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Wang Tingting* and Wang Xiaonan

On the Golomb's conjecture and Lehmer's numbers

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Abstract: Let p be an odd prime. For each integer a with $1 \leq a \leq p - 1$, it is clear that there exists one and only one \bar{a} with $1 \leq \bar{a} \leq p - 1$ such that $a \cdot \bar{a} \equiv 1 \pmod{p}$. Let $N(p)$ denote the set of all primitive roots $a \pmod{p}$ with $1 \leq a \leq p - 1$ in which a and \bar{a} are of opposite parity. The main purpose of this paper is using the analytic method and the estimate for the hybrid exponential sums to study the solvability of the congruence $a + b \equiv 1 \pmod{p}$ with $a, b \in N(p)$, and give a sharper asymptotic formula for the number of the solutions of the congruence equation.

Keywords: D. H. Lehmer's numbers, Golomb's conjecture, The hybrid exponential sums, Congruence equation, Primitive roots

MSC: 11L03, 11L07

1 Introduction

Let p be an odd number. For each integer a with $1 \leq a \leq p - 1$, it is clear that there exists one and only one \bar{a} with $1 \leq \bar{a} \leq p - 1$ such that $a \cdot \bar{a} \equiv 1 \pmod{p}$. Let $M(p)$ denote the set of cases in which a and \bar{a} are of opposite parity. For convenience, we call such a number a with $(2, a + \bar{a}) = 1$ as a D. H. Lehmer's number. Professor D. H. Lehmer [1] asked us to study the properties of $|M(p)|$ or at least to say something nontrivial about it, where $|M(p)|$ denotes the number of all elements in $M(p)$. It is known that $|M(p)| \equiv 2$ or $0 \pmod{4}$ when $p \equiv \pm 1 \pmod{4}$. Some related works can be found in references [2-5]. For example, W. Zhang [2] proved that for any odd $q > 3$, one has the asymptotic formula

$$|M(q)| = \frac{1}{2}\phi(q) + O\left(q^{\frac{1}{2}} \cdot d^2(q) \cdot \ln^2 q\right),$$

where $\phi(q)$ is Euler function, and $d(q)$ is Dirichlet divisor function.

Now we let $N(p)$ denote the set of all primitive roots $a \pmod{p}$ with $1 \leq a \leq p - 1$ in which a and \bar{a} are of opposite parity. For any integer d with $1 \leq d \leq p - 1$, Golomb [6] posed some problems related to primitive roots, one of which is that whether there exist two primitive roots $a, b \pmod{p}$ such that $a + b \equiv d \pmod{p}$. Some related works can also be found in [7-9].

In this paper, we consider the following problem involving Golomb's conjecture and Lehmer's numbers: Do there exist two primitive roots a and $b \in N(p)$ such that

$$a + b \equiv 1 \pmod{p} ? \tag{1}$$

Regarding this problem, it seems that it has not been studied yet, or at least we have not seen any related result before. The problem is difficult and important, because it has close relations with the Golomb's conjecture and the Lehmer's numbers.

*Corresponding Author: Wang Tingting: College of Science, Northwest A&F University, Yangling, Shaanxi 712100, China,

E-mail: ttwang@nwsuaf.edu.cn

Wang Xiaonan: College of Science, Northwest A&F University, Yangling, Shaanxi 712100, China

The main purpose of this paper is using the analytic method and the estimate for the hybrid exponential sums to study the solvability of the congruence equation $a + b \equiv 1 \pmod p$ with $a, b \in N(p)$, and give a sharper asymptotic formula for (1). That is, we shall prove the following:

Theorem. *Let p be an odd prime and $S(p)$ denote the number of all solutions of (1). Then we have the asymptotic formula*

$$S(p) = \frac{1}{4} \frac{\phi^2(p-1)}{p-1} + O\left(\frac{\phi^2(p-1)}{p^{\frac{5}{4}}} \cdot 4^{\omega(p-1)} \cdot \ln^2 p\right),$$

where $\phi(n)$ denotes the Euler function, and $\omega(n)$ denotes the number of all distinct prime divisors of n .

2 Several lemmas

In this section, we shall prove several lemmas, which will be used in the proof of our theorem. Hereinafter, we shall use many properties of Gauss sums and the hybrid exponential sums, all of these can be found in references [10-11], so they will not be repeated here. First we have the following:

Lemma 2.1. *Let p be an odd prime. Then for any integer c with $(c, p) = 1$, we have the identity*

$$\frac{\phi(p-1)}{p-1} \sum_{h|p-1} \frac{\mu(h)}{\phi(h)} \sum_{\substack{k=1 \\ (h,k)=1}}^h e\left(\frac{k \operatorname{ind} c}{h}\right) = \begin{cases} 1, & \text{if } c \text{ is a primitive root of } p, \\ 0, & \text{otherwise,} \end{cases}$$

where $\operatorname{ind} c$ denotes the index of c relative to some fixed primitive root of p , $\mu(n)$ is the Möbius function.

Proof. See Proposition 2.2 of reference [12]. □

Lemma 2.2. *Let p be an odd prime, $R_1(x)$ and $R_2(x)$ are two rational functions mod p , χ be any a non-principal character mod p , and at least one of the conditions 1) $\chi(R_1(x)) \neq \text{const}$; 2) $\chi(R_2(x)) \neq \text{const}$ is satisfied. Then we have the estimate*

$$\sum_{a=0}^{p-1} \chi(R_1(a)) e\left(\frac{R_2(a)}{p}\right) \ll p^{\frac{1}{2}},$$

where $e(y) = e^{2\pi iy}$.

Proof. See Lemma A of [10]. □

Lemma 2.3. *Let p be an odd prime, χ_1 and χ_2 are two characters mod p . Then for any integers m and n with $(mn, p) = 1$, we have the estimate*

$$\sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) e\left(\frac{ma + n\bar{a}}{p}\right) \ll p^{\frac{3}{4}}.$$

Proof. If $\chi_1 = \chi_2$, then from Lemma 2.2 we may immediately deduce the estimate

$$\sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) e\left(\frac{ma + n\bar{a}}{p}\right) \ll p^{\frac{1}{2}}. \tag{2}$$

So without loss of generality we can assume that $\chi_1 \neq \chi_2$. From Lemma 2.2 we have

$$\left| \sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) e\left(\frac{ma + n\bar{a}}{p}\right) \right|^2$$

$$\begin{aligned}
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_1(a) \chi_2(a-1) \bar{\chi}_1(b) \bar{\chi}_2(b-1) e\left(\frac{m(a-b) + n(\bar{a}-\bar{b})}{p}\right) \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_1(a) \chi_2(ab-1) \bar{\chi}_2(b-1) e\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{p}\right) \\
 &= p-2 + \sum_{a=2}^{p-1} \chi_1(a) \sum_{b=2}^{p-1} \chi_2\left(\frac{ab-1}{b-1}\right) e\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{p}\right) \\
 &\ll p + \sum_{a=2}^{p-1} \left| \sum_{b=2}^{p-1} \chi_2\left(\frac{ab-1}{b-1}\right) e\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{p}\right) \right| \ll p^{\frac{3}{2}}.
 \end{aligned} \tag{3}$$

Combining (2) and (3) we can deduce the estimate

$$\sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) e\left(\frac{ma + n\bar{a}}{p}\right) \ll p^{\frac{3}{4}}.$$

This proves Lemma 2.3. □

Lemma 2.4. *Let p be an odd prime, χ_1 and χ_2 are two characters mod p . Then we have the estimate*

$$\sum_{a=1}^{p-2} \chi_1(a-1) \chi_2(a) (-1)^{a+\bar{a}} \ll p^{\frac{3}{4}} \cdot \ln^2 p.$$

Proof. From the trigonometrical identity

$$\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p, & \text{if } (n, p) = p, \\ 0, & \text{if } (n, p) = 1. \end{cases}$$

we have

$$\begin{aligned}
 &\sum_{a=1}^{p-1} \chi_1(a-1) \chi_2(a) (-1)^{a+\bar{a}} \\
 &= \frac{1}{p^2} \sum_{\substack{a=1 \\ ab \equiv 1 \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \chi_1(a-1) \chi_2(a) \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} (-1)^{c+d} \sum_{r=0}^{p-1} \sum_{s=0}^{p-1} e\left(\frac{r(a-c)}{p}\right) e\left(\frac{s(b-d)}{p}\right) \\
 &= \frac{1}{p^2} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \sum_{a=1}^{p-1} \chi_1(a-1) \chi_2(a) e\left(\frac{ra + s\bar{a}}{p}\right) \left(\sum_{c=1}^{p-1} (-1)^c e\left(\frac{-rc}{p}\right) \right) \\
 &\quad \times \left(\sum_{d=1}^{p-1} (-1)^d e\left(\frac{-sd}{p}\right) \right) \\
 &\ll \frac{1}{p^2} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_1(a-1) \chi_2(a) e\left(\frac{ra + s\bar{a}}{p}\right) \right| \\
 &\quad \times \left| \sum_{c=1}^{p-1} (-1)^c e\left(\frac{-rc}{p}\right) \right| \cdot \left| \sum_{d=1}^{p-1} (-1)^d e\left(\frac{-sd}{p}\right) \right|.
 \end{aligned} \tag{4}$$

Note that the identity

$$\left| \sum_{c=1}^{p-1} (-1)^c e\left(\frac{-rc}{p}\right) \right| = \left| \frac{\sin \frac{r\pi}{p}}{\cos \frac{r\pi}{p}} \right| \ll \frac{p}{|p-2r|},$$

From (4) and Lemma 2.3 we have the estimate

$$\left| \sum_{a=1}^{p-1} \chi_1(a-1) \chi_2(a) (-1)^{a+\bar{a}} \right| \ll \frac{1}{p^2} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} p^{\frac{3}{4}} \frac{p}{|p-2r|} \cdot \frac{p}{|p-2s|} \ll p^{\frac{3}{4}} \cdot \ln^2 p.$$

This proves Lemma 2.4. □

Lemma 2.5. *Let p be an odd prime, χ_1 and χ_2 are two characters mod p such that at least one of them is not principal character mod p . Then we have*

$$\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a-1) \ll \sqrt{p}.$$

Proof. If χ_1 is the principal character mod p , then from the orthogonality of characters χ mod p we have

$$\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a-1) = \sum_{a=1}^{p-1} \chi_2(a-1) = \sum_{a=1}^{p-1} \chi_2(a) - \chi_2(p-1) = -\chi_2(-1). \tag{5}$$

If χ_2 is the principal character mod p , then we have

$$\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a-1) = \sum_{a=2}^{p-1} \chi_1(a) = \sum_{a=1}^{p-1} \chi_1(a) - \chi_1(1) = -1. \tag{6}$$

If both χ_1 and χ_2 are not principal characters mod p , then from the properties of Gauss sums we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi_1(a)\chi_2(a-1) &= \sum_{a=1}^{p-1} \chi_1(a) \frac{1}{\tau(\bar{\chi}_2)} \sum_{b=1}^{p-1} \bar{\chi}_2(b) e\left(\frac{b(a-1)}{p}\right) \\ &= \frac{1}{\tau(\bar{\chi}_2)} \sum_{b=1}^{p-1} \bar{\chi}_2(b) e\left(\frac{-b}{p}\right) \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ab}{p}\right) \\ &= \frac{\tau(\chi_1)}{\tau(\bar{\chi}_2)} \sum_{b=1}^{p-1} \bar{\chi}_2(b) \bar{\chi}_1(b) e\left(\frac{-b}{p}\right) = \bar{\chi}_1 \bar{\chi}_2(-1) \frac{\tau(\chi_1) \tau(\bar{\chi}_1 \bar{\chi}_2)}{\tau(\bar{\chi}_2)}. \end{aligned} \tag{7}$$

For any character χ mod p , note that the estimate $|\tau(\chi)| \leq \sqrt{p}$, from (5), (6) and (7) we may immediately deduce Lemma 2.5. □

3 Proof of the theorem

In this section, we will use the lemmas in section two to complete the proof of our theorem. Note that if $a + b \equiv 1 \pmod{p}$, then $a + b = p + 1$ and $\bar{b} = \overline{p + 1 - a} = p - \overline{a - 1}$. So from Lemma 2.1 and the definition of $N(p)$ we have

$$\begin{aligned} S(p) &= \sum_{\substack{a \in N(p) \\ p+1-a \in N(p)}} 1 \\ &= \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{h \\ (h,k)=1}} \sum_{\substack{r \\ (r,s)=1}} \sum_{a=1}^{p-1} \frac{1}{2} \left(1 - (-1)^{a+\bar{a}}\right) \\ &\quad \times \frac{1}{2} \left(1 - (-1)^{p+1-a+\overline{p+1-a}}\right) \chi(a, k; h) \chi(p + 1 - a, s; r) \\ &= \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{h \\ (h,k)=1}} \sum_{\substack{r \\ (r,s)=1}} \sum_{a=1}^{p-1} \chi(a, k; h) \chi(1 - a, s; r) \\ &\quad - \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{h \\ (h,k)=1}} \sum_{\substack{r \\ (r,s)=1}} \sum_{a=1}^{p-1} \chi(a, k; h) \end{aligned}$$

$$\begin{aligned}
 & \times \chi(1-a, s; r)(-1)^{a+\bar{a}} \\
 & - \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h) \\
 & \times \chi(1-a, s; r)(-1)^{p+1-a+\overline{p+1-a}} \\
 & - \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h) \\
 & \times \chi(1-a, s; r)(-1)^{\bar{a}+\overline{a-1}} \\
 \equiv & A_1 + A_2 + A_3 + A_4. \tag{8}
 \end{aligned}$$

Now we estimate A_1, A_2, A_3 and A_4 in (8) respectively. Note that the identity

$$\sum_{d|p-1} |\mu(d)| = 2^{\omega(p-1)},$$

where $\omega(n)$ denotes the number of all distinct prime divisors of n . From Lemma 2.5 and the definition of A_1 we have

$$\begin{aligned}
 A_1 &= \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h)\chi(1-a, s; r) \\
 &= \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{\substack{h|p-1 \\ (h,r) \neq (1,1)}} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h)\chi(1-a, s; r) \\
 & \quad + \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \cdot (p-2) \\
 &= \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \cdot (p-2) + O\left(\frac{\phi^2(p-1)}{p^{\frac{3}{2}}} \sum_{\substack{h|p-1 \\ (h,r) \neq (1,1)}} \sum_{r|p-1} |\mu(r)| \cdot |\mu(h)|\right) \\
 &= \frac{1}{4} \frac{\phi^2(p-1)}{p-1} + O\left(\frac{\phi^2(p-1)}{p^{\frac{3}{2}}} \cdot 4^{\omega(p-1)}\right). \tag{9}
 \end{aligned}$$

From Lemma 2.4 and the definitions of A_2 and A_3 we have

$$\begin{aligned}
 A_2 &= \frac{-1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h) \\
 & \times \chi(1-a, s; r)(-1)^{a+\bar{a}} \\
 &= \frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \chi(-1, s; r) \\
 & \times \sum_{a=1}^{p-1} \chi(a, k; h)\chi(a-1, s; r)(-1)^{a+\bar{a}} \\
 &\ll \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} |\mu(r)| \cdot |\mu(h)| \cdot p^{\frac{3}{4}} \cdot \ln^2 p \\
 &\ll \frac{\phi^2(p-1)}{p^{\frac{5}{4}}} \cdot 4^{\omega(p-1)} \cdot \ln^2 p. \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 A_3 &= -\frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h) \\
 &\quad \times \chi(1-a, s; r)(-1)^{p+1-a+\overline{p+1-a}} \\
 &= -\frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \chi(-1, k; h) \\
 &\quad \times \sum_{a=1}^{p-1} \chi(a, s; r)\chi(a-1, k; h)(-1)^{a+\overline{a}} \\
 &\ll \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} |\mu(r)| \cdot |\mu(h)| \cdot p^{\frac{3}{4}} \cdot \ln^2 p \\
 &\ll \frac{\phi^2(p-1)}{p^{\frac{5}{4}}} \cdot 4^{\omega(p-1)} \cdot \ln^2 p. \tag{11}
 \end{aligned}$$

For all $1 \leq a \leq p-2$, note that $\overline{a+1} \equiv \overline{a} \cdot \overline{a+1} \equiv 1 - \overline{a} + \overline{1} \pmod{p}$, so that $\overline{a+1} = p+1 - \overline{a} + \overline{1}$. From this and Lemma 2.4 we have

$$\begin{aligned}
 &\sum_{a=1}^{p-1} \chi(a, k; h)\chi(1-a, s; r)(-1)^{\overline{a}+\overline{a-1}} = \sum_{a=1}^{p-2} \chi(a+1, k; h)\chi(-a, s; r)(-1)^{\overline{a+1}+\overline{a}} \\
 &= \chi(-1, s; r) \sum_{a=1}^{p-2} \chi(a+1, k; h)\chi(a, s; r)(-1)^{\overline{a+1}+\overline{a}} \\
 &= \chi(-1, s; r) \sum_{a=1}^{p-2} \chi(a+1, k; h)\chi(a, s; r)(-1)^{p+1-\overline{a+1}+\overline{a}} \\
 &= \chi(-1, s; r) \sum_{a=1}^{p-2} \chi(\overline{a}+1, k; h)\overline{\chi}(a, s; r)(-1)^{a+\overline{a+1}} \\
 &= -\chi(-1, s; r) \sum_{a=1}^{p-1} \chi(a, k; h)\overline{\chi}(a-1, k; h)\overline{\chi}(a-1, s; r)(-1)^{a+\overline{a}} \\
 &\ll p^{\frac{3}{4}} \cdot \ln^2 p. \tag{12}
 \end{aligned}$$

Applying (12) and the definition of A_4 we have

$$\begin{aligned}
 A_4 &= -\frac{1}{4} \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} \frac{\mu(h)\mu(r)}{\phi(h)\phi(r)} \sum_{\substack{k=1 \\ (h,k)=1}}^h \sum_{\substack{s=1 \\ (r,s)=1}}^r \sum_{a=1}^{p-1} \chi(a, k; h) \\
 &\quad \times \chi(1-a, s; r)(-1)^{\overline{a}+\overline{a-1}} \\
 &\ll \frac{\phi^2(p-1)}{(p-1)^2} \sum_{h|p-1} \sum_{r|p-1} |\mu(h)| \cdot |\mu(r)| \cdot p^{\frac{3}{4}} \cdot \ln^2 p \\
 &\ll \frac{\phi^2(p-1)}{p^{\frac{5}{4}}} \cdot 4^{\omega(p-1)} \cdot \ln^2 p. \tag{13}
 \end{aligned}$$

Now combining (8), (9), (10), (11) and (13) we may immediately deduce the asymptotic formula

$$S(p) = \frac{1}{4} \frac{\phi^2(p-1)}{p-1} + O\left(\frac{\phi^2(p-1)}{p^{\frac{5}{4}}} \cdot 4^{\omega(p-1)} \cdot \ln^2 p\right).$$

This completes the proof of our theorem.

Competing interests

The authors declare that they have no competing interests.

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