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Research Article

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Value distributions of solutions to complex linear differential equations in angular domains

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Abstract: In this paper we study the iterated order and oscillations of the solutions to some complex linear differential equations in angular domains. Our theorems improve some recent results.

Keywords: Meromorphic function, Iterated order, Complex differential equation, Angular domain

MSC: 30D10, 34M05

1 Introduction and main results

In this article, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory in the unit disk $\Delta = \{z : |z| < 1\}$, in an angular region, and in the complex plane $\mathbb{C}$ respectively; see [1–5]. The order $\rho(f)$ and lower order $\mu(f)$ of f which is meromorphic in $\mathbb{C}$ or Δ are defined as follows:

$$\rho_{\mathbb{C}}(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_{\Delta}(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)},$$
$$\mu_{\mathbb{C}}(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu_{\Delta}(f) = \liminf_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}.$$

We call a meromorphic function is adimissable in the unit disk if it satisfies

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} = \infty.$$

Definition 1.1. *The iterated n-order $\rho_{n,\Delta}(f)$ of a meromorphic function $f(z)$ in Δ is defined by*

$$\rho_{n,\Delta}(f) = \limsup_{r\to 1^-} \frac{\log^{[n]} T(r,f)}{-\log(1-r)}, \tag{1}$$

where $\log^{[1]} r = \log r$ *and* $\log^{[n+1]} r = \log(\log^{[n]} r)$, $n \in \mathbb{N}$.

Definition 1.2. *The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by*

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is non-admissible}, \\ \min\{n \in \mathbb{N} : \rho_{n,\Delta}(f) < \infty\} & \text{if } f \text{ is admissible}, \\ \infty & \text{if } \rho_{n,\Delta}(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

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Definition 1.3. *For $n \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$, the iterated n-convergent exponent of the sequence of a-point in Δ of a meromorphic function f in Δ is defined by*

$$\lambda_{n,\Delta}(f-a) = \limsup_{r\to 1^-} \frac{\log^{[n]} N(r, \frac{1}{f-a})}{-\log(1-r)}$$

and $\overline{\lambda}_{n,\Delta}(f-a)$, the iterated n-convergent exponent of the sequence of distinct a-point in Δ of a meromorphic function f in Δ is defined by

$$\overline{\lambda}_{n,\Delta}(f-a) = \limsup_{r\to 1^-} \frac{\log^{[n]} \overline{N}(r, \frac{1}{f-a})}{-\log(1-r)}.$$

The growth and oscillation of solutions to higher-order linear differential equations in $\mathbb{C}$ and in Δ have been well studied by many authors. In the paper [6], Cao and Yi studied the properties of solutions to the arbitrary order linear differential equations in Δ of the form

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{2}$$

where $A_0(\not\equiv 0), A_1, \cdots, A_k$ are analytic in Δ. In fact, they got the following theorem.

Theorem 1.4. *Let $0 < p < \infty$ and $i(A_0) = p$. If $\max\{i(A_j) : j = 1, 2, \cdots, k\} < p$ or $\max\{\rho_{p,\Delta}(A_j) : j = 1, 2, \cdots, k\} < \rho_{p,\Delta}(A_0)$, then $i(f) = p + 1$ and $\rho_{p,\Delta}(A_0) \le \rho_{p+1,\Delta}(f)$ holds for all solutions $f \not\equiv 0$ of equation* (2).

In what follows, we give some notations and definitions of a meromorphic function in an angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$. In this paper, Ω usually denotes the angular domain $\Omega(\alpha, \beta)$ and $\Omega_\varepsilon = \{z : \alpha + \varepsilon < \arg z < \beta - \varepsilon\}$, where $0 < \varepsilon < \frac{\beta-\alpha}{2}$. Let $f(z)$ be a meromorphic function on $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}$. Recall the definition of Ahlfors-Shimizu characteristic in an angular domain; see [5, pp.66]. Set $\Omega(r) = \Omega(\alpha, \beta) \cap \{z : 0 < |z| < r\}$. Define

$$\mathcal{S}(r, \Omega, f) = \frac{1}{\pi} \iint\limits_{\Omega(r)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\rho, \;\; \mathcal{T}(r, \Omega, f) = \int_1^r \frac{\mathcal{S}(t, \Omega, f)}{t} dt.$$

The order and lower order of f on Ω are defined by

$$\rho_\Omega(f) = \limsup_{r\to\infty} \frac{\log \mathcal{T}(r, \Omega, f)}{\log r}, \;\; \mu_\Omega(f) = \liminf_{r\to\infty} \frac{\log \mathcal{T}(r, \Omega, f)}{\log r}.$$

We remark that the above definitions is reasonable because $\mathcal{T}(r, \mathbb{C}, f) = T(r, f) + O(1)$; see [1, pp.20].

Definition 1.5. *The iterated n-order $\rho_{n,\Omega}(f)$ of a meromorphic function $f(z)$ in an angular region Ω is defined by*

$$\rho_{n,\Omega}(f) = \limsup_{r\to\infty} \frac{\log^{[n]} \mathcal{T}(r, \Omega, f)}{\log r}, \tag{3}$$

where $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r), n \in \mathbb{N}$.

Remark 1. *It is obvious that $\rho_{1,\Omega}(f) = \rho_\Omega(f)$.*

Motivated by the definition of a convergent exponent of a-value points of f in Ω in [5, p. 93], we give the following definition.

Definition 1.6. *For $n \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$, the iterated n-convergent exponent of the sequence of a-point in Ω of a meromorphic function f in Ω is defined by*

$$\lambda_{n,\Omega}(f-a) = \limsup_{r\to\infty} \frac{\log^{[n]} N(r, \Omega, \frac{1}{f-a})}{\log r}$$

and $\overline{\lambda}_{n,\Omega}(f-a)$*, the iterated n-convergent exponent of the sequence of distinct a-point in* Ω *of a meromorphic function* f *in* Ω *is defined by*

$$\overline{\lambda}_{n,\Omega}(f-a)=\limsup_{r\to\infty}\frac{\log^{[n]}\overline{N}(r,\Omega,\frac{1}{f-a})}{\log r}.$$

The first purpose of this paper is to study the iterated growth order of solutions to complex linear differential equations in an angular domain. In fact, we obtain the results as follows:

Theorem 1.7. *Let* $A(z)$ *be analytic in angular region* $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$ *satisfying either*

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,\Omega_\varepsilon,A)}{r^\omega\log r}=\infty,$$

or

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}\mathcal{T}(r,\Omega_\varepsilon,A)}{\log r}=\infty,(n\geq 2),$$

where $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}, 0<\varepsilon<\frac{\beta-\alpha}{2}, \omega=\pi/(\beta-\alpha)$*. Then, all solutions* $f\not\equiv 0$ *of the equation* $f^{(k)}+A(z)f=0$ *have the order* $\rho_{n,\Omega}(f)=+\infty$.

Theorem 1.8. *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$ *be analytic in an angular region* $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$*. If for any small* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$, $\rho_{1,\Omega}(A_j)<\rho_{1,\Omega_\varepsilon}(A_0)-\omega$ *and* $\rho_{n,\Omega}(A_j)<\rho_{n,\Omega_\varepsilon}(A_0)$ $(n\geq 2, j=1,2,\cdots,k-1)$*, then all solutions* $f\not\equiv 0$ *of equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0 \quad (4)$$

have the order $\rho_{n+1,\Omega}(f)\geq\rho_{n,\Omega_\varepsilon}(A_0)$*. In particular,* $\rho_{n,\Omega}(f)=+\infty$ *if* $\rho_{n,\Omega_\varepsilon}(A_0)>0$.

Theorem 1.9. *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$ *and* $g(z)$ *be analytic in an angular region* $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}(0<\beta-\alpha<2\pi)$*. Suppose that* $f\not\equiv 0$ *is a solution of equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=g(z), \quad (5)$$

such that, for $n\geq 2$, $\max\{\rho_{n,\Omega}(A_j),\rho_{n,\Omega}(g)\}<\rho_{n,\Omega_\varepsilon}(f)$ *and, for* $n=1$, $\max\{\rho_{1,\Omega}(A_j),\rho_{1,\Omega}(g)\}<\rho_{1,\Omega_\varepsilon}(f)-\omega$ *. Then* $\rho_{n,\Omega_\varepsilon}(f)\leq\overline{\lambda}_{n,\Omega}(f)=\lambda_{n,\Omega}(f)$ *and* $\overline{\lambda}_{n,\Omega_\varepsilon}(f)=\lambda_{n,\Omega_\varepsilon}(f)\leq\rho_{n,\Omega}(f)$ *for any positive integer* n.

Recalling the Nevanlinna theory in an angular domain and following the terms in [1], we set

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t};$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta;$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_n-\alpha);$$

$$D_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f),$$

where $\omega=\pi/(\beta-\alpha)$, and $b_n=|b_n|e^{i\beta_n}$ are poles of $f(z)$ in $\Omega(\alpha,\beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Thus, the order and lower order of f on Ω can also be defined by

$$\rho_{\alpha,\beta}(f)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r},\quad \mu_{\alpha,\beta}(f)=\liminf_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r}. \quad (6)$$

For $a \in \mathbb{C} \cup \{\infty\}$, the convergence exponent of the sequence of a-point in $\Omega(\alpha, \beta)$ of a meromorphic function f is defined by

$$\tau_{\alpha,\beta}(f-a) = \limsup_{r\to+\infty} \frac{\log C_{\alpha,\beta}(r, \frac{1}{f-a})}{\log r}. \tag{7}$$

According to the inequality, see [5, Theorem 2.4.7],

$$S_{\alpha,\beta}(r, f) \le 2\omega^2 \frac{\mathcal{T}(r, \Omega, f)}{r^\omega} + \omega^3 \int_1^r \frac{\mathcal{T}(t, \Omega, f)}{t^{\omega+1}} dt + O(1),$$

if $\rho_\Omega(f) < \infty$, then $\rho_{\alpha,\beta}(f) < \infty$.

We consider q pairs of real numbers $\{\alpha_j, \beta_j\}$ such that

$$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \alpha_q < \beta_q \le \pi \tag{8}$$

and the angular domains $X = \cup_{j=1}^q \{z : \alpha_j \le \arg z \le \beta_j\}$. For a function f meromorphic in the complex plane $\mathbb{C}$, we define the order of f on X as

$$\rho_X(f) = \limsup_{r\to\infty} \frac{\log \mathcal{T}(r, X, f)}{\log r}.$$

It is obvious that $\rho_{\alpha_j,\beta_j}(f) \le \rho_X(f) \le \sum_{j=1}^q \rho_{\alpha_j,\beta_j}(f)$, $j = 1, 2, \cdots, q$ and $\rho_X(f) = +\infty$ if and only if there exists at least one $1 \le j_0 \le q$ such that $\rho_{\alpha_{j_0},\beta_{j_0}}(f) = +\infty$.

In [7], Wu considered the growth of solutions to higher order linear homogeneous differential equations in angular domains. The following theorem was obtained.

Theorem 1.10. *Let A_0 be a meromorphic function in $\mathbb{C}$ with finite lower order $\mu < \infty$ and nonzero order $0 < \rho \le \infty$ and $\delta = \delta(\infty, A_0) > 0$. For q pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying* (8) *and*

$$\sum_{j=1}^q (\alpha_{j+1} - \beta_j) < \frac{4}{\lambda} \arcsin \sqrt{\frac{\lambda}{2}}, \tag{9}$$

where $\lambda > 0$ with $\mu \le \lambda \le \rho$. If $A_j(z)(j = 1, 2, \cdots, n)$ are meromorphic functions in $\mathbb{C}$ with $T(r, A_j) = o(T(r, A_0))$, every solution $f \not\equiv 0$ to the equation

$$A_n f^{(n)} + A_{n-1} f^{(n-1)} + \cdots + A_0 f = 0 \tag{10}$$

has the order $\rho_X(f) = +\infty$ in $X = \cup_{j=1}^q \{z : \alpha_j \le \arg z \le \beta_j\}$.

For the derivatives of the nonzero solutions to the equation in the above theorem, we can get the following result easily.

Theorem 1.11. *Let A_0 be a meromorphic function in $\mathbb{C}$ with finite lower order $\mu < \infty$ and nonzero order $0 < \rho \le \infty$ and $\delta = \delta(\infty, A_0) > 0$. For q pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying* (8) *and*

$$\sum_{j=1}^q (\alpha_{j+1} - \beta_j) < \frac{4}{\lambda} \arcsin \sqrt{\frac{\lambda}{2}}, \tag{11}$$

where $\lambda > 0$ with $\mu \le \lambda \le \rho$. If $A_j(z)(j = 1, 2, \cdots, n)$ are meromorphic functions in $\mathbb{C}$ with $T(r, A_j) = o(T(r, A_0))$, the derivatives of every solution $f \not\equiv 0$ to the equation (10) *have the order $\rho_X(f^{(p)}) = +\infty$ in $X = \cup_{j=1}^q \{z : \alpha_j \le \arg z \le \beta_j\}$, where p is a natural number.*

The last result relates to the convergence exponent of the sequence of a-point of the solutions of equation (8) in the angular domain X.

Theorem 1.12. *Let A_0 be an entire function in $\mathbb{C}$ with finite lower order $\mu < \infty$ and nonzero order $0 < \rho \leq \infty$ and $\delta = \delta(\infty, A_0) > 0$. For q pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying* (8) *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\lambda}\arcsin\sqrt{\frac{\lambda}{2}}, \tag{12}$$

where $\lambda > 0$ with $\mu \leq \lambda \leq \rho$, if $A_j(z)(j = 1, 2, \cdots, n)$ are entire functions in $\mathbb{C}$ with $T(r, A_j) = o(T(r, A_0))$, then every solution $f \not\equiv 0$ to the equation (10) *satisfies $\tau_X(f - a) = +\infty$ in $X = \cup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$ for $a \neq 0$.*

2 Preliminary lemmas

Lemma 2.1 ([8])**.** *The transformation*

$$\zeta(z) = \frac{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} - 1}{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} + 1}, \quad \theta_0 = \frac{\alpha + \beta}{2} \tag{13}$$

maps the angular domain $X = \{z : \alpha < \arg z < \beta\}, (0 < \beta - \alpha < 2\pi)$ conformally onto the unit disk $\{\zeta : |\zeta| < 1\}$ in the ζ-plane, and maps $z = e^{i\theta_0}$ to $\zeta = 0$. The image of $X_\varepsilon = \{z : 1 \leq |z| \leq r, \alpha + \varepsilon \leq \arg z \leq \beta - \varepsilon\}, (0 < \varepsilon < \frac{\beta-\alpha}{2})$ in the ζ-plane is contained in the disk $\Delta_h := \{\zeta : |\zeta| < h\}$, where

$$h = 1 - \frac{\varepsilon}{\beta - \alpha} r^{-\frac{\pi}{\beta-\alpha}}.$$

On the other hand, the inverse image of the disk $\Delta_h := \{\zeta : |\zeta| < h\}, h < 1$ in the z-plane is contained in $X \cap \{z : |z| \leq r\}$, where

$$r = \left(\frac{2}{1-h}\right)^{(\beta-\alpha)/\pi}.$$

The inverse transformation of (13) *is*

$$z(\zeta) = e^{i\theta_0}\left(\frac{1+\zeta}{1-\zeta}\right)^{(\beta-\alpha)/\pi}. \tag{14}$$

Remark 2. *Note that the conformal mapping* (13) *is univalent, then we get*

$$n\left(r, \Omega_\varepsilon, \frac{1}{f(z) - a}\right) \leq n\left(1 - \eta r^{-\omega}, \Delta, \frac{1}{f(z(\zeta)) - a}\right)$$

and

$$n(h, \Delta, \frac{1}{f(z(\zeta)) - a}) \leq n\left(\left(\frac{2}{1-h}\right)^{\frac{1}{\omega}}, \Omega, \frac{1}{f(z) - a}\right)$$

by Lemma 2.1, the notations η, ω here are similar as that in the following Lemma 2.2. Thus, by the Definition 1.3 and 1.6 we conclude that $\frac{1}{\omega}\lambda_{n,\Omega_\varepsilon}(f(z) - a) \leq \lambda_{n,\Delta}(f(z(\zeta)) - a) \leq \frac{1}{\omega}\lambda_{n,\Omega}(f(z) - a)$.

Using Lemma 2.1, the following Lemma 2.2 was proved in [9].

Lemma 2.2. *Let $f(z)$ be meromorphic in angular region Ω. For any small $\varepsilon > 0$, write $\omega = \frac{\pi}{\beta-\alpha}, \eta = \frac{\varepsilon}{\beta-\alpha}$. Then the following inequalities hold:*

$$\mathcal{T}(r, \mathbb{C}, f(z(\zeta))) \leq 2\mathcal{T}\left(\left(\frac{2}{1-r}\right)^{\frac{1}{\omega}}, \Omega, f(z)\right) + O(1), \tag{15}$$

$$\mathcal{T}(r, \Omega_\varepsilon, f(z)) \leq \frac{r^\omega}{\omega\eta}\mathcal{T}\left(1 - \eta r^{-\omega}, \mathbb{C}, f(z(\zeta))\right) + O(1), \tag{16}$$

where $z = z(\zeta)$ is the inverse transformation of (13). *Consequently,*

$$\rho_{\Delta}(f(z(\zeta))) \le \frac{1}{\omega}\rho_{\Omega}(f(z)), \quad \rho_{\Omega_{\varepsilon}}(f(z)) \le (\rho_{\Delta}(f(z(\zeta))) + 1)\omega. \tag{17}$$

Remark 3. *From* (15), (16) *and Definition 1.1, 1.5, we obtain, for $n \ge 2$,*

$$\frac{1}{\omega}\rho_{n,\Omega_{\varepsilon}}(f(z)) \le \rho_{n,\Delta}(f(z(\zeta))) \le \frac{1}{\omega}\rho_{n,\Omega}(f(z)). \tag{18}$$

Lemma 2.3 ([9, 10]). *Let $f(z)$ be meromorphic in $\Omega = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$ and $z = z(\zeta)$ be the inverse transformation of* (13). *Write $F(\zeta) = f(z(\zeta)), \psi(\zeta) = f^{(l)}(z(\zeta))$. Then*

$$\psi(\zeta) = \sum_{j=1}^{l} \alpha_j F^{(j)}(\zeta), \tag{19}$$

where the coefficients α_j are the polynomials (with numerical coefficients) in the variables $V(\zeta)(= \frac{1}{z'(\zeta)}), V'(\zeta), V''(\zeta), \cdots$. Moreover, we have $T(r, \alpha_j) = O(\log(1-r)^{-1}), j = 1, 2, \cdots, l$.

Lemma 2.4 ([11]). *Let $A_0, A_1, \cdots, A_{k-1}$ and $F(\not\equiv)$ be analytic function in Δ and let $f(z)$ be a solution of equation*

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_1(z)f'(z) + A_0(z)f(z) = F(z) \tag{20}$$

such that $\max\{\rho_{n,\Delta}(F), \rho_{n,\Delta}(A_j), j = 0, 1, \cdots, k-1\} < \rho_{n,\Delta}(f)$. *Then $\overline{\lambda}_{n,\Delta}(f) = \lambda_{n,\Delta}(f) = \rho_{n,\Delta}(f)$.*

Lemma 2.5 ([12]). *Let $\varphi(r)$ be a nondecreasing, continuous function on $\mathbb{R}^+$, and let*

$$0 < \rho < \limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r}$$

and $H = \{r \in \mathbb{R}^+ : |\varphi(r)| \ge r^{\rho}\}$. Then

$$\overline{\log dens} H = \limsup_{r\to\infty} \frac{\int_{H\cap[1,r]} \frac{1}{t}dt}{\log r} > 0.$$

Lemma 2.6 ([5, Theorem 2.6.5]). *Let $f(z)$ be a meromorphic function in $\overline{\Omega}(\alpha, \beta)$. Then for $\tau > 1$ and a natural number p, we have*

$$S_{\alpha+\eta,\beta-\eta}(r, f) \le K(S_{\alpha,\beta}(\tau r, f^{(p)}) + \log^+ r + 1), \tag{21}$$

where η is such that $0 < 2\eta < \beta - \alpha$ and K is a constant only depending on τ, η, α and β.

It is important and necessary to determine the relations between $C_{\alpha,\beta}(r, f)$ and $N(r, \Omega, f)$, which will be helpful in characterizing meromorphic functions in an angle in terms of the number of points of some values.

Lemma 2.7. *[5] Let $f(z)$ be a meromorphic function on $\overline{\Omega}(\alpha, \beta)$. Then the following inequalities hold:*

$$C_{\alpha,\beta}(r, f) \le 4\omega\frac{N(r)}{r^{\omega}} + 2\omega^2 \int_1^r \frac{N(t)}{t^{\omega+1}}dt \tag{22}$$

and

$$C_{\alpha,\beta}(r, f) \ge 2\omega\sin(\omega\delta)\frac{N_0(r)}{r^{\omega}} + 2\omega^2\sin(\omega\delta)\int_1^r \frac{N_0(t)}{t^{\omega+1}}dt, \tag{23}$$

where $N(t) = N(t, \Omega, f) = \int_1^r \frac{n(t,\Omega,f)}{t}dt$, $n(t, \Omega, f)$ is the number of poles of $f(z)$ in $\Omega \cap \{z : 1 < |z| \le t\}$ and $N_0(t) = N(t, \Omega_{\delta}, f) = \int_1^t \frac{n(t,\Omega_{\delta},f)}{t}dt$ and $\Omega_{\delta} = \Omega(\alpha+\delta, \beta-\delta)$. The above two inequalities still hold for $\overline{C}$ and $\overline{N}$ in the place of C and N.

Note that we may replace the integrated counting function $N(r,\Omega,\frac{1}{f-a})$ with unintegrated counting function $n(r,\Omega,\frac{1}{f-a})$ in the definition of the convergent exponent, because, see [5, pp. 39],

$$N\left(r,\Omega,\frac{1}{f-a}\right)=\int_0^r\frac{n(t,\Omega,\frac{1}{f-a})-n(0,\Omega,\frac{1}{f-a})}{t}dt+n\left(0,\Omega,\frac{1}{f-a}\right)\log r$$
$$\geq\int_1^r\frac{n(t,\Omega,\frac{1}{f-a})}{t}dt\geq n\left(dr,\Omega,\frac{1}{f-a}\right)\log\frac{1}{d},\quad 0<d<1. \tag{24}$$

for $1<dr$ and

$$N\left(r,\Omega,\frac{1}{f-a}\right)=\int_1^r\frac{n(t,\Omega,\frac{1}{f-a})}{t}dt+O(1)\leq n\left(t,\Omega,\frac{1}{f-a}\right)\log r+O(1). \tag{25}$$

Lemma 2.8 ([1]). *Suppose that $f(z)$ is a nonconstant meromorphic function in an angular domain $\Omega(\alpha,\beta)$ with $0<\beta-\alpha\leq 2\pi$. Then:*

(1) *([1, Chapter 1]) for any complex number $a\in\mathbb{C}$*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+O(1), \tag{26}$$

(2) *([1, p.138]) for any $r<R$,*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R\frac{\log^+T(t,f)}{t^{1+\omega}}dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\right\} \tag{27}$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right) \tag{28}$$

where $\omega=\frac{\pi}{\beta-\alpha}$, and K is a positive constant not depending on r and R.

Lemma 2.9 ([5, Corollary 2.2.2]). *Let $f(z)$ be an analytic function on $\overline{\Omega}(\alpha,\beta)$ with $0<\beta-\alpha\leq 2\pi$. Then we have*

$$S_{\alpha,\beta}(r,f)\leq\frac{2\omega}{\pi}\int_1^r\frac{\log^+M(t,\Omega,f)}{t^{1+\omega}}dt+\frac{4}{\pi}\frac{\log^+M(r,\Omega,f)}{r^{\omega}}, \tag{29}$$

where $M(r,\Omega,f)=\max\{|f(te^{i\theta})|:\alpha\leq\theta\leq\beta,1\leq t\leq r\}$ and K is a positive constant.

Let $f(z)$ be a non-constant entire function and $M(r,f)$ the maximum of $|f(z)|$ on the circle $|z|=r$, that is $M(r,f)=\max_{|z|=r}|f(z)|$. We have the following relations between $M(r,f)$ and $T(r,f)$.

Lemma 2.10 ([4, Theorem 1.4]). *Suppose $f(z)$ is a non-constant entire function. Then for $0\leq r<R<+\infty$, we have*

$$T(r,f)\leq\log^+M(r,f)\leq\frac{R+r}{R-r}T(r,f). \tag{30}$$

3 Proof of Theorems

Proof of Theorem 1.7. Suppose that $f\not\equiv 0$ is a solution of $f^{(k)}+A(z)f=0$ in Ω. Then, by Lemma 2.3, $F(\zeta)=f(z(\zeta))$ is a solution of the differential equation

$$\alpha_kF^{(k)}(\zeta)+\alpha_{k-1}F^{(k-1)}(\zeta)+\cdots+\alpha_1F^{'}(\zeta)+B(\zeta)=0 \tag{31}$$

in Δ, where $\alpha_j\,(j=1,2,\cdots,k)$ are described in Lemma 2.3 and $T(r,\alpha_j)=O(\log(1-r)^{-1})$, that is $i(\alpha_j)=0$ by Definition 1.2, and $B(\zeta)=A(z(\zeta))$. If

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,\Omega_\varepsilon,A)}{r^{\omega}\log r}=\infty,$$

by (16) and Definition 1.2, we obtain $i(B(z(\zeta)))\ge 1$. Thus, by Theorem 1.4, we get $i(F)=i(B(z(\zeta)))+1\ge 2$, that is $\rho_{1,\Delta}(F)=+\infty$.

If

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}\mathcal{T}(r,\Omega_\varepsilon,A)}{\log r}=\infty,\,(n\ge 2),$$

by (16) and Definition 1.2, we obtain $i(B(z(\zeta)))\ge n$. Thus, by Theorem 1.4, we get $i(F)=i(B(z(\zeta)))+1\ge n+1$, that is $\rho_{n,\Delta}(F)=+\infty$. Combining these results with (17), (18) leads to $\rho_{n,\Omega}(f)=+\infty$. □

Proof of Theorem 1.8. Suppose that $f\not\equiv 0$ is a solution of (4). From (19), we have

$$\begin{aligned}
&f^{(k)}(z(\zeta))+A_{k-1}(z(\zeta))f^{(k-1)}(z(\zeta))+\cdots+A_1(z(\zeta))f'(z(\zeta))+A_0(z(\zeta))f(z(\zeta))\\
&=\sum_{j=1}^{k}\alpha_{j,k}F^{(j)}(\zeta)+A_{k-1}(z(\zeta))\sum_{j=1}^{k-1}\alpha_{j,k-1}F^{(j)}(\zeta)+\cdots+A_1(z(\zeta))\alpha_{1,1}F'(z(\zeta))+A_0(z(\zeta))f(z(\zeta))\\
&=\alpha_{k,k}F^{(k)}(z(\zeta))+(\alpha_{k-1,k}+A_{k-1}(z(\zeta))\alpha_{k-1,k-1})F^{(k-1)}(z(\zeta))\\
&\quad+\left(\alpha_{k-2,k}+\sum_{m=k-2}^{k-1}A_m(z(\zeta))\alpha_{k-2,m}\right)F^{(k-2)}(z(\zeta))+\cdots+\left(\alpha_{1,k}+\sum_{m=1}^{k-1}A_m(z(\zeta))\alpha_{1,m}\right)F'(z(\zeta))\\
&\quad+A_0(z(\zeta))F(z(\zeta))
\end{aligned}\tag{32}$$

Set $B_k(\zeta)=\alpha_{k,k}, B_j(\zeta)=\alpha_{j,k}+\sum_{m=j}^{k-1}A_m(z(\zeta))\alpha_{j,m},(j=1,2,\cdots,k-1), B_0(\zeta)=A_0(z(\zeta))$, then $F(\zeta)=f(z(\zeta))$ is a solution of the differential equation

$$B_k(\zeta)F^{(k)}(\zeta)+B_{k-1}(\zeta)F^{(k-1)}(\zeta)+\cdots+B_1(\zeta)F'(\zeta)+B_0(\zeta)F(\zeta)=0\tag{33}$$

in Δ. Since $T(r,\alpha_{j,m})=O(\log(1-r)^{-1})$ $(1\le j\le m\le k)$ by Lemma 2.3, it follows that

$$T(r,B_j(\zeta))\le\sum_{m=j}^{k-1}T(r,A_m(z(\zeta)))+O(\log(1-r)^{-1}),\, j=1,2,\cdots,k-1.\tag{34}$$

If for any small $0<\varepsilon<\frac{\beta-\alpha}{2}$, $\rho_{1,\Omega}(A_j)<\rho_{1,\Omega_\varepsilon}(A_0)-\omega$, the conclusion holds by [9, Theorem 1.8]. If $\rho_{n,\Omega}(A_j)<\rho_{n,\Omega_\varepsilon}(A_0)$ $(n\ge 2, j=1,2,\cdots,k-1)$, it follows from (34) and (18) that

$$\rho_{n,\Delta}(B_j(\zeta))\le\max\{\rho_{n,\Delta}(A_j(z(\zeta)))\}\le\frac{1}{\omega}\max\{\rho_{n,\Omega}(A_j(z))<\frac{1}{\omega}\rho_{n,\Omega_\varepsilon}(A_0(z)).\tag{35}$$

By $B_0(\zeta)=A_0(z(\zeta))$ and (18), we get

$$\frac{1}{\omega}\rho_{n,\Omega_\varepsilon}(A_0(z))\le\rho_{n,\Delta}(B_0(\zeta)).\tag{36}$$

Combining (35), (36) and $\rho_{n,\Delta}(B_k(\zeta))=0$, we deduce that $\rho_{n,\Delta}(B_j(\zeta))<\rho_{n,\Delta}(B_0(\zeta))$, $j=1,2,\cdots,k$. Thus, by Theorem 1.4, we obtain $i(F(\zeta))=n+1$ and $\rho_{n+1,\Delta}(F(\zeta))>\rho_{n,\Delta}(B_0(\zeta))$. It follows from (18), we get $\rho_{n+1,\Omega}(f)\ge\rho_{n,\Omega_\varepsilon}(A_0)$. □

Proof of Theorem 1.9. Suppose that $f\not\equiv 0$ is a solution of (10). Set $F(\zeta)=f(z(\zeta))$ and $G(\zeta)=f(z(\zeta))$ by (14). By (19) we also have (32) and denote $B_j(\zeta)$ as in the proof of Theorem 1.8. Thus, $F(\zeta)=f(z(\zeta))$ is a solution of the nonhomogeneous differential equation

$$B_k(\zeta)F^{(k)}(\zeta)+B_{k-1}(\zeta)F^{(k-1)}(\zeta)+\cdots+B_1(\zeta)F'(\zeta)+B_0(\zeta)F(\zeta)=G(\zeta)\tag{37}$$

in Δ. For $n \geq 2$, by similar arguments as in the proof of Theorem 1.8 we get

$$\rho_{n,\Delta}(B_j(\zeta)) \leq \max\{\rho_{n,\Delta}(A_j(z(\zeta)))\} \leq \frac{1}{\omega}\max\{\rho_{n,\Omega}(A_j(z)) < \frac{1}{\omega}\rho_{n,\Omega_\varepsilon}(f(z)) \leq \rho_{n,\Delta}(F(\zeta)) \tag{38}$$

for $j = 1, 2, \cdots, k-1$. Note that

$$\rho_{n,\Delta}(G(\zeta)) \leq \frac{1}{\omega}\rho_{n,\Omega}(g(z)) < \frac{1}{\omega}\rho_{n,\Omega_\varepsilon}(f(z)) \leq \rho_{n,\Delta}(F(\zeta)) \tag{39}$$

and $\rho_{n,\Delta}(B_k(\zeta)) = 0$, we have $\max\{\rho_{n,\Delta}(B_j(\zeta)), \rho_{n,\Delta}(G(\zeta))\} < \rho_{n,\Delta}(F(\zeta))$ for $j = 1, 2, \cdots, k$.

Hence, from Lemma 2.4, we have $\overline{\lambda}_{n,\Delta}(F) = \lambda_{n,\Delta}(F) = \rho_{n,\Delta}(F)$. For $n = 1$, we can easily obtain

$$\begin{aligned}
\rho_{1,\Delta}(B_j) &\leq \frac{1}{\omega}\max\{\rho_{1,\Omega}(A_j)\} < \frac{1}{\omega}(\rho_{1,\Omega_\varepsilon}(f) - \omega) \leq \rho_{1,\Delta}(f);\\
\rho_{1,\Delta}(B_0) &\leq \frac{1}{\omega}\rho_{1,\Omega}(A_0) < \frac{1}{\omega}(\rho_{1,\Omega_\varepsilon}(f) - \omega) \leq \rho_{1,\Delta}(f);\\
\rho_{1,\Delta}(G) &\leq \frac{1}{\omega}\rho_{1,\Omega}(g(z)) < \frac{1}{\omega}(\rho_{1,\Omega_\varepsilon}(f) - \omega) \leq \rho_{1,\Delta}(f)
\end{aligned} \tag{40}$$

and $\rho_{1,\Delta}(B_k(\zeta)) = 0$. Thus, $\max\{\rho_{1,\Delta}(B_j), \rho_{1,\Delta}(G)\} < \rho_{1,\Delta}(f)$. Applying Lemma 16 to (37), we deduce $\overline{\lambda}_{1,\Delta}(F) = \lambda_{1,\Delta}(F) = \rho_{1,\Delta}(F)$.

Finally, by Remark 2 and 3, we obtain $\rho_{n,\Omega_\varepsilon}(f) \leq \overline{\lambda}_{n,\Omega}(f) = \lambda_{n,\Omega}(f)$ and $\overline{\lambda}_{n,\Omega_\varepsilon}(f) = \lambda_{n,\Omega_\varepsilon}(f) \leq \rho_{n,\Omega}(f)$ for any positive integer n. □

Proof of Theorem 1.11. Suppose that $\rho_X(f^{(p)}) < +\infty$, then for any $j = 1, 2, \cdots, q$, we have $\rho_{\alpha_j,\beta_j}(f^{(p)}) < +\infty$. By the Definition 6 and Lemma 2.6, we know that $\rho_{\alpha_j,\beta_j}(f) < +\infty$ for $j = 1, 2, \cdots, q$. Since $\rho_{\alpha_j,\beta_j}(f) \leq \rho_X(f) \leq \sum_{j=1}^{q}\rho_{\alpha_j,\beta_j}(f)$, we have $\rho_X(f) < +\infty$. It is a contradiction to Theorem 1.10. □

Proof of Theorem 1.12. Suppose that $f(z) \not\equiv 0$ is a solution of equation (10) under the hypotheses of Theorem 1.12. Set $g(z) = f(z) - a$, substitute it into (10) to obtain

$$A_n g^{(n)} + A_{n-1}g^{(n-1)} + \cdots + A_0 g = -aA_0 \tag{41}$$

and rewrite it as

$$\frac{1}{g} = -\frac{1}{aA_0}\left(A_n\frac{g^{(n)}}{g} + A_{n-1}\frac{g^{(n-1)}}{g} + \cdots + A_1\frac{g'}{g} + A_0\right). \tag{42}$$

Appling Wiman-Valiron theory to (42), similarly as in [3, p.130], we know that

$$\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r} \leq \rho(A_0). \tag{43}$$

Therefore, for sufficiently large r, we have

$$\log T(r,f) \leq r^{\rho(A_0)+1}. \tag{44}$$

Since $T(r,g) = T(r,f) + O(1)$, then we have

$$\log T(r,g) \leq r^{\rho(A_0)+1}. \tag{45}$$

By Theorem 1.10, we know that there exist an angular domain $\Omega_0(\alpha_0, \beta_0) \subset X$ satisfying $\rho_{\alpha_0,\beta_0}(g) = \rho_{\alpha_0,\beta_0}(f) = +\infty$, and then $\rho(g) = +\infty$. From (2) of Lemma 2.8, for $\varepsilon \in (0, \frac{\pi}{2(\rho(A_0)+1)})$ and $\Omega_\theta(\theta-\varepsilon, \theta+\varepsilon) \subset \Omega_0(\alpha_0, \beta_0)$, we can deduce that

$$A_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{g'}{g}\right) \leq O\left(\int_1^{2r}\frac{\log^+ T(t,g)}{t^{1+\frac{\pi}{2\varepsilon}}}dt\right) = O\left(\int_1^{2r}\frac{t^{\rho(A_0)+1}}{t^{1+\frac{\pi}{2\varepsilon}}}dt\right) = O(1) \tag{46}$$

and

$$B_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{g'}{g}\right) \leq \frac{2\pi}{\varepsilon}r^{-\frac{\pi}{2\varepsilon}}m\left(r, \frac{g'}{g}\right) = \frac{2\pi}{\varepsilon}r^{-\frac{\pi}{2\varepsilon}}O(\log(rT(r,g))) = O(r^{\rho(A_0)+1-\frac{\pi}{2\varepsilon}}) = O(1) \tag{47}$$

outside a set of finite linear measure. By using Lemma 1.6 in [4, p.35], it is easy to see that (45) still holds for $g^{(j)}, j = 1, 2, \cdots, n$ in place of g. Similarly as above, (46) and (47) also hold by using $g^{(j)}$ instead of g. Therefore, according to the definition of $D_{\alpha,\beta}(r, g)$, we can deduce, for each $s = 1, 2, \cdots, n$,

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{g^{(s)}}{g}\right) \leq \sum_{j=1}^{s} D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{g^{(j)}}{g^{(j-1)}}\right) + O(1) = O(1), \tag{48}$$

with an exceptional set of finite linear measure. Combining (42) with (48) and from Lemma 2.9 and 2.10, we get

$$\begin{aligned} D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) &\leq \sum_{i=1}^{n} D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{g^{(i)}}{g}\right) + \sum_{i=0}^{n} D_{\theta-\varepsilon,\theta+\varepsilon}(r, A_i) + D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, -\frac{1}{aA_0}\right) + O(1) \\ &\leq 2\sum_{i=0}^{n} S_{\theta-\varepsilon,\theta+\varepsilon}(r, A_i) + O(1) \\ &\leq 2\sum_{i=0}^{n}\left(\frac{1}{\varepsilon}\int_1^r \frac{\log^+ M(t, A_j)}{t^{1+\frac{\pi}{2\varepsilon}}}dt + \frac{4}{\pi}\frac{\log^+ M(r, A_j)}{r^{\frac{\pi}{2\varepsilon}}}\right) + O(1) \\ &\leq 2\sum_{i=0}^{n}\left(\frac{3}{\varepsilon}\int_1^r \frac{T(t, A_j)}{t^{1+\frac{\pi}{2\varepsilon}}}dt + \frac{12}{\pi}\frac{T(r, A_j)}{r^{\frac{\pi}{2\varepsilon}}}\right) + O(1) \\ &\leq O(T(r, A_0)). \end{aligned} \tag{49}$$

If $g(z)$ has a zero at $z_0 \in \Omega_\theta(\theta-\varepsilon, \theta+\varepsilon)$ of multiplicity $m(> n)$, then from (42) we know z_0 is a zero of aA_0 of multiplicity at least $m - k$. Hence we have

$$N\left(r, \Omega_\theta, \frac{1}{g}\right) \leq n\overline{N}\left(r, \Omega_\theta, \frac{1}{g}\right) + N\left(r, \Omega_\theta, \frac{1}{aA_0}\right). \tag{50}$$

By Lemma 2.7, we obtain

$$\begin{aligned} C_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) &\leq n\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) + C_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{aA_0}\right) \\ &\leq n\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) + S_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{aA_0}\right) \\ &\leq n\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) + O(T(r, A_0)). \end{aligned} \tag{51}$$

Combining (49) with (51) and utilizing Lemma 2.8, we deduce that

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) \leq n\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{g}\right) + O(T(r, A_0)). \tag{52}$$

Given positive constants ζ satisfying $\rho(A_0) + 1 < \zeta < +\infty$ and set $H = \{r \in \mathbb{R}^+ : S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) \geq r^\zeta\}$, by applying Lemma 2.5 to $S_{\theta-\varepsilon,\theta+\varepsilon}(r, g)$, we get

$$\overline{\log dens} H = \limsup_{r\to\infty} \frac{\int_{H\cap[1,r]} \frac{1}{t}dt}{\log r} > 0.$$

Hence,

$$\frac{T(r, A_0)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r, g)} \leq \frac{r^{\rho(A_0)+\delta}}{r^\zeta} \to 0 \tag{53}$$

holds when r belongs to the infinite logarithmical measure set H. From (52), (53) and the definition of $\tau_{\alpha,\beta}(f)$, we know that $\tau_{\theta-\varepsilon,\theta+\varepsilon}(f - a) = +\infty$. Thus, $\tau_X(f - a) = +\infty$. □

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