $x, y \in I^1$ and $x \leq y$. Since α is order-preserving, $x\alpha \leq y\alpha$. It follows that $x\overline{\alpha} = (x\alpha)^\circ \leq (y\alpha)^\circ = y\overline{\alpha}$ by (4). This shows that $\overline{\alpha}$ is order-preserving. Dually, $\overline{\beta}$ is also order-preserving. Moreover, for all $x \in I^1$, we have

$$x\overline{\alpha} = (x\alpha)^\circ = (x^\circ\alpha)^\circ = x^\circ\overline{\alpha}$$

as $x^\circ\alpha = x\alpha$ for all $x \in I^1$. Dually, we have $y\overline{\beta} = y^\circ\overline{\beta}$ for all $y \in \Lambda^1$. Now let $x \in I^1$. For $u \in \Lambda^1$, since $\sigma_{x\alpha} \leq \beta\rho_x\alpha$, we have $u\sigma_{x\alpha} \leq u\beta\rho_x\alpha$. That is, $u(x\alpha) \leq (x(u\beta))\alpha$. By (4), $(u(x\alpha))^\circ \leq ((x(u\beta))\alpha)^\circ$. This implies that

$$u\sigma_{x\overline{\alpha}} = u(x\overline{\alpha}) = u(x\alpha)^\circ = u^\circ(x\alpha)^\circ = (u(x\alpha))^\circ \leq ((x(u\beta))\alpha)^\circ$$
$$= ((x(u\beta)^\circ)\alpha)^\circ = (x(u\beta)^\circ)\overline{\alpha} = (u\beta)^\circ(\rho_x\overline{\alpha}) = u(\overline{\beta}\rho_x\overline{\alpha})$$

by Lemma 3.4. This shows that $\sigma_{x\overline{\alpha}} \leq \overline{\beta}\rho_x\overline{\alpha}$ for all $x \in I^1$. Dually, $\rho_{y\overline{\beta}} \leq \overline{\alpha}\sigma_y\overline{\beta}$ for all $y \in \Lambda^1$. Thus $(\overline{\alpha}, \overline{\beta}) \in C$. This implies that " $-$ " is also well-defined.

We next show that the identities (1-8) in Table 1 are satisfied. By symmetry, the identities $(1)'$-$(6)'$ also hold. Let $(\alpha,\beta),(\gamma,\delta)\in C$.

(1) For $x\in I^1$, we have $\sigma_{x\alpha}\le\beta\rho_x\alpha$ by the fact that $(\alpha,\beta)\in C$. So

$$x\alpha=1\sigma_{x\alpha}\le 1(\beta\rho_x\alpha)=(x(1\beta))\alpha.$$

Since I is a left normal band, we have $x(1\beta)\le x$. This shows that $(x(1\beta))\alpha\le x\alpha$ as α is order-preserving. So $x\alpha=(x(1\beta))\alpha$. Observe that $v^\circ\alpha=v\alpha$ for all $v\in I^1$ by the fact that $(\alpha,\beta)\in C$, it follows that $(x(1\beta))\alpha=(x(1\beta))^\circ\alpha$. Thus

$$x\alpha=(x(1\beta))\alpha=(x(1\beta))^\circ\alpha=(x^\circ(1\beta)^\circ)\alpha=((1\beta)^\circ x)^\circ\alpha=x(\alpha^+\diamond\alpha) \tag{6}$$

by lemma 3.4. We have shown that $\alpha^+\diamond\alpha=\alpha$. Dually, $\beta\star\beta^+=\beta$. Thus

$$(\alpha,\beta)^+(\alpha,\beta)=(\alpha^+,\beta^+)(\alpha,\beta)=(\alpha^+\diamond\alpha,\beta\star\beta^+)=(\alpha,\beta).$$

(2) For $x\in I^1$, by Lemma 3.4 we have

$$x(\alpha^+\diamond\gamma^+)=(x\alpha^+)^\circ\gamma^+=((1\beta)^\circ x)^\circ\gamma^+=((1\beta)^\circ x^\circ)\gamma^+=(1\delta)^\circ((1\beta)^\circ x^\circ)=(1\delta)^\circ(1\beta)^\circ x^\circ \tag{7}$$

and

$$\begin{aligned}x((\alpha^+\diamond\gamma^+)\diamond\alpha^+)&=(x(\alpha^+\diamond\gamma^+))^\circ\alpha^+=((1\delta)^\circ(1\beta)^\circ x^\circ)^\circ\alpha^+\\&=((1\delta)^\circ(1\beta)^\circ x^\circ)\alpha^+=(1\beta)^\circ(1\delta)^\circ(1\beta)^\circ x^\circ=(1\delta)^\circ(1\beta)^\circ x^\circ.\end{aligned}$$

This implies that $\alpha^+\diamond\gamma^+=\alpha^+\diamond\gamma^+\diamond\alpha^+$. Dually, $\beta^+\star\delta^+\star\beta^+=\delta^+\star\beta^+$. Thus

$$(\alpha,\beta)^+(\gamma,\delta)^+(\alpha,\beta)^+=(\alpha,\beta)^+(\gamma,\delta)^+.$$

(3) For all $x\in I^1$, we have $x(\alpha^+\diamond\gamma^+)=(1\delta)^\circ(1\beta)^\circ x^\circ$ by (7) and

$$x(\alpha^+\diamond\gamma^+)^+=(1(\delta^+\star\beta^+))^\circ x=((1\beta)(1\delta)^\circ)^\circ x=((1\delta)^\circ(1\beta)^\circ)x=(1\delta)^\circ(1\beta)^\circ x^\circ$$

by Lemma 3.4. This yields that $\alpha^+\diamond\gamma^+=(\alpha^+\diamond\gamma^+)^+$. Dually, $(\delta^+\star\beta^+)^+=\delta^+\star\beta^+$. Thus $((\alpha,\beta)^+(\gamma,\delta)^+)^+=(\alpha,\beta)^+(\gamma,\delta)^+$.

(4) For all $x\in I^1$, we have $x(\alpha\diamond\gamma)^+=(1(\delta\star\beta))^\circ x=((1\delta)^\circ\beta)^\circ x$ and

$$x(\alpha\diamond\gamma^+)^+=(1(\delta^+\star\beta))^\circ x=((1\delta^+)^\circ\beta)^\circ x=((1\delta)^\circ\beta)^\circ x.$$

This implies that $(\alpha\diamond\gamma)^+=(\alpha\diamond\gamma^+)^+$. Dually, $(\delta\star\beta)^+=(\delta^+\star\beta)^+$. Thus

$$((\alpha,\beta)(\gamma,\delta))^+=((\alpha,\beta)(\gamma,\delta)^+)^+.$$

(5) For all $x\in I^1$, by Lemma 3.4 we have

$$x(\alpha^+)^*=x^\circ(1\alpha^+)=x^\circ(1\beta)^\circ=(1\beta)^\circ x^\circ=(1\beta)^{\circ\circ}x^\circ=(1\overline{\beta})^\circ x^\circ=(1\overline{\beta})^\circ x=x\overline{\alpha}^+. \tag{8}$$

This shows that $(\alpha^+)^*=\overline{\alpha}^+$. Dually, $(\beta^+)^*=\overline{\beta}^+$. Thus $((\alpha,\beta)^+)^*=\overline{(\alpha,\beta)}^+$.

(6) For all $x\in I^1$, by Lemma 3.4 and (8) we have

$$x\overline{\alpha}^+=(1\beta)^\circ x^\circ=((1\beta)^\circ x)^\circ=(x\alpha^+)^\circ=x\overline{\alpha^+}.$$

So $\overline{\alpha}^+=\overline{\alpha^+}$. Dually, $\overline{\beta}^+=\overline{\beta^+}$. Thus $\overline{(\alpha,\beta)}^+=\overline{(\alpha,\beta)^+}$.

(7) For all $x\in I^1$, by Lemma 3.4 and (6), we have

$$x(\alpha^+\diamond\overline{\alpha}\diamond\alpha^*)=(((1\beta)^\circ x^\circ)\alpha)^\circ(1\alpha)=(((1\beta)^\circ x)^\circ\alpha)^\circ(1\alpha)=(x\alpha)^\circ(1\alpha)=(x\alpha)(1\alpha).$$

Since $x \leq 1$ and α is order-preserving, we have $(x\alpha)(1\alpha) = x\alpha$. This shows that $x(\alpha^+ \diamond \overline{\alpha} \diamond \alpha^*) = x\alpha$. So $\alpha^+ \diamond \overline{\alpha} \diamond \alpha^* = \alpha$. Dually, $\beta^* \star \overline{\beta} \star \beta^+ = \beta$. Thus

$$(\alpha, \beta) = (\alpha, \beta)^+ \overline{(\alpha, \beta)} (\alpha, \beta)^*.$$

(8) For all $x \in I^1$, by Lemma 3.4 we have

$$x(\alpha^* \diamond \gamma^+) = (x\alpha^*)^\circ \gamma^+ = (x^\circ(1\alpha))^\circ \gamma^+ = (1\delta)^\circ x^\circ (1\alpha)^\circ$$

and

$$x(\overline{\alpha}^* \diamond \overline{\gamma}^+) = (x\overline{\alpha}^*)^\circ \overline{\gamma}^+ = (x^\circ(1\overline{\alpha}))^\circ \overline{\gamma}^+$$
$$= (x^\circ(1\alpha)^\circ)^\circ \overline{\gamma}^+ = (x^\circ(1\alpha)^\circ)\overline{\gamma}^+ = (1\overline{\delta})^\circ x^\circ (1\alpha)^\circ = (1\delta)^\circ x^\circ (1\alpha)^\circ.$$

This shows that $\alpha^* \diamond \gamma^+ = \overline{\alpha}^* \diamond \overline{\gamma}^+$. Dually, $\delta^+ \star \beta^* = \overline{\delta}^+ \star \overline{\beta}^*$. Thus

$$(\alpha, \beta)^* (\gamma, \delta)^+ = \overline{(\alpha, \beta)}^* \, \overline{(\gamma, \delta)}^+.$$

We have shown that $(C, \cdot, ^+, ^*, ^-)$ is a generalized Ehresmann semigroup. □

Remark 3.10. *In the case that $I = \Lambda = E^\circ$ is a semilattice, the above semigroup $C_{(I,\Lambda,E^\circ)}$ is exactly the Ehresmann semigroup C_{E° constructed in [6].*

Corollary 3.11. *On the semigroup $(C, \cdot, ^+, ^*, ^-)$, we have*

$$I_C = \{(\theta_e, \tau_e) | e \in I\}, \Lambda_C = \{(\eta_f, \xi_f) | f \in \Lambda\}, E_C^\circ = \{(\theta_e, \tau_e) | e \in E^\circ\}.$$

Proof. In view of (5), it follows that

$$(\alpha, \beta)^+ = (\alpha^+, \beta^+) = (\theta_{1\beta}, \tau_{1\beta})$$

for all $(\alpha, \beta) \in C$. This gives $I_C \subseteq \{(\theta_e, \tau_e) | e \in I\}$. Conversely, for $e \in I$, we have $(\theta_e, \tau_e) \in C$ by Lemma 3.7. So

$$(\theta_e, \tau_e) = (\theta_{1\tau_e}, \tau_{1\tau_e}) = (\theta_e, \tau_e)^+ \in I_C.$$

Thus $I_C = \{(\theta_e, \tau_e) | e \in I\}$. Dually, $\Lambda_C = \{(\eta_f, \xi_f) | f \in \Lambda\}$.

For $(\alpha, \beta) \in C$, we have $1\beta \in I$ and $(1\beta)^\circ \in E^\circ$. By (5), we have

$$\overline{(\alpha, \beta)}^+ = (\overline{\alpha}, \overline{\beta})^+ = (\overline{\alpha}^+, \overline{\beta}^+) = (\theta_{1\overline{\beta}}, \tau_{1\overline{\beta}}) = (\theta_{(1\beta)^\circ}, \tau_{(1\beta)^\circ}) \in \{(\theta_e, \tau_e) | e \in E^\circ\}.$$

This shows that $E_C^\circ \subseteq \{(\theta_e, \tau_e) | e \in E^\circ\}$. Conversely, for $e \in E^\circ$, by (5) again we have

$$(\theta_e, \tau_e) = (\theta_{e^\circ}, \tau_{e^\circ}) = (\theta_{(1\tau_e)^\circ}, \tau_{(1\tau_e)^\circ})$$
$$= (\theta_{1\overline{\tau_e}}, \tau_{1\overline{\tau_e}}) = (\overline{\theta_e}^+, \overline{\tau_e}^+) = (\overline{\theta_e}, \overline{\tau_e})^+ = \overline{(\theta_e, \tau_e)}^+ \in E_C^\circ.$$

Thus $E_C^\circ = \{(\theta_e, \tau_e) | e \in E^\circ\}$. □

We say that two admissible triples (I, Λ, E°) and (J, Π, F°) are *isomorphic* if there exist an isomorphism φ from I onto J and an isomorphism ψ from Λ onto Π such that

$$\varphi|_{E^\circ} = \psi|_{E^\circ}, E^\circ\varphi = F^\circ.$$

If this is the case, then one can easily show that $C_{(I,\Lambda,E^\circ)}$ is (2,1,1,1)-isomorphic to $C_{(J,\Pi,F^\circ)}$. Moreover, we have the following.

Corollary 3.12. *Let (I, Λ, E°) be an admissible triple. Then (I, Λ, E°) is isomorphic to the admissible triple $(I_C, \Lambda_C, E_C^\circ)$ of $C = C_{(I,\Lambda,E^\circ)}$.*

Proof. By Corollary 3.11, we can define the following surjective mappings

$$\varphi : I \to I_C, e \mapsto (\theta_e, \tau_e), \quad \psi : \Lambda \to \Lambda_C, f \mapsto (\eta_f, \xi_f).$$

On the other hand, if $e, g \in I$ and $(\theta_e, \tau_e) = (\theta_g, \tau_g)$, then $\tau_e = \tau_g$ and so $e = 1\tau_e = 1\tau_g = g$. Thus φ is also injective. Finally, let $e, g \in I$ and $x \in I^1$. Then we have

$$x(\theta_e \diamond \theta_g) = g^\circ(e^\circ x)^\circ = g^\circ e^\circ x^\circ = (eg)^\circ x^\circ = (eg)^\circ x = \theta_{eg}$$

by Lemma 3.4. This shows that $\theta_e \diamond \theta_g = \theta_{eg}$. Dually, $\tau_g \star \tau_e = \tau_{eg}$. Thus

$$(eg)\varphi = (\theta_{eg}, \tau_{eg}) = (\theta_e \diamond \theta_g, \tau_g \star \tau_e) = (\theta_e, \tau_e)(\theta_g, \tau_g) = (e\varphi)(g\varphi).$$

This implies that φ is a morphism. Dually, ψ is also a morphism. Furthermore, by Lemma 3.7 and Corollary 3.11, we can see that $\varphi|_{E^\circ} = \psi|_{E^\circ}$ and $E^\circ\varphi = E^\circ_C$. □

Corollary 3.13. *Let (I, Λ, E°) be an admissible triple and D a (2,1,1,1)-subalgebra of $C = C_{(I,\Lambda,E^\circ)}$ containing E°_C. Then D is fundamental. In particular, C itself is fundamental.*

Proof. Let D be a (2,1,1,1)–subalgebra of C and $(\alpha, \beta), (\gamma, \delta) \in D$ such that (α, β) is μ_D-related to (γ, δ). Then $(\alpha, \beta)^+ = (\gamma, \delta)^+$ and $(\alpha, \beta)^* = (\gamma, \delta)^*$ whence $\alpha^+ = \gamma^+, \beta^+ = \delta^+$ and $\alpha^* = \gamma^*, \beta^* = \delta^*$. By $\alpha^* = \gamma^*$, we get $1\alpha = 1\alpha^* = 1\gamma^* = 1\gamma$. Now let $x \in I$. Then $x^\circ \in E^\circ$. By Corollary 3.11, $(\theta_{x^\circ}, \tau_{x^\circ}) \in E^\circ_C \subseteq D$. Since μ_D is a semigroup congruence on D, it follows that

$$(\theta_{x^\circ} \diamond \alpha, \beta \star \tau_{x^\circ}) = (\theta_{x^\circ}, \tau_{x^\circ})(\alpha, \beta)\ \mu_D\ (\theta_{x^\circ}, \tau_{x^\circ})(\gamma, \delta) = (\theta_{x^\circ} \diamond \gamma, \delta \star \tau_{x^\circ})$$

whence $(\theta_{x^\circ} \diamond \alpha)^* = (\theta_{x^\circ} \diamond \gamma)^*$. This implies that

$$x^\circ\alpha = (x^\circ)^\circ\alpha = (1\theta_{x^\circ})^\circ\alpha = 1(\theta_{x^\circ} \diamond \alpha) = 1(\theta_{x^\circ} \diamond \alpha)^*$$

$$= 1(\theta_{x^\circ} \diamond \gamma)^* = 1(\theta_{x^\circ} \diamond \gamma) = (1\theta_{x^\circ})^\circ\gamma = (x^\circ)^\circ\gamma = x^\circ\gamma.$$

On the other hand, we have $x\alpha = x^\circ\alpha$ and $x\gamma = x^\circ\gamma$ by the fact $(\alpha, \beta), (\gamma, \delta) \in C$. This implies that $x\alpha = x\gamma$. Thus $\alpha = \gamma$. Dually, $\beta = \delta$. Therefore μ_D is the identity relation on D. That is, D is fundamental. □

4 A representation of generalized Ehresmann semigroups

In this section, we always assume that $(S, \cdot, ^+, ^*, ^-)$ is a generalized Erhesmann semigroup. Then we have the admissible triple $(I_S, \Lambda_S, E^\circ_S)$ of S and the semigroup $C_{(I_S,\Lambda_S,E^\circ_S)}$ by Remark 3.3 and Theorem 3.9. The aim of this section is to show that there exists a $(2, 1, 1, 1)$- homomorphism $\Phi : S \to C_{(I_S,\Lambda_S,E^\circ_S)}$ whose kernel is μ_S. *To accommodate with the notations of Section 3, we use the notations from Section 3 for the admissible triple $(I_S, \Lambda_S, E^\circ_S)$ throughout this section.*

We first consider some properties of the admissible triple $(I_S, \Lambda_S, E^\circ_S)$ of S and the semigroup $C_{(I_S,\Lambda_S,E^\circ_S)}$. Denote $C_{(I_S,\Lambda_S,E^\circ_S)}$ by C for convenience. In view of Remark 3.3, in the admissible triple $(I_S, \Lambda_S, E^\circ_S)$, for all $i \in I_S$ and $\lambda \in \Lambda_S$, we have

$$i^\circ = i^* \text{ and } \lambda^\circ = \lambda^+.$$

For $a \in S$, there are functions

$$\alpha_a : I^1_S \to \Lambda_S, \quad \beta_a : \Lambda^1_S \to I_S$$

given by

$$x\alpha_a = (xa)^*, \quad x\beta_a = (ax)^+.$$

Lemma 4.1. *With above notation, we have the following results:*
(a) For all $a \in S$, $\alpha_a \in \mathcal{O}(I^1_S \to \Lambda_S)$ and $\beta_a \in \mathcal{O}(\Lambda^1_S \to I_S)$.

(b) *For all* $x \in I_S^1$ *and* $y \in \Lambda_S^1$, $x\alpha_a = x^\circ\alpha_a$ *and* $y\beta_a = y^\circ\beta_a$.
(c) *For all* $a \in S$ *and* $x \in I_S^1$ *and* $y \in \Lambda_S^1$, $\sigma_{x\alpha_a} \leq \beta_a\rho_x\alpha_a$ *and* $\rho_{y\beta_a} \leq \alpha_a\sigma_y\beta_a$.
(d) *For all* $a, b \in S$ *and* $x \in I_S^1$ *and* $y \in \Lambda_S^1$, $(x\alpha_a)^\circ\alpha_b = x\alpha_{ab}$ *and* $(y\beta_b)^\circ\beta_a = y\beta_{ab}$.

Proof. (a) Let $x, y \in I_S^1$ with $x \leq y$. Then $xy = yx = x$. It follows that $x\alpha_a = (xa)^* = (xya)^* \leq (ya)^* = y\alpha_a$ by Lemma 2.7 (b), whence $\alpha_a \in \mathcal{O}(I_S^1 \to \Lambda_S)$. Dually, $\beta_a \in \mathcal{O}(\Lambda_S^1 \to I_S)$.

(b) Observe that $x^\circ = x^*$ for all $x \in I_S$, it follows that

$$x\alpha_a = (xa)^* = (x^*a)^* = x^*\alpha_a = x^\circ\alpha_a$$

by the identity $(4)'$ in Table 1. Moreover, since $1^\circ = 1$, we have $1\alpha_a = 1^\circ\alpha_a$. Thus $x\alpha_a = x^\circ\alpha_a$ for all $x \in I_S^1$. Dually, $y\beta_a = y^\circ\beta_a$ for all $y \in \Lambda_S^1$.

(c) Let $u \in \Lambda_S^1$. Then

$$u\sigma_{x\alpha_a} = u(x\alpha_a) = u(xa)^* \in \Lambda_S, \quad u(\beta_a\rho_x\alpha_a) = (x(au)^+a)^* \in \Lambda_S.$$

By Lemma 2.7 (a),

$$u(xa)^*a^* = u(xa)^*, \quad (x(au)^+a)^*a^* = (x(au)^+a)^*.$$

Since Λ_S is a right normal band, we have

$$u(xa)^* \cdot (x(au)^+a)^* = u(xa)^* \cdot (x(au)^+a)^* \cdot a^*$$

$$= (x(au)^+a)^* \cdot u(xa)^* \cdot a^* = (x(au)^+a)^* \cdot u(xa)^*.$$

On the other hand, since $u(xa)^* \in \Lambda_S$, we obtain that $(u(xa)^*)^* = u(xa)^*$ by Lemma 2.6 (a). Using the identity $(3)'$, $(4)'$, (1), $(4)'$ and the fact that Λ_S is a right normal band successively, we get

$$(x(au)^+a)^* \cdot u(xa)^* = (x(au)^+a)^* \cdot (u(xa)^*)^* = ((x(au)^+a)^* \cdot (u(xa)^*)^*)^* =$$

$$(x(au)^+au(xa)^*)^* = (xau(xa)^*)^* = ((xa)^*u(xa)^*)^* = (u(xa)^*)^* = u(xa)^*.$$

Thus

$$u\sigma_{x\alpha_a} = u(xa)^* \leq (x(au)^+a)^* = u(\beta_a\rho_x\alpha_a)$$

for all $u \in \Lambda_S^1$. That is, $\sigma_{x\alpha_a} \leq \beta_a\rho_x\alpha_a$. Dually, $\rho_{y\beta_a} \leq \alpha_a\sigma_y\beta_a$.

(d) For $x \in I_S^1$, we have $x\alpha \in \Lambda_S$ and $(x\alpha)^\circ = (x\alpha_a)^+ = ((xa)^*)^+$. Using Lemma 2.7 (a), the identities (6), $(6)'$, Lemma 2.6 (a), (b), the identity $(4)'$ and Lemma 2.7 (a) in that order, we have

$$(x\alpha_a)^\circ\alpha_b = (((xa)^*)^+)\alpha_b = (((xa)^*)^+b)^* = (\overline{((xa)^*)^+}\overline{b})^*b^*$$

$$= (\overline{(xa)^*}^+\overline{b})^*b^* = ((\overline{xa}^*)^+\overline{b})^*b^* = (\overline{xa}^*\overline{b})^*b^* = (\overline{xa}\overline{b})^*b^* = (xab)^* = x\alpha_{ab}.$$

Dually, $(y\beta_b)^\circ\beta_a = y\beta_{ab}$. □

Now we can state our main result in this section.

Theorem 4.2. *Define* $\Phi : S \to C, a \mapsto (\alpha_a, \beta_a)$. *Then* Φ *is a* $(2, 1, 1, 1)$*-homomorphism whose kernel is* μ_S. *Moreover,*
(a) $\Phi|_{I_S}$ *is a* $(2, 1, 1, 1)$*-isomorphism from* I_S *to* I_C.
(b) $\Phi|_{\Lambda_S}$ *is a* $(2, 1, 1, 1)$*-isomorphism from* Λ_S *to* Λ_C.
(c) $\Phi|_{E_S^\circ}$ *is a* $(2, 1, 1, 1)$*-isomorphism from* E_S° *to* E_C°.

Proof. By (a), (b) and (c) of Lemma 4.1, $(\alpha_a, \beta_a) \in C$ for all $a \in S$. Observe that $(\alpha_a, \beta_a)(\alpha_b, \beta_b) = (\alpha_a \diamond \alpha_b, \beta_b \star \beta_a)$ in C and

$$x(\alpha_a \diamond \alpha_b) = (x\alpha_a)^\circ \alpha_b = x\alpha_{ab}, y(\beta_b \star \alpha_a) = (y\beta_b)^\circ \beta_a = y\beta_{ab}$$

for all $x \in I_S^1$ and $y \in \Lambda_S^1$ by Lemma 4.1 (d). It follows that

$$(a\Phi)(b\Phi) = (\alpha_a, \beta_a)(\alpha_b, \beta_b) = (\alpha_{ab}, \beta_{ab}) = (ab)\Phi.$$

Thus Φ preserves the binary operation.

Let $x \in I_S^1$. Then

$$x\alpha_{a^+} = (xa^+)^* = (xa^+)^\circ = x^\circ (a^+)^\circ = (a^+)^\circ x^\circ = (a^+)^\circ x = (1\beta_a)^\circ x = x\alpha_a^+$$

by Lemma 3.4. This shows that $\alpha_{a^+} = \alpha_a^+$. Dually, $\beta_{a^+} = \beta_a^+$. So

$$a^+\Phi = (\alpha_{a^+}, \beta_{a^+}) = (\alpha_a^+, \beta_a^+) = (\alpha_a, \beta_a)^+. \tag{9}$$

This shows that Φ preserves " $+$ ". Dually, Φ preserves " $*$ ".

Let $x \in I_S^1$. If $x = 1$, then

$$x\alpha_{\overline{a}} = 1\alpha_{\overline{a}} = \overline{a}^* = (a^*)^+ = (1\alpha_a)^+ = (1\alpha_a)^\circ = 1\overline{\alpha_a} = x\overline{\alpha_a}$$

by the identity $(5)'$ in Table 1. If $x \in I_S$, using the identity $(4)'$, Lemma 2.6 (a), Lemma 2.7 (c), the identity $(5)'$ in that order, we have

$$x\alpha_{\overline{a}} = (x\overline{a})^* = (x^*\overline{a})^* = (\overline{x}\,\overline{a})^* = \overline{xa}^* = ((xa)^*)^+ = ((xa)^*)^\circ = (x\alpha_a)^\circ = x\overline{\alpha_a}.$$

This shows that $\alpha_{\overline{a}} = \overline{\alpha_a}$. Dually, $\beta_{\overline{a}} = \overline{\beta_a}$. So

$$\overline{a}\Phi = (\alpha_{\overline{a}}, \beta_{\overline{a}}) = (\overline{\alpha_a}, \overline{\beta_a}) = \overline{(\alpha_a, \beta_a)} = \overline{a\Phi}.$$

Thus Φ preserves " $-$ ". We have proved that Φ is a $(2,1,1,1)$-homomorphism and so

$$\ker\Phi = \{(a,b) \in S \times S \mid \alpha_a = \alpha_b, \beta_b = \beta_a\}$$

is a $(2,1,1,1)$-congruence on S. If $(a,b) \in \ker\Phi$, then we have $a^* = 1\alpha_a = 1\alpha_b = b^*$. Dually, $a^+ = b^+$. This shows that

$$\ker\Phi \subseteq \{(a,b) \in S \times S \mid a^+ = b^+, a^* = b^*\}.$$

Let σ be a semigroup congruence on S and $a\sigma b$ such that

$$\sigma \subseteq \{(a,b) \in S \times S \mid a^+ = b^+, a^* = b^*\}.$$

Then for all $x \in I_S^1$, we have $xa\sigma xb$, whence $x\alpha_a = (xa)^* = (xb)^* = x\alpha_b$. This shows that $\alpha_a = \alpha_b$. Dually, $\beta_a = \beta_b$. Thus $(a,b) \in \ker\Phi$. We have shown that $\ker\Phi$ is the largest semigroup congruence contained in

$$\{(a,b) \in S \times S \mid a^+ = b^+, a^* = b^*\}.$$

That is, $\mu_S = \ker\Phi$.

(a) Since $I_S = \{a^+ \mid a \in S\}$ and

$$a^+\Phi = (\alpha_a, \beta_a)^+ \in I_C = \{(\alpha,\beta)^+ \mid (\alpha,\beta) \in C\}$$

by (9), it follows that $I_S\Phi \subseteq I_C$. Now, let $(\alpha,\beta)^+ = (\alpha^+, \beta^+) \in I_C$ where $(\alpha,\beta) \in C$. Then $1\beta \in I_S$. For $x \in I_S^1$, we have

$$x\alpha_{1\beta} = (x(1\beta))^* = (x(1\beta))^\circ = x^\circ(1\beta)^\circ = (1\beta)^\circ x^\circ = (1\beta)^\circ x = x\alpha^+$$

by Lemma 3.4, which implies that $\alpha_{1\beta} = \alpha^+$. Dually, $\beta_{1\beta} = \beta^+$. Thus

$$(1\beta)\Phi = (\alpha_{1\beta}, \beta_{1\beta}) = (\alpha^+, \beta^+) = (\alpha, \beta)^+.$$

This gives $I_S\Phi = I_C$ and so $\Phi|_{I_S}$ is surjective. If $i, j \in I_S$ and $i\Phi = j\Phi$, then $(i, j) \in \ker\Phi$, this implies $i = i^+ = j^+ = j$ by Lemma 2.6 (a) and the fact that

$$\ker\Phi \subseteq \{(a, b) \in S \times S | a^+ = b^+, a^* = b^*\}.$$

it follows that $\Phi|_{I_S}$ is also injective. In view of Lemma 2.6, I_S and I_C are (2,1,1,1)-subalgebras of S and C, respectively. By the first part of the theorem, $\Phi|_{I_S}$ is a (2, 1, 1, 1)-isomorphism from I_S to I_C.

(b) This is the dual of (a).

(c) This follows from items (a) and (b). □

A (2,1,1,1)-subalgebra D of a generalized Ehresmann semigroup $(S, \cdot, ^+, ^*, ^-)$ is called *quasi-full* if $I_S \cup \Lambda_S \subseteq D$. Combining Corollaries 3.12 and 3.13 and Theorem 4.2, we obtain the main result of this paper.

Theorem 4.3. *Let (I, Λ, E°) be a given admissible triple. Then $(S, \cdot, ^+, ^*, ^-)$ is a fundamental generalized Ehresmann semigroup whose admissible triple is isomorphic to (I, Λ, E°) if and only if it is (2,1,1,1)-isomorphic to a quasi-full (2,1,1,1)-subalgebra of $C_{(I,\Lambda,E^\circ)}$.*

Considering the case that $I = \Lambda = E^\circ$ is a semilattice, by Remark 3.10 we have the following corollary which is Theorem 3.2 in [6] substantially.

Corollary 4.4. *Let E be a given semilattice. Then $(S, \cdot, ^*, ^+)$ is a fundamental Ehresmann semigroup whose distinguished semilattice is isomorphic to E if and only if it is (2,1,1)-isomorphic to a (2,1,1)-subalgebra of C_E containing the distinguished semilattice of C_E .*

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