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Some applications of the Archimedean copulas in the proof of the almost sure central limit theorem for ordinary maxima

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Abstract: Our goal is to state and prove the almost sure central limit theorem for maxima (M_n) of $X_1, X_2, ..., X_n$, $n \in \mathbb{N}$, where (X_i) forms a stochastic process of identically distributed r.v.'s of the continuous type, such that, for any fixed n, the family of r.v.'s $(X_1, ..., X_n)$ has the Archimedean copula C^{Ψ}.

Keywords: Almost sure central limit theorems, ordinary maxima, Archimedean copulas, generator of copula, processes defined by Archimedean copulas

MSC: 60F15, 60F05, 60E05

1 Introduction and Preliminaries

Beginning from the celebrated papers by Brosamler [1] and Schatte [2], the almost sure versions of limit theorems have been studied by a large number of authors. These types of limit theorems are commonly known as the almost sure central limit theorems (ASCLTs). The following property is explored in the investigations concerning the ASCLTs. Namely, suppose that: $X_1, X_2, ..., X_i, ...$ are some r.v.'s, $f_1, f_2, ..., f_i, ...$ stand for some real-valued, measurable functions, defined on $\mathbb{R}, \mathbb{R}^2, ..., \mathbb{R}^i, ...$, respectively; we seek conditions under which the strong convergence below is satisfied for some nondegenerate cdf H

$$\lim_{N\to\infty} \frac{1}{D_N} \sum_{n=1}^{N} d_n I\left(f_n\left(X_1, ..., X_n\right) \le x\right) = H\left(x\right) \quad a.s. \text{ for all } x \in C_H, \tag{1}$$

where: $\{d_n\}$ is some sequence of weights, $D_N = \sum_{n=1}^{N} d_n$, I denotes the indicator function, and: $a.s.$, C_H stand for the almost sure convergence and the set of continuity points of function H, respectively.

The topics pertaining to the ASCLTs have attracted an immense attention since the publication of the two above mentioned papers and a great deal of works devoted to the proofs of (1) for various classes of functions f_n and random sequences (X_i) have appeared throughout the last two decades or so. We cite in this context the articles by: Berkes and Csáki [3], Chen and Lin [4], Cheng et al. [5], Csáki and Gonchigdanzan [6], Dudziński [7], Dudziński and Górka [8], Gonchigdanzan and Rempała [9], Ho and Hsing [10], Lacey and Philipp [11], Matuła [12], Mielniczuk [13], Peligrad and Shao [14], Stadtmüller [15], and Zhao et al. [16], among others. Functions f_n included different kinds of functions of r.v.'s, e.g.: partial sums (see: [3], [7], [11]-[14]), products of partial sums (see [9]), maxima (see: [3]-[6]), extreme order statistics (see [15]), maxima of sums (see: [3], [8]), and - jointly - maxima and sums as

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well (see [16]). It is worth noting that not only the indicator functions need to be considered with regard to this issue - see, e.g., Fazekas and Rychlik [17]; we discuss the functional almost sure central limit theorem in this case.

Our principal objective is to prove the property in (1) with: $d_n = 1/n$, $D_N \sim \log N$, $f_n(X_1, ..., X_n) = M_n$, where $M_n := \max(X_1, ..., X_n)$. The assumptions imposed in our assertions are strictly connected with the notions of the so-called Archimedean copulas and their generators. For this reason, we shall introduce some definitions and properties related to the copulas, and to the Archimedean copulas in particular. Let us start with a general definition of copula.

Definition 1.1. *A d-dimensional function C: $[0,1]^d \to [0,1]$, $d \geq 2$, defined on the unit cube $[0,1]^d$, is a d-dimensional copula if C is a joint cdf of a d-dimensional random vector with uniform-$[0,1]$ marginals, i.e.,*

$$C(v_1, v_2, ..., v_d) = P(V_1 \leq v_1, V_2 \leq v_2, ..., V_d \leq v_d) \quad \text{for any } v_i \in [0,1],$$

where all of the r.v.'s V_i, $i = 1, 2, ..., d$, have an uniform-$[0,1]$ cdf.

The theoretical groundwork for an area concerning the applications of copulas has been laid in the papers by Sklar [18]-[19], where the following celebrated claim has been stated among some other valuable results.

Theorem 1.2 (Sklar's theorem)**.** *For a given multivariate (joint) cdf F of a random vector $(X_1, ..., X_d)$ with marginal cdfs $F_1, ..., F_d$, $d \geq 2$, there exists a unique copula C satisfying*

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{2}$$

This copula is unique if the F_i's, $i = 1, ..., d$, are continuous.
Conversely, for a given copula C: $[0,1]^d \to [0,1]$ and the marginal cdfs $F_1, ..., F_d$, relation (2) defines a multivariate distribution of $(X_1, ..., X_d)$ with margins X_i having the cdfs F_i, $i = 1, ..., d$, respectively.

In view of Sklar's theorem, we may treat a copula as a structure describing the dependence between the coordinates of the random vector $(X_1, ..., X_d)$. Indeed, (2) means that C couples the marginal cdfs F_i into the joint cdf F. Simultaneously, due to Sklar's proposition, we are also able to decouple the dependence structure into the corresponding marginals.

In our investigations leading to the proof of the ASCLT for some order statistics, we are concerned with a special class of copulas, commonly known as the Archimedean copulas. Before we define the Archimedean copula, we will introduce the notion of copula's generator.

Definition 1.3. *Suppose that $d \geq 2$ and Ψ: $[0,1] \to [0,\infty]$ is a strictly decreasing, convex function satisfying the conditions $\Psi(0) = \infty$ and $\Psi(1) = 0$. Let for $v_i \in [0,1]$, $i = 1, ..., d$,*

$$C^{\Psi}(v_1, ..., v_d) = \Psi^{-1}\left(\sum_{i=1}^{d} \Psi(v_i)\right). \tag{3}$$

The function Ψ is called a generator of C^{Ψ}.

If $d \geq 3$, C^{Ψ} is, on the whole, not a copula. However, the following statement from Kimberling [20] gives a necessary and sufficient condition under which C^{Ψ} is a copula for all $d \geq 2$.

Theorem 1.4. *Choose $d \geq 2$. The function $C^{\Psi}(v_1, ..., v_d)$ in (3) is a copula iff a generator Ψ has an inverse Ψ^{-1}, which is completely monotonic on $[0,\infty)$, i.e.,*

$$(-1)^j \frac{d^j}{dz^j} \Psi^{-1}(z) \geq 0 \quad \text{for all } j \in \mathbb{N} \text{ and } z \in [0,\infty).$$

We are now in a position to define the class of Archimedean copulas.

Definition 1.5. *If Ψ^{-1} is completely monotonic on $[0,\infty)$, we say that C^{Ψ} given by (3) is the so-called Archimedean copula.*

In our research, we study the situation when the investigated sequence of r.v.'s (X_i) is a stochastic process defined as follows. Namely, we assume that, for every $i \in \mathbb{N}$, a r.v. X_i has a marginal cdf F of the continuous type and that, for any sequence $(t_1, t_2, ..., t_n)$, of natural numbers, the n-dimensional distribution of $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is defined by a certain Archimedean copula $C^{\Psi_n} = C^{\Psi}$ having a generator $\Psi_n = \Psi$, not depending on n. It means that, for any $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$,

$$P\left(X_{t_1} \le x_1, X_{t_2} \le x_2, ..., X_{t_n} \le x_n\right) = C^{\Psi}\left(F\left(x_1\right), ..., F\left(x_n\right)\right) = \Psi^{-1}\left(\sum_{i=1}^{n} \Psi\left(F\left(x_i\right)\right)\right), \tag{4}$$

where the mapping Ψ: $[0,1] \to [0,\infty]$ - called a generator of C^{Ψ} - is a strictly decreasing, convex function, satisfying $\Psi(0) = \infty$ and $\Psi(1) = 0$, whose inverse function Ψ^{-1} is completely monotonic on $[0,\infty]$, i.e., $(-1)^j \dfrac{d^j}{dz^j}\Psi^{-1}(z) \ge 0$ for all $j \in \mathbb{N}$ and any $z \in [0,\infty]$.

Then, it can be shown that, there exists a r.v. $\Theta > 0$ such that Ψ^{-1} is the Laplace transform of Θ, i.e.,

$$\Psi^{-1}(z) = E\left\{\exp\left(-\Theta \cdot z\right)\right\} \quad \text{for any } z \in [0,\infty], \tag{5}$$

where - here and throughout the whole paper - E denotes the expected value with respect to Θ.

We assume that, for any $x \in \mathbb{R}$ and $\theta \in \operatorname{supp}\Theta$,

$$P\left(X_i \le x \mid \Theta = \theta\right) = \left(G(x)\right)^{\theta}, \ i = 1, 2, ..., n, \tag{6}$$

where G satisfies

$$G(x) = \exp\left\{-\Psi\left(F(x)\right)\right\}. \tag{7}$$

It is known (see Marshall and Olkin [21] and Frees and Valdez [22]) that under the assumptions above, $X_1, ..., X_n$ are conditionally independent given Θ.

The purpose of our note is to prove the ASCLT for (M_n) - an appropriate sequence of the maxima among $X_1, ..., X_n$, $n \in \mathbb{N}$. Before we give the statement of our main result, we will introduce some additional conditions and notations. Thus, we also assume that, for some numerical sequence (u_n):

$$\lim_{n\to\infty} n\left(1 - F\left(u_n\right)\right) = \tau \quad \text{for some } \tau \in [0,\infty), \tag{8}$$

$$\limsup_{n\to\infty} n^{\beta}\Psi\left(F\left(u_n\right)\right) \le C \quad \text{for some } \beta \ge 2 \text{ and } C > 0, \tag{9}$$

and that

$$\mu := E(\Theta) < \infty. \tag{10}$$

The rest of our paper is structured as follows. In Section 2, we formulate our main result, which is the ASCLT for the ordinary maxima (M_n) obtained from the processes of identically distributed r.v.'s of the continuous type, such that the corresponding multidimensional distributions are determined by the Archimedean copula. In Section 3, some auxiliary results necessary for the proof of the established ASCLT are stated and proved. The complete proof of our ASCLT is given in Section 4. Additionally, in Section 5, some application of the basic claim is depicted.

2 Main result

Our principal assertion is the following ASCLT for (M_n) - an appropriate sequence of the maxima among $X_1, ..., X_n$, $n \in \mathbb{N}$.

Theorem 2.1. *Suppose that: (X_i) is a stationary sequence of identically distributed r.v.'s of the continuous type, with a common cdf F, and that, for any fixed $n \in \mathbb{N}$, the family of r.v.'s $(X_1, ..., X_n)$ has the Archimedean copula C^{Ψ}.*

Assume in addition that (X_i), a numerical sequence (u_n) and a generator Ψ of C^{Ψ} satisfy the conditions in (6)-(10), respectively. Then,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(M_n\le u_n\right)=e^{-\tau}\quad a.s.,$$

where τ is such as in (8).

3 Auxiliary results

In this section, we will state and prove some auxiliary results, which we make an extensive use of in the proof of our main result. The first of these results is the following claim.

Lemma 3.1. *Let the natural numbers m, n satisfy the condition $1\le m<n$ and M_n $(M_{m,n})$ denote a sequence of maxima among $X_1,...,X_n$ $(X_{m+1},...,X_n)$. Then, under the assumptions of Theorem 2.1,*

$$E\left|I\left(M_n\le u_n\right)-I\left(M_{m,n}\le u_n\right)\right|\le\frac{m}{n}.$$

Proof. First, observe that

$$\begin{aligned}E\left|I\left(M_n\le u_n\right)-I\left(M_{m,n}\le u_n\right)\right| &= P\left(M_{m,n}\le u_n\right)-P\left(M_n\le u_n\right)\\ &= EP\left(M_{m,n}\le u_n|\,\Theta\right)-EP\left(M_n\le u_n|\,\Theta\right),\end{aligned}\tag{11}$$

where Θ is a r.v. satisfying (5)-(6).

Due to (6) and the fact that $X_1,...,X_n$ are conditionally independent given Θ, we obtain:

$$EP\left(M_{m,n}\le u_n|\,\Theta\right)=E\left\{\left(G\left(u_n\right)\right)^{\Theta}\right\}^{n-m},\tag{12}$$

$$EP\left(M_n\le u_n|\,\Theta\right)=E\left\{\left(G\left(u_n\right)\right)^{\Theta}\right\}^{n}.\tag{13}$$

Consequently, by (11)-(13), we get

$$E\left|I\left(M_n\le u_n\right)-I\left(M_{m,n}\le u_n\right)\right|\le E\left[\left\{\left(G\left(u_n\right)\right)^{\Theta}\right\}^{n-m}-\left\{\left(G\left(u_n\right)\right)^{\Theta}\right\}^{n}\right].$$

This and the relation that $z^{n-m}-z^n\le m/n$, for all $z\in[0,1]$, imply

$$E\left|I\left(M_n\le u_n\right)-I\left(M_{m,n}\le u_n\right)\right|\le\frac{m}{n},$$

which completes the proof of Lemma 3.1. □

Our second auxiliary result is the following lemma.

Lemma 3.2. *Let the natural numbers m, n satisfy the condition $1\le m<n$ and M_m $(M_{m,n})$ denote a sequence of maxima among $X_1,...,X_m$ $(X_{m+1},...,X_n)$. Then, under the assumptions of Theorem 2.1,*

$$\left|Cov\left(I\left(M_m\le u_m\right),I\left(M_{m,n}\le u_n\right)\right)\right|\le\frac{m}{n}+\frac{C_1}{n^{\gamma}}\quad \text{for some } \gamma\ge 1 \text{ and } C_1>0,$$

Proof. Clearly, we have

$$\begin{aligned}&\left|Cov\left(I\left(M_m\le u_m\right),I\left(M_{m,n}\le u_n\right)\right)\right|\\ &=\left|P\left(M_m\le u_m,M_{m,n}\le u_n\right)-P\left(M_m\le u_m\right)P\left(M_{m,n}\le u_n\right)\right|\\ &=\left|EP\left(M_m\le u_m,M_{m,n}\le u_n|\,\Theta\right)-EP\left(M_m\le u_m|\,\Theta\right)EP\left(M_{m,n}\le u_n|\,\Theta\right)\right|.\end{aligned}\tag{14}$$

Since (6) holds and $X_1, ..., X_n$ are conditionally independent given Θ, we obtain:

$$EP\left(M_m \le u_m, M_{m,n} \le u_n | \Theta\right) = E\left[\left\{G\left(u_m\right)^{\Theta}\right\}^m \left\{G\left(u_n\right)^{\Theta}\right\}^{n-m}\right], \tag{15}$$

$$EP\left(M_m \le u_m | \Theta\right) = E\left\{G\left(u_m\right)^{\Theta}\right\}^m. \tag{16}$$

By (14)-(16) and the relation in (12), we get

$$\begin{aligned} &\left|Cov\left(I\left(M_m \le u_m\right), I\left(M_{m,n} \le u_n\right)\right)\right| \\ &\le E\left[\left\{G\left(u_m\right)^{\Theta}\right\}^m \left\{G\left(u_n\right)^{\Theta}\right\}^{n-m}\right] - E\left\{G\left(u_m\right)^{\Theta}\right\}^m E\left\{\left(G\left(u_n\right)\right)^{\Theta}\right\}^{n-m}. \end{aligned} \tag{17}$$

Thus, it is easy to check that

$$\begin{aligned} \left|Cov\left(I\left(M_m \le u_m\right), I\left(M_{m,n} \le u_n\right)\right)\right| = &\left|E\left[\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(\left\{G\left(u_n\right)^{\Theta}\right\}^{n-m} - \left\{G\left(u_n\right)^{\Theta}\right\}^n\right)\right]\right. \\ &-E\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(E\left\{G\left(u_n\right)^{\Theta}\right\}^{n-m} - E\left\{G\left(u_n\right)^{\Theta}\right\}^n\right) \\ &\left.+E\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(\left\{G\left(u_n\right)^{\Theta}\right\}^n - E\left\{G\left(u_n\right)^{\Theta}\right\}^n\right)\right|. \end{aligned}$$

Therefore, we may write that

$$\begin{aligned} \left|Cov\left(I\left(M_m \le u_m\right), I\left(M_{m,n} \le u_n\right)\right)\right| \le &\ E\left[\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(\left\{G\left(u_n\right)^{\Theta}\right\}^{n-m} - \left\{G\left(u_n\right)^{\Theta}\right\}^n\right)\right] \\ &+E\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(E\left\{G\left(u_n\right)^{\Theta}\right\}^{n-m} - E\left\{G\left(u_n\right)^{\Theta}\right\}^n\right) \\ &+\left|E\left\{G\left(u_m\right)^{\Theta}\right\}^m \left(\left\{G\left(u_n\right)^{\Theta}\right\}^n - E\left\{G\left(u_n\right)^{\Theta}\right\}^n\right)\right| \\ &=: A_1 + A_2 + A_3. \end{aligned} \tag{18}$$

The properties that: $z^{n-m} - z^n \le m/n$, if $z \in [0,1]$, and $0 \le \left\{G\left(u_m\right)^{\Theta}\right\} \le 1$ immediately imply

$$A_1 + A_2 \le m/n. \tag{19}$$

Furthermore, it follows from the Cauchy-Schwarz inequality that

$$A_3 \le \sqrt{E\left(\left\{G\left(u_n\right)^{\Theta}\right\}^n - E\left[\left\{G\left(u_n\right)^{\Theta}\right\}^n\right]\right)^2} = \sqrt{E\left\{G\left(u_n\right)^{\Theta}\right\}^{2n} - \left[E\left\{G\left(u_n\right)^{\Theta}\right\}^n\right]^2}.$$

Hence, using (7) and (5), we have

$$\begin{aligned} A_3 &\le \sqrt{E\left\{\exp\left(-\Theta 2\Psi\left(F\left(u_n\right)\right)\right)\right\} - \left[E\left\{\exp\left(-\Theta\Psi\left(F\left(u_n\right)\right)\right)\right\}\right]^2} \\ &= \sqrt{\Psi^{-1}\left(2\Psi\left(F\left(u_n\right)\right)\right) - \left[\Psi^{-1}\left(\Psi\left(F\left(u_n\right)\right)\right)\right]^2}. \end{aligned} \tag{20}$$

Applying the fact that $|z - xy| \le |z - x|\,|y| + |y - 1|\,|z|$, for any x, y and z, together with the properties that $0 \le \Psi^{-1} \le 1$ and $\Psi^{-1}(0) = 1$, we may write

$$\begin{aligned} \Psi^{-1}\left(2\Psi\left(F\left(u_n\right)\right)\right) - \left[\Psi^{-1}\left(\Psi\left(F\left(u_n\right)\right)\right)\right]^2 \le &\left|\Psi^{-1}\left(2\Psi\left(F\left(u_n\right)\right)\right) - \Psi^{-1}\left(\Psi\left(F\left(u_n\right)\right)\right)\right| \\ &+\left|\Psi^{-1}\left(\Psi\left(F\left(u_n\right)\right)\right) - \Psi^{-1}(0)\right|. \end{aligned}$$

The last relation and the fact that Ψ^{-1} is a Lipschitz function yield

$$\Psi^{-1}\left(2\Psi\left(F\left(u_n\right)\right)\right) - \left[\Psi^{-1}\left(\Psi\left(F\left(u_n\right)\right)\right)\right]^2 \le 2L\Psi\left(F\left(u_n\right)\right), \tag{21}$$

where $L > 0$ denotes an appropriate Lipschitz constant.

By (20), (21) and assumption (9), we obtain that there exists $\beta \geq 2$ such that

$$A_3 \leq \sqrt{2L\Psi\left(F\left(u_n\right)\right)} = \sqrt{2Ln^{\beta}\Psi\left(F\left(u_n\right)\right)}/n^{\beta/2} = \mathcal{O}\left(1/n^{\beta/2}\right) \quad \text{if } n \to \infty. \tag{22}$$

Therefore, putting $\gamma := \beta/2$, we obtain

$$A_3 \leq \frac{C_1}{n^{\gamma}} \quad \text{for some } \gamma \geq 1 \text{ and } C_1 > 0. \tag{23}$$

Combining (18), (19) and (23), we get a desired result from Lemma 3.2. □

The following lemma will also be needed for the proof of our main result.

Lemma 3.3. *Under the assumptions of Theorem 2.1 on (X_i), (u_n) and Ψ, we have*

$$\lim_{n\to\infty} P\left(M_n \leq u_n\right) = e^{-\tau},$$

where τ is such as in (8).

Proof. Firstly, we will show that assumption (9) yields the condition

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P\left(X_1 > u_n, X_j > u_n\right) = 0. \tag{24}$$

Observe that, for any $j \geq 2$,

$$\begin{aligned} &\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \\ &= \left|EP\left(X_1 > u_n, X_j > u_n \middle| \Theta\right) - EP\left(X_1 > u_n \middle| \Theta\right) EP\left(X_j > u_n \middle| \Theta\right)\right| \\ &= \left|E\left\{P\left(X_1 > u_n \middle| \Theta\right) P\left(X_j > u_n \middle| \Theta\right)\right\} - EP\left(X_1 > u_n \middle| \Theta\right) EP\left(X_j > u_n \middle| \Theta\right)\right|, \end{aligned}$$

where the last relation follows from the fact that $X_1, ..., X_n$ are conditionally independent given Θ.

Consequently, we obtain

$$\begin{aligned} &\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \\ &= \left|EP\left(X_1 > u_n \middle| \Theta\right)\left\{P\left(X_j > u_n \middle| \Theta\right) - EP\left(X_j > u_n \middle| \Theta\right)\right\}\right| \\ &\leq EP\left(X_1 > u_n \middle| \Theta\right)\left|P\left(X_j > u_n \middle| \Theta\right) - EP\left(X_j > u_n \middle| \Theta\right)\right|. \end{aligned}$$

As in addition, a r.v. $\left|P\left(X_j > u_n \middle| \Theta\right) - EP\left(X_j > u_n \middle| \Theta\right)\right|$ is bounded above by 2, we may write that

$$\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \leq 2EP\left(X_1 > u_n \middle| \Theta\right). \tag{25}$$

Furthermore, using assumption (6), we obtain

$$EP\left(X_1 > u_n \middle| \Theta\right) = E\left\{1 - \left(G\left(u_n\right)\right)^{\Theta}\right\}. \tag{26}$$

Thus, by (25)-(26), we get

$$\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \leq 2E\left\{1 - \left(G\left(u_n\right)\right)^{\Theta}\right\}. \tag{27}$$

Due to (7) and the property that $1 - \exp(-x) \leq x$, for any $x \in \mathbb{R}$, we have

$$E\left\{1 - \left(G\left(u_n\right)\right)^{\Theta}\right\} = E\left\{1 - \exp\left(-\Theta\Psi\left(F\left(u_n\right)\right)\right)\right\} \leq E\left\{\Theta\Psi\left(F\left(u_n\right)\right)\right\} = \left(\Psi\left(F\left(u_n\right)\right)\right) E\left(\Theta\right). \tag{28}$$

In view of (27)-(28), we obtain

$$\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \leq 2E\left(\Theta\right) \cdot \Psi\left(F\left(u_n\right)\right). \tag{29}$$

Derivation (29), assumption (10) and condition (9) yield

$$\begin{aligned}&\left|P\left(X_1>u_n,X_j>u_n\right)-P\left(X_1>u_n\right)P\left(X_j>u_n\right)\right|\leq 2\mu\Psi\left(F\left(u_n\right)\right)\\&\leq\frac{C\mu}{n^\beta}\quad\text{for some }\beta\geq 2\text{ and some }C,\mu>0.\end{aligned}\tag{30}$$

By virtue of (30), we have

$$\begin{aligned}&\left|P\left(X_1>u_n,X_j>u_n\right)-P\left(X_1>u_n\right)P\left(X_j>u_n\right)\right|\\&=\left|P\left(X_1>u_n,X_j>u_n\right)-\left(1-F\left(u_n\right)\right)^2\right|\\&\leq\frac{C\mu}{n^\beta}\quad\text{for some }\beta\geq 2\text{ and some }C,\mu>0.\end{aligned}\tag{31}$$

Combining (31) with (8), we obtain

$$P\left(X_1>u_n,X_j>u_n\right)\leq\left(1-F\left(u_n\right)\right)^2+\frac{C\mu}{n^\beta}\leq\frac{C_1}{n^2}+\frac{C\mu}{n^\beta}\quad\text{for some }\beta\geq 2\text{ and some }C_1,C,\mu>0.$$

Therefore, putting $C_2:=C_1\vee C\mu$, we get

$$P\left(X_1>u_n,X_j>u_n\right)\leq\frac{2C_2}{n^2}\quad\text{for any }j\geq 2.\tag{32}$$

Thus, in view of (32),

$$n\sum_{j=2}^{[n/k]}P\left(X_1>u_n,X_j>u_n\right)\leq n\sum_{j=2}^{[n/k]}\frac{2C_2}{n^2}\leq n\frac{n}{k}\frac{2C_2}{n^2}=\frac{2C_2}{k}\to 0\quad\text{if }k\to\infty.\tag{33}$$

Employing (33), we immediately obtain a desired relation in (24).

In the second stage of our proof, we will show that assumption (9) implies the following property: for any integers $i_1<...<i_p<j_1<...<j_{p'}\leq n$, for which $j_1-i_p\geq l_n$, we have

$$\left|F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right)-F_{i_1...i_p}\left(u_n\right)F_{j_1,...,j_{p'}}\left(u_n\right)\right|\leq\alpha_{n,l_n},\tag{34}$$

where:

$$F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right):=P\left(X_{i_1}\leq u_n,...X_{i_p}\leq u_n,X_{j_1}\leq u_n,...,X_{j_{p'}}\leq u_n\right),$$

and $\alpha_{n,l_n}\to 0$ as $n\to\infty$, for some sequence $l_n=o(n)$.

Let us notice that, since $|z-xy|\leq|z-x||y|+|y-1||z|$, for any x, y and z, we may write as follows

$$\begin{aligned}&\left|F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right)-F_{i_1...i_p}\left(u_n\right)F_{j_1,...,j_{p'}}\left(u_n\right)\right|\\&\leq\left|F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right)-F_{i_1...i_p}\left(u_n\right)\right|+\left|F_{j_1,...,j_{p'}}\left(u_n\right)-1\right|.\end{aligned}\tag{35}$$

This and property (4) yield:

$$F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right)=\Psi^{-1}\left(\left(p+p'\right)\Psi\left(F\left(u_n\right)\right)\right),\tag{36}$$

$$F_{i_1...i_p}\left(u_n\right)=\Psi^{-1}\left(p\Psi\left(F\left(u_n\right)\right)\right),\tag{37}$$

$$F_{j_1,...,j_{p'}}\left(u_n\right)=\Psi^{-1}\left(p'\Psi\left(F\left(u_n\right)\right)\right).\tag{38}$$

Thus, in view of (35)-(38), we get

$$\begin{aligned}&\left|F_{i_1...i_p,\,j_1,...,j_{p'}}\left(u_n\right)-F_{i_1...i_p}\left(u_n\right)F_{j_1,...,j_{p'}}\left(u_n\right)\right|\\&\leq\left|\Psi^{-1}\left(\left(p+p'\right)\Psi\left(F\left(u_n\right)\right)\right)-\Psi^{-1}\left(p\Psi\left(F\left(u_n\right)\right)\right)\right|\\&+\left|\Psi^{-1}\left(p'\Psi\left(F\left(u_n\right)\right)\right)-\Psi^{-1}(0)\right|.\end{aligned}$$

This and the fact that Ψ^{-1} is a Lipschitz function imply

$$\left|F_{i_1\ldots i_p,\, j_1,\ldots,j_{p'}}\left(u_n\right)-F_{i_1\ldots i_p}\left(u_n\right)F_{j_1,\ldots,j_{p'}}\left(u_n\right)\right|\leq 2Lp'\Psi\left(F\left(u_n\right)\right), \tag{39}$$

where $L>0$ stands for an appropriate Lipschitz constant.

On the other hand, due to (9), we obtain

$$0\leq 2Lp'\Psi\left(F\left(u_n\right)\right)=\frac{2Lp'n^{\beta}\Psi\left(F\left(u_n\right)\right)}{n^{\beta}}=\mathcal{O}\left(1/n^{\beta}\right)\quad\text{for some }\beta\geq 2.$$

Therefore,

$$\lim_{n\to\infty}\left|F_{i_1\ldots i_p,\, j_1,\ldots,j_{p'}}\left(u_n\right)-F_{i_1\ldots i_p}\left(u_n\right)F_{j_1,\ldots,j_{p'}}\left(u_n\right)\right|=0,$$

which yields (34).

Finally, since the conditions in (24) and (34) hold, a desired convergence straightforwardly follows from Theorem 5.3.1 in Leadbetter et al. [23]. □

4 Proof of the main result

The objective of this section is to present the proof of Theorem 2.1.

Proof of Theorem 2.1. First, we will show that the following property holds

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(I\left(M_n\leq u_n\right)-P\left(M_n\leq u_n\right)\right)=0\quad a.s. \tag{40}$$

In view of Lemma 3.1 in Csáki and Gonchigdanzan [6], in order to prove (40), it is enough to show that

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n\leq u_n\right)\right)=\mathcal{O}\left(\frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}}\right)\quad\text{if }N\to\infty, \tag{41}$$

for some $\varepsilon>0$.

Let

$$\xi_n:=I\left(M_n\leq u_n\right). \tag{42}$$

Then,

$$\begin{aligned}
&Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n\leq u_n\right)\right)=Var\left(\sum_{n=1}^{N}\frac{1}{n}\xi_n\right)\\
&\leq\sum_{n=1}^{N}\frac{1}{n^2}Var\left(\xi_n\right)+2\sum_{1\leq m<n\leq N}\frac{1}{mn}\left|Cov\left(\xi_m,\xi_n\right)\right|\\
&=:\sum\nolimits_1+\sum\nolimits_2.
\end{aligned} \tag{43}$$

It is clear that

$$\sum\nolimits_1\leq\sum_{n=1}^{N}\frac{1}{n^2}<\infty. \tag{44}$$

Thus, it remains to estimate the second component $\sum_2$ in (43). Observe that

$$\begin{aligned}
\left|Cov\left(\xi_m,\xi_n\right)\right|&=\left|Cov\left(I\left(M_m\leq u_m\right),I\left(M_n\leq u_n\right)\right)\right|\\
&\leq\left|Cov\left(I\left(M_m\leq u_m\right),I\left(M_n\leq u_n\right)-I\left(M_{m,n}\leq u_n\right)\right)\right|\\
&\qquad+\left|Cov\left(I\left(M_m\leq u_m\right),I\left(M_{m,n}\leq u_n\right)\right)\right|,
\end{aligned}$$

which implies

$$|Cov\left(\xi_m,\xi_n\right)| \le 2E\left|I\left(M_n \le u_n\right) - I\left(M_{m,n} \le u_n\right)\right| + \left|Cov\left(I\left(M_m \le u_m\right), I\left(M_{m,n} \le u_n\right)\right)\right|.$$

Thus, by Lemmas 3.1 and 3.2, there exists a positive constant C_1 such that

$$|Cov\left(\xi_m,\xi_n\right)| \le 3\frac{m}{n} + C_1\frac{1}{n^{\gamma}} \quad \text{for some } \gamma \ge 1. \tag{45}$$

It follows from (45) and a definition of $\sum_2$ in (43) that

$$\sum\nolimits_2 \le 3\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\frac{1}{mn}\frac{m}{n} + C_1\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\frac{1}{mn}\frac{1}{n^{\gamma}} =: \sum\nolimits_3 + \sum\nolimits_4. \tag{46}$$

Obviously, we have

$$\sum\nolimits_3 \le 3\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\frac{1}{n^2} \le 3\sum_{m=1}^{N-1}\frac{1}{m} = \mathcal{O}(\log N) \quad \text{if } N\to\infty, \tag{47}$$

where the penultimate relation follows from the fact that $\sum_{n=m+1}^{N}\frac{1}{n^{\delta+1}} \le \frac{1}{\delta}\frac{1}{m^{\delta}}$ for any $\delta > 0$.

Furthermore, using the property mentioned in the previous line, we immediately obtain

$$\sum\nolimits_4 = C_1\sum_{m=1}^{N-1}\frac{1}{m}\sum_{n=m+1}^{N}\frac{1}{n^{\gamma+1}} \le \frac{C_1}{\gamma}\sum_{m=1}^{N-1}\frac{1}{m^{1+\gamma}} < \infty. \tag{48}$$

In view of (46)-(48), we get

$$\sum\nolimits_2 = \mathcal{O}(\log N) \quad \text{if } N\to\infty. \tag{49}$$

Combining (43), (44) and (49), we have

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n \le u_n\right)\right) = \mathcal{O}(\log N) \quad \text{if } N\to\infty.$$

Thus, the relation in (41) is fulfilled and (40) holds true.

Finally, the convergence in (40), Lemma 3.3 and the regularity property of logarithmic mean imply the result established in Theorem 2.1. □

5 Application of the main result

In this section, some example of application of Theorem 2.1 is given.

Theorem 5.1. *Let:*

$$a_n = \frac{1}{(2\log n)^{1/2}},\ b_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}, a_1 \in \mathbb{R}_+, b_1 \in \mathbb{R}. \tag{50}$$

Suppose that (X_i) is a stationary, standard normal sequence satisfying (9) and that: for any fixed $n \in \mathbb{N}$, the random vector $(X_1, ..., X_n)$ has the Gumbel copula C^{Ψ} with a generator of the form $\Psi(t) = (-\ln t)^{\alpha}$ for some $\alpha \ge \beta$, where β is such as in (9), and that the conditions in (6)-(7) are satisfied. Then, the claim of Theorem 2.1 holds true with $u_n := u_n(x) = a_n x + b_n$ and $\tau = e^{-x}$, i.e., for any $x \in \mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(M_n \le a_n x + b_n\right) = \exp\left(-e^{-x}\right) \quad a.s.$$

Proof. Put $u_n := u_n(x) = a_n x + b_n$, where a_n, b_n are defined as in (50) and x is a fixed real number. It may be checked that (see, e.g., the derivations in Leadbetter et al. [23])

$$\lim_{n\to\infty} n(1-\Phi(u_n)) = e^{-x}. \tag{51}$$

Hence, the condition in (8) is satisfied with $u_n = a_n x + b_n$ and $\tau = e^{-x}$. Furthermore, as C^{Ψ} is the Archimedean copula - the property in (5) occurs as well.

Our aim now is to show that condition (24) is satisfied. It straightforwardly follows from the derivations in the first part of the proof of Lemma 3.3 (with F replaced by the standard normal cdf Φ) that:

$$\begin{aligned}
&\left|P\left(X_1 > u_n, X_j > u_n\right) - P\left(X_1 > u_n\right) P\left(X_j > u_n\right)\right| \\
&= \left|P\left(X_1 > u_n, X_j > u_n\right) - (1-\Phi(u_n))^2\right| \\
&\le \frac{C\mu}{n^{\beta}} \quad \text{for some } \beta \ge 2 \text{ and some } C, \mu > 0,
\end{aligned}$$

$$P\left(X_1 > u_n, X_j > u_n\right) \le (1-\Phi(u_n))^2 + \frac{C\mu}{n^{\beta}} \le \frac{C_1}{n^2} + \frac{C\mu}{n^{\beta}} \le \frac{2C_2}{n^2}, \text{ where } C_2 := C_1 \vee C\mu,$$

and

$$n\sum_{j=2}^{[n/k]} P\left(X_1 > u_n, X_j > u_n\right) \le n\frac{n}{k}\frac{2C_2}{n^2} = \frac{2C_2}{k} \to 0 \quad \text{if } k \to \infty.$$

Consequently, condition (24) is fulfilled.

In addition, since the random vector $(X_1, ..., X_n)$ has the Gumbel copula C^{Ψ} with a generator $\Psi(t) = (-\ln t)^{\alpha}$ for some $\alpha \ge \beta \ge 2$ and the $X_i's$ have the standard normal cdf, we get, in view of (51),

$$\begin{aligned}
n^{\beta}\Psi(F(u_n)) &= n^{\beta}\Psi(\Phi(u_n)) \sim n^{\beta}\Psi\left(1 - \frac{e^{-x}}{n}\right) = n^{\beta}\left\{\ln\left(\frac{n}{n-e^{-x}}\right)\right\}^{\alpha} \\
&= n^{\beta-\alpha}\left\{n\ln\left(\frac{n}{n-e^{-x}}\right)\right\}^{\alpha} = n^{\beta-\alpha}\left\{\ln\left(1+\frac{e^{-x}}{n-e^{-x}}\right)^{n}\right\}^{\alpha} \\
&\sim n^{\beta-\alpha}e^{-x\alpha} = \mathcal{O}(1) \quad \text{if } n \to \infty, \text{ since } \alpha \ge \beta,
\end{aligned}$$

and condition (9) is satisfied.

Consequently, we obtain that all the assumpions of Theorem 5.3.1 in Leadbetter et al. [23] and Theorem 2.1 hold true. Therefore, applying the latter assertion with $u_n = u_n(x) = a_n x + b_n$ and $\tau = \tau(x) = e^{-x}$, we have the conclusion of Theorem 5.1. □

References

[1] Brosamler G., An almost everywhere central limit theorem, Math. Proc. Cambridge Philos. Soc., 1988, 104, 561–574.
[2] Schatte P., On the central limit theorem with almost sure convergence, Probab. Math. Statist., 1991, 11, 237-246.
[3] Berkes I., Csáki E., A universal result in almost sure central limit theory, Stoch. Proc. Appl., 2001, 94, 105–134.
[4] Chen S., Lin Z., Almost sure max-limits for nonstationary Gaussian sequence, Statist. Probab. Lett., 2006, 76, 1175–1184.
[5] Cheng S., Peng L., Qi Y., Almost sure convergence in extreme value theory, Math. Nachr., 1998, 190, 43–50.
[6] Csáki E., Gonchigdanzan K., Almost sure limit theorems for the maximum of stationary Gaussian sequences, Statist. Probab. Lett., 2002, 58, 195–203.
[7] Dudziński M., A note on the almost sure central limit theorem for some dependent random variables, Statist. Probab. Lett., 2003, 61, 31–40.
[8] Dudziński M., Górka P., The almost sure central limit theorems for the maxima of sums under some new weak dependence assumptions, Acta Math. Sin., English Series, 2013, 29, 429-448.
[9] Gonchigdanzan K., Rempała G., A note on the almost sure limit theorem for the product of partial sums, Appl. Math. Lett., 2006, 19, 191-196.
[10] Ho H.C., Hsing T., On the asymptotic joint distribution of the sum and maximum of stationary normal random variables, J. Appl. Probab., 1996, 33, 138-145.

[11] Lacey M., Philipp W., A note on the almost sure central limit theorem, Statist. Probab. Lett., 1990, 9, 201–205.
[12] Matuła P., Convergence of weighted averages of associated random variables, Probab. Math. Statist., 1996, 16, 337-343.
[13] Mielniczuk J., Some remarks on the almost sure central limit theorem for dependent sequences, In: Limit theorems in Probability and Statistics II (I. Berkes, E. Csáki, M. Csörgő, eds.), Bolyai Institute Publications, Budapest, 2002, 391-403.
[14] Peligrad M., Shao Q., A note on the almost sure central limit theorem for weakly dependent random variables, Statist. Probab. Lett., 1995, 22, 131–136.
[15] Stadtmüller U., Almost sure versions of distributional limit theorems for certain order statistics, Statist. Probab. Lett., 2002, 58, 413–426.
[16] Zhao S., Peng Z., Wu S., Almost Sure Convergence for the Maximum and the Sum of Nonstationary Guassian Sequences, J. Inequal. Appl., 2010:856495, 14 pages, doi:10.1155/2010/856495.
[17] Fazekas I., Rychlik Z., Almost sure functional limit theorems, Ann. Univ. Mariae Curie-Skłodowska, Sect. A: Mathematica, 2002, 56, 1-18.
[18] Sklar A., Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris, 1959, 8, 229-231.
[19] Sklar A., Random variables, distribution functions and copulas - a personal look backward and forward, In: Rüschendorf, L., Schweizer, B., Taylor, M.D. (Eds.), Distributions with Fixed Marginals and Related Topics. Institute of Mathematical Statistics, Hayward, CA, 1996, 1–14.
[20] Kimberling C.H., A probabilistic interpretation of complete monotonicity, Aequationes Mathematicae, 1974, 10, 152–164.
[21] Marshall A.W., Olkin I., Families of multivariate distributions, JASA, 1988, 83, 834–841.
[22] Frees E.W., Valdez E.A., Understanding relationships using copulas, North Amer. Actuarial J., 1998, 2, 1-25.
[23] Leadbetter M.R., Lindgren G., Rootzén H., Extremes and Related Properties of Random Sequences and Processes, Springer, New York, 1983.