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New error bounds for linear complementarity problems of weakly chained diagonally dominant B-matrices

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Abstract: Some new error bounds for the linear complementarity problems are obtained when the involved matrices are weakly chained diagonally dominant B-matrices. Numerical examples are given to show the effectiveness of the proposed bounds.

Keywords: Error bounds, Linear complementarity problems, Weakly chained diagonally dominant matrices, B-matrices

MSC: 90C33, 60G50, 65F35

1 Introduction

Given a real matrix $M = [m_{ij}] \in R^{n\times n}$ and $q \in R^n$, the linear complementarity problem $LCP(M,q)$ is to find a vector $x \in R^n$ satisfying

$$(Mx+q)^T x = 0, \qquad Mx+q \geq 0, \qquad x \geq 0,$$

or to prove that no such vector x exists. The $LCP(M,q)$ has many applications such as finding Nash equilibrium point of a bimatrix game, the network equilibrium problems and the free boundary problems for journal bearing etc, see [1–3].

As is known, the $LCP(M,q)$ has a unique solution for any vector $q \in R^n$ if and only if M is a P-matrix [2]. Here, a matrix M is called a P-matrix if all its principal minors are positive [4].

For the $LCP(M,q)$, one of the interesting problems is to estimate

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty,$$

which can be used to bound the error $\|x-x^*\|_\infty$ [5], that is

$$\|x-x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where x^* is the solution of the $LCP(M,q)$, $r(x) = min\{x, Mx+q\}$, $D = diag(d_i)$ with $0 \leq d_i \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. If the matrix M for the $LCP(M,q)$ satisfies certain structures, various bounds for $\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty$ can be derived, see [6–13].

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Definition 1.1 ([4]). *A real matrix $A = [a_{ij}] \in R^{n\times n}$ is called a B-matrix if for any $i, j \in N = \{1, 2, \cdots, n\}$,*

$$\sum_{k\in N} a_{ik} > 0, \qquad \frac{1}{n}\left(\sum_{k\in N} a_{ik}\right) > a_{ij}, \qquad j \neq i.$$

Definition 1.2 ([14]). *A complex matrix $A = [a_{ij}] \in C^{n\times n}$ is called a weakly chained diagonally dominant ($wcdd$) matrix if A is diagonally dominant, i.e.,*

$$a_{ii} \geq r_i(A) = \sum_{j=1,\neq i}^{n} |a_{ij}|, \qquad \forall i \in N,$$

and for each $i \in J(A)$, there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{i_1i_2}, \cdots, a_{i_rj}$ with $j \in J(A) = \{i \in N : a_{ii} > r_i(A)\} \neq \emptyset$.

Definition 1.3 ([13]). *A real matrix $M = [m_{ij}] \in R^{n\times n}$ is called a weakly chained diagonally dominant ($wcdd$) B-matrix if it can be written in form $M = B^+ + C$ with B^+ a $wcdd$ matrix whose diagonal entries are all positive.*

When M is a B-matrix as a subclass of P-matrices, García-Esnaola *et al.* [9] provided an upper bound for $\max\limits_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$.

Theorem 1.4 ([9]). *Let $M = [m_{ij}] \in R^{n\times n}$ be a B-matrix with the form*

$$M = B^+ + C,$$

where

$$B^+ = [b_{ij}] = \begin{bmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{bmatrix},$$

and $r_i^+ = \max\{0, m_{ij} | j \neq i\}$. Then

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{n-1}{min\{\beta, 1\}}, \tag{1}$$

where $\beta = \min\limits_{i\in N}\{\beta_i\}$ and $\beta_i = b_{ii} - \sum\limits_{j\neq i} |b_{ij}|$.

As shown in [12], the bound (1) will be inaccurate when the matrix M has very small value of $\min\limits_{i\in N}\left\{b_{ii} - \sum\limits_{j\neq i}|b_{ij}|\right\}$, see [12, 13]. To improve the bound (1), Li *et al.* [12] gave the following bound for $\max\limits_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B-matrix.

Theorem 1.5 ([12]). *Let $M = [m_{ij}] \in R^{n\times n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is defined as in Theorem 1.4. Then*

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \sum_{i=1}^{n} \frac{n-1}{min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1}\left(1 + \frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n} |b_{jk}|\right), \tag{2}$$

where $\bar{\beta}_i = b_{ii} - \sum\limits_{j=i+1}^{n} |b_{ij}| l_i(B^+)$, $l_k(B^+) = \max\limits_{k\leq i\leq n}\left\{\frac{1}{|b_{ii}|}\sum\limits_{j=k,\neq i}^{n} |b_{ij}|\right\}$ and

$$\prod_{j=1}^{i-1}\left(1 + \frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n} |b_{jk}|\right) = 1 \; if \; i = 1.$$

Recently, Li *et al.* [13] gave the following bound for $\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty$ when M is a $wcdd$ B-matrix.

Theorem 1.6 ([13]). *Let $M=[m_{ij}]\in R^{n\times n}$ be a $wcdd$ B-matrix with the form $M=B^{+}+C$, where $B^{+}=[b_{ij}]$ is defined as in Theorem 1.4. Then*

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \le \sum_{i=1}^{n}\left(\frac{n-1}{min\{\tilde{\beta}_i,1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\tilde{\beta}_j}\right), \tag{3}$$

where $\tilde{\beta}_i=b_{ii}-\sum\limits_{j=i+1}^{n}|b_{ij}|>0$ and $\prod\limits_{j=1}^{i-1}\frac{b_{jj}}{\tilde{\beta}_j}=1$ if $i=1$.

Since a B-matrix is a $wcdd$ B-matrix [13], thus the bound (3) also holds for B-matrix M.

Now, some notations are given, which will be used later. Given a matrix $A=[a_{ij}]\in R^{n\times n}$, let

$$u_i(A)=\frac{1}{|a_{ii}|}\sum_{j=i+1}^{n}|a_{ij}|,\quad u_n(A)=0,$$

$$l_{ki}(A)=\frac{1}{|a_{kk}|}\sum_{j=i,\neq k}^{n}|a_{kj}|,\quad l_i(A)=\max_{i\le k\le n}\{l_{ki}(A)\},\quad l_n(A)=0,$$

$$t_{ki}(A)=\begin{cases}\dfrac{|a_{ki}|}{|a_{kk}|-\sum\limits_{j=i+1,\neq k}^{n}|a_{kj}|}, & a_{ki}\neq 0\\ 0, & a_{ki}=0\end{cases},$$

$$t_i(A)=\begin{cases}\max\limits_{i+1\le k\le n}\{t_{ki}\}, & 1\le i\le n-1\\ 0, & i=n\end{cases},$$

$$b_k(A)=\max_{k+1\le i\le n}\left\{\frac{\sum\limits_{j=k,\neq i}^{n}|a_{ij}|}{|a_{ii}|}\right\},\quad b_n(A)=1,$$

$$p_k(A)=\max_{k+1\le i\le n}\left\{\frac{|a_{ik}|+\sum\limits_{j=k+1,\neq i}^{n}|a_{ij}|b_k(A)}{|a_{ii}|}\right\},\quad p_n(A)=1.$$

The rest of this paper is organized as follows: In Section 2, we present some new bounds for $\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty$ when M is a $wcdd$ B-matrix. Numerical examples are presented to illustrate the corresponding theoretical results in Section 3.

2 Main results

In this section, some new upper bounds of $\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty$ for $wcdd$ B-matrix M are provided. We first list some lemmas which will be used later.

Lemma 2.1 ([15]). *Let $A=[a_{ij}]\in R^{n\times n}$ be a $wcdd$ M-matrix with $u_k(A)p_k(A)<1$ $(\forall k\in N)$. Then*

$$\|A^{-1}\|_\infty \le \max\left\{\sum_{i=1}^{n}\left(\frac{1}{a_{ii}(1-u_i(A)t_i(A))}\prod_{j=1}^{i-1}\frac{u_j(A)}{1-u_j(A)t_j(A)}\right),\right.$$
$$\left.\sum_{i=1}^{n}\left[\frac{p_i(A)}{a_{ii}(1-u_i(A)t_i(A))}\prod_{j=1}^{i-1}\left(1+\frac{u_j(A)p_j(A)}{1-u_j(A)t_j(A)}\right)\right]\right\},$$

where

$$\prod_{j=1}^{i-1}\frac{u_j(A)}{1-u_j(A)p_j(A)}=1,\quad \prod_{j=1}^{i-1}\left(1+\frac{u_j(A)p_j(A)}{1-u_j(A)t_j(A)}\right)=1,\quad if\ i=1.$$

Lemma 2.2 ([12])**.** *Let $\gamma>0$ and $\eta\geq 0$. Then for any $a\in[0,1]$,*

$$\frac{1}{1-a+\gamma a}\leq\frac{1}{\min\{\gamma,1\}}$$

and

$$\frac{\eta a}{1-a+\gamma a}\leq\frac{\eta}{\gamma}.$$

Theorem 2.3. *Let $M=[m_{ij}]\in R^{n\times n}$ be a $wcdd$ B-matrix with the form $M=B^{+}+C$, where $B^{+}=[b_{ij}]$ is defined as in Theorem 1.4. If for each $i\in N$,*

$$\hat{\beta}_i=b_{ii}-\sum_{j=i+1}^{n}|b_{ij}|t_i(B^{+})>0,$$

then

$$\begin{aligned}&\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty\\ &\leq(n-1)\max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\right.\\ &\left.\sum_{i=1}^{n}\frac{p_i(B^{+})}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^{+})}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\},\end{aligned}\tag{4}$$

where

$$\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)=1,\quad \prod_{j=1}^{i-1}\left(1+\frac{p_j(B^{+})}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)=1,\quad if\ i=1.$$

Proof. Let $M_D=I-D+DM$. Then

$$M_D=I-D+DM=I-D+D(B^{+}+C)=B_D^{+}+C_D,$$

where $B_D^{+}=I-D+DB^{+}$. From Theorem 2 in [13], we know that B_D^{+} is a $wcdd$ M-matrix with positive diagonal elements, $C_D=DC$, and

$$\|M_D^{-1}\|_\infty\leq\|(I+(B_D^{+})^{-1}C_D)^{-1}\|_\infty\|(B_D^{+})^{-1}\|_\infty\leq(n-1)\|(B_D^{+})^{-1}\|_\infty.\tag{5}$$

By Lemma 2.1, we have

$$\begin{aligned}\|(B_D^{+})^{-1}\|_\infty\leq\max&\left\{\sum_{i=1}^{n}\frac{1}{(1-d_i+b_{ii}d_i)(1-u_i(B_D^{+})t_i(B_D^{+}))}\prod_{j=1}^{i-1}\frac{u_j((B_D^{+}))}{1-u_j((B_D^{+}))t_j(B_D^{+})},\right.\\ &\left.\sum_{i=1}^{n}\frac{p_i(B_D^{+})}{(1-d_i+b_{ii}d_i)(1-u_i((B_D^{+}))t_i(B_D^{+}))}\prod_{j=1}^{i-1}\left(1+\frac{u_j(B_D^{+})p_j(B_D^{+})}{1-u_j(B_D^{+})t_j(B_D^{+})}\right)\right\}.\end{aligned}$$

By Lemma 2.2, we can easily get the following results: for each $i,j,k\in N$,

$$u_i(B_D^{+})=\frac{\sum\limits_{j=i+1}^{n}|b_{ij}|d_i}{1-d_i+b_{ii}d_i}\leq\frac{\sum\limits_{j=i+1}^{n}|b_{ij}|}{b_{ii}}=u_i(B^{+}),\tag{6}$$

$$\begin{aligned} t_k(B_D^+) &= \max_{i+1 \le k \le n} \left\{ \frac{|b_{ki}| d_k}{1 - d_k + b_{kk} d_k - \sum\limits_{j=i+1, \ne k}^{n} |b_{kj}| d_k} \right\} \\ &\le \max_{i+1 \le k \le n} \left\{ \frac{|b_{ki}|}{b_{kk} - \sum\limits_{j=i+1, \ne k}^{n} |b_{kj}|} \right\} = t_k(B^+), \end{aligned} \tag{7}$$

$$b_k(B_D^+) = \max_{k+1 \le i \le n} \left\{ \frac{\sum\limits_{j=k, \ne i}^{n} |b_{ij}| d_i}{1 - d_i + b_{ii} d_i} \right\} \le \max_{k+1 \le i \le n} \left\{ \frac{\sum\limits_{j=k, \ne i}^{n} |b_{ij}|}{b_{ii}} \right\} = b_k(B^+), \tag{8}$$

$$\begin{aligned} p_k(B_D^+) &= \max_{k+1 \le i \le n} \left\{ \frac{|b_{ik}| d_i + \sum\limits_{j=k+1, \ne i}^{n} |b_{ij}| d_i b_k(B_D^+)}{1 - d_i + b_{ii} d_i} \right\} \\ &\le \max_{k+1 \le i \le n} \left\{ \frac{|b_{ik}| + \sum\limits_{j=k+1, \ne i}^{n} |b_{ij}| b_k(B_D^+)}{b_{ii}} \right\} \\ &\le \max_{k+1 \le i \le n} \left\{ \frac{|b_{ik}| + \sum\limits_{j=k+1, \ne i}^{n} |b_{ij}| b_k(B^+)}{b_{ii}} \right\} \\ &= p_k(B^+). \end{aligned} \tag{9}$$

Furthermore, by (6), (7), (8) and (9), we have

$$\begin{aligned} \frac{1}{(1 - d_i + b_{ii} d_i)(1 - u_i(B_D^+) t_i(B_D^+))} &= \frac{1}{1 - d_i + b_{ii} d_i - \sum\limits_{j=i+1}^{n} |b_{ij}| d_i t_i(B_D^+)} \\ &\le \frac{1}{\min\left\{ b_{ii} - \sum\limits_{j=i+1}^{n} |b_{ij}| t_i(B^+), 1 \right\}} \\ &= \frac{1}{\min\left\{ \hat{\beta}_i, 1 \right\}}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{u_i(B_D^+)}{1 - u_i(B_D^+) t_i(B_D^+)} &= \frac{\sum\limits_{j=i+1}^{n} |b_{ij}| d_i}{1 - d_i + b_{ii} d_i - \sum\limits_{j=i+1}^{n} |b_{ij}| d_i t_i(B_D^+)} \\ &\le \frac{\sum\limits_{j=i+1}^{n} |b_{ij}|}{b_{ii} - \sum\limits_{j=i+1}^{n} |b_{ij}| t_i(B^+)} \\ &= \frac{1}{\hat{\beta}_i} \sum_{j=i+1}^{n} |b_{ij}|. \end{aligned} \tag{11}$$

From (10) and (11), we obtain

$$\|(B_D^+)^{-1}\|_\infty \le \max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\right.$$
$$\left.\sum_{i=1}^{n}\frac{p_i(B^+)}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^+)}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\}. \tag{12}$$

Therefore, the result in (4) follows from (5) and (12). □

Since a B-matrix is a $wcdd$ B-matrix, then by Theorem 2.3, we can obtain the following upper bound of $\max\limits_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty$ for B-matrix M.

Corollary 2.4. *Let $M=[m_{ij}]\in R^{n\times n}$ be a B-matrix with the form $M=B^++C$, where $B^+=[b_{ij}]$ is defined as in Theorem 1.4. Then*

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \le (n-1)\max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\right.$$
$$\left.\sum_{i=1}^{n}\frac{p_i(B^+)}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^+)}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\}, \tag{13}$$

where $\hat{\beta}_i$ is defined as in Theorem 2.3.

We next give a comparison of the bounds in (3) and (4) as follows.

Theorem 2.5. *Let $M=[m_{ij}]\in R^{n\times n}$ be a $wcdd$ B-matrix with the form $M=B^++C$, where $B^+=[b_{ij}]$ is defined as in Theorem 1.4. Let $\bar{\beta}_i$, $\tilde{\beta}_i$ and $\hat{\beta}_i$ be defined as in Theorem 1.5, Theorem 1.6 and Theorem 2.3, respectively. Then*

$$(n-1)\max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\ \sum_{i=1}^{n}\frac{p_i(B^+)}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^+)}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\}$$
$$\le \sum_{i=1}^{n}\left[\frac{n-1}{\min\{\bar{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right]$$
$$\le \sum_{i=1}^{n}\left(\frac{n-1}{min\{\tilde{\beta}_i,1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\tilde{\beta}_j}\right). \tag{14}$$

Proof. Since B^+ is a $wcdd$ matrix with positive diagonal elements, thus $p_i(B^+)\le 1(\forall i\in N)$, and for any $k\in N, i+1\le k\le n, 1\le i\le n-1$, if $b_{ki}\ne 0$, then

$$l_{ki}(B^+)-t_{ki}(B^+) = \frac{|b_{ki}|+\sum\limits_{j=i+1,\ne k}^{n}|b_{kj}|}{|b_{kk}|} - \frac{|b_{ki}|}{|b_{kk}|-\sum\limits_{j=i+1,\ne k}^{n}|b_{kj}|}$$
$$= \frac{\sum\limits_{j=i+1,\ne k}^{n}|b_{kj}|(|b_{kk}|-|b_{ki}|-\sum\limits_{j=i+1,\ne k}^{n}|b_{kj}|)}{|b_{kk}|(|b_{kk}|-\sum\limits_{j=i+1,\ne k}^{n}|b_{kj}|)}$$
$$\ge 0.$$

If $b_{ki} = 0$, then $l_{ki}(B^+) \geq t_{ki}(B^+) = 0$. Hence, for any $i \in N$, we have

$$0 \leq t_i(B^+) \leq l_i(B^+) < 1. \tag{15}$$

By (15), for each $i \in N$,

$$\bar{\beta}_i = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| l_i(B^+) \leq b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| t_i(B^+) = \hat{\beta}_i, \tag{16}$$

then by (16), for each $i \in N$,

$$\frac{1}{\min\{\hat{\beta}_i, 1\}} \leq \frac{1}{\min\{\bar{\beta}_i, 1\}}, \tag{17}$$

and for $j = 1, 2, \ldots, n-1$,

$$1 + \frac{1}{\hat{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}| \leq 1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}|, \tag{18}$$

From (17) and (18), we have

$$\begin{aligned}
&(n-1)\max\left\{\sum_{i=1}^{n} \frac{1}{min\{\hat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(\frac{1}{\hat{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}|\right), \right. \\
&\left. \sum_{i=1}^{n} \frac{p_i(B^+)}{min\{\hat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{p_j(B^+)}{\hat{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}|\right)\right\} \\
&\leq \sum_{i=1}^{n} \left[\frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}|\right)\right].
\end{aligned} \tag{19}$$

Otherwise, note that

$$\tilde{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}|, \qquad \bar{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}| l_i(B^+),$$

and $l_k(B_D^+) \leq l_k(B^+) < 1$. Hence, for each $i \in N$, $\tilde{\beta}_i \leq \bar{\beta}_i$, and

$$\frac{1}{min\{\bar{\beta}_i, 1\}} \leq \frac{1}{min\{\tilde{\beta}_i, 1\}}. \tag{20}$$

In the meantime, for $j = 1, 2, \ldots, n-1$,

$$1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}| \leq 1 + \frac{1}{\tilde{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}| = \frac{1}{\tilde{\beta}_j} \left(\tilde{\beta}_j + \sum_{k=j+1}^{n} |b_{jk}|\right) = \frac{b_{jj}}{\tilde{\beta}_j}. \tag{21}$$

From (20) and (21), we obtain

$$\sum_{i=1}^{n} \frac{n-1}{min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^{n} |b_{jk}|\right) \leq \sum_{i=1}^{n} \left(\frac{n-1}{min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j}\right). \tag{22}$$

The result in (14) follows from (19) and (22). □

Adopting the same procedure as in the proof of Theorem 2.4 in [9], we can provide the following new bound for the constant $\beta_p(M)$ when M is a P-matrix, where

$$\beta_p(M) = \max_{d \in [0,1]^n} \|(I - D + DM)^{-1} D\|_p,$$

$D = diag(d_i)$ with $0 \leq d_i \leq 1$ $(i \in N)$, and $\| \bullet \|_p$ is the matrix norm induced by the vector norm for $p \geq 1$. The constant is used to measure the sensitivity of the solution of the P-matrix linear complementarity problem [1] .

Theorem 2.6. *Let $M = [m_{ij}] \in R^{n\times n}$ be a $wcdd$ B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is defined as in Theorem 1.4. Then*

$$\beta_\infty(M) \le (n-1)\max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\right.$$
$$\left.\sum_{i=1}^{n}\frac{p_i(B^+)}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^+)}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\}, \tag{23}$$

where $\hat{\beta}_i > 0$ is defined as in Theorem 2.3.

Corollary 2.7. *Let $M = [m_{ij}] \in R^{n\times n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is defined as in Theorem 1.4. Then*

$$\beta_\infty(M) \le (n-1)\max\left\{\sum_{i=1}^{n}\frac{1}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right),\right.$$
$$\left.\sum_{i=1}^{n}\frac{p_i(B^+)}{min\{\hat{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(1+\frac{p_j(B^+)}{\hat{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right)\right\}. \tag{24}$$

3 Numerical examples

In this section, we present numerical examples to illustrate the advantages of our derived results.

Example 3.1. *Consider the family of B-matrices in [12]:*

$$M_k = \begin{bmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1\frac{k}{k+1} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{bmatrix},$$

where $k \ge 1$. Then $M_k = B_k^+ + C_k$, where

$$B_k^+ = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1\frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{bmatrix}.$$

By Theorem 1.4, we have

$$\max_{d\in[0,1]^4}\|(I-D+DM_k)^{-1}\|_\infty \le 30(k+1) \to +\infty, \quad if\ k \to +\infty.$$

By Theorem 1.5, we get

$$\max_{d\in[0,1]^4}\|(I-D+DM_k)^{-1}\|_\infty < 15.2675.$$

By Corollary 1 of [13], we have

$$\max_{d\in[0,1]^4}\|(I-D+DM_k)^{-1}\|_\infty \le \sum_{i=1}^{4}\left(\frac{3}{min\{\tilde{\beta}_i,1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\tilde{\beta}_j}\right) \approx 15.2675.$$

By Corollary 2.4, we have

$$\max_{d\in[0,1]^4}\|(I-D+DM_k)^{-1}\|_\infty < 9.6467.$$

Example 3.2. *Consider the $wcdd$ B-matrix in [13]:*

$$M = \begin{bmatrix} 1.5 & 0.2 & 0.4 & 0.5 \\ -0.1 & 1.5 & 0.5 & 0.1 \\ 0.5 & -0.1 & 1.5 & 0.1 \\ 0.4 & 0.4 & 0.8 & 1.8 \end{bmatrix}.$$

Thus $M = B^+ + C$, where

$$B^+ = \begin{bmatrix} 1 & -0.3 & -0.1 & 0 \\ -0.6 & 1 & 0 & -0.4 \\ 0 & -0.6 & 1 & -0.4 \\ -0.4 & -0.4 & 0 & 1 \end{bmatrix}.$$

By Theorem 1.6, we get

$$\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_\infty \leq 41.1111.$$

By Theorem 2.3, we obtain

$$\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_\infty \leq 21.6667.$$

This example shows that the bound in Theorem 2.3 is sharper than that in Theorem 1.6.

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