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Research Article

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The join of split graphs whose completely regular endomorphisms form a monoid

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Abstract: In this paper, completely regular endomorphisms of the join of split graphs are investigated. We give conditions under which all completely regular endomorphisms of the join of two split graphs form a monoid.

Keywords: Endomorphism, Monoid, Join of split graphs, Completely regular

MSC: 05C25, 20M20

1 Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years, much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained (cf. [1–4]). The endomorphism monoids of graphs have valuable applications (cf. [1]) and are related to automata theory (cf. [2, 5]). Let X be a graph. Denote by $End(X)$ the set of all endomorphisms of X. It is known that $End(X)$ forms a monoid with respect to composition of mappings. We call $End(X)$ the endomorphism monoid of X. An element a of a semigroup S is said to be *regular* if there exists $x \in S$ such that $axa = a$. Let $f \in End(X)$. Then f is called a *regular endomorphism* of X if it is a regular element in $End(X)$. Denote by $rEnd(X)$ the set of all regular endomorphisms of X. For a monoid S, the composition of its two regular elements is not regular in general. So it is natural to ask: Under what conditions does the set $rEnd(X)$ form a monoid for a graph X? In [6], Hou, Gu and Shang characterized the regular endomorphisms of the join of split graphs and the conditions under which the regular endomorphisms of the join of split graphs form a monoid were given. The endomorphism monoids of split graphs and the joins of split graphs were studied by several authors (cf. [7–13]).

An element a of a semigroup S is said to be completely regular if $a = axa$ and $xa = ax$ hold for some $x \in S$. Let $f \in End(X)$ for a graph X. Then f is called a *completely regular endomorphism* of X if it is a completely regular element in $End(X)$. Denote by $cEnd(X)$ the set of all completely regular endomorphisms of graph X. In general, the composition of its two completely regular elements is also not completely regular for a monoid S. So it is natural to ask: Under what conditions does the set $cEnd(X)$ form a monoid for X? However, it seems difficult to obtain a general answer to this question. Therefore a natural strategy for work towards answering this question is to find various kinds of conditions for various kinds of graphs. In [14], completely regular endomorphisms of split graphs were characterized and the conditions under which the completely regular endomorphisms of split graphs form a monoid were given. In this paper we give an answer to this question in the range of the joins of split graphs.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent

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in X, the edge joining them is denoted by $\{x_1, x_2\}$. For a vertex v of X, denote by $N_X(v)$ (or just by $N(v)$) the set $\{x \in V(X) | \{x, v\} \in E(X)\}$. The cardinality of $N_X(v)$ is called the *degree* of v in X and is denoted by $d_X(v)$ (or just $d(v)$). A subgraph H is called an *induced subgraph* of X if for any $a, b \in H$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph X is called *complete* if $\{a, b\} \in E(X)$ for any $a, b \in V(X)$. We denote by K_n (or just K) a complete graph with n vertices. A *clique* of a graph X is a maximal complete subgraph of X. A subset $K \subseteq V(X)$ is said to be *complete* if $\{a, b\} \in E(X)$ for any two vertices $a, b \in K$ with $a \neq b$. A subset $S \subseteq V(X)$ is said to be *independent* if $\{a, b\} \notin E(X)$ for any two vertices $a, b \in S$. A graph X is called a *split graph* if its vertex set $V(X)$ can be partitioned into disjoint (non-empty) sets K and S such that K is a complete set and S is an independent set. In this paper, we always assume that a split graph X has a fixed partition $V(X) = K \cup S$, where K is a maximum complete set and S is an independent set. Since K is a maximum complete set of X, $0 \leq d_X(y) \leq n - 1$ for any $y \in S$. Let X and Y be two graphs. The *join* of X and Y, denoted by $X + Y$, is a graph such that $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y) \cup \{\{x, y\} | x \in V(X), y \in V(Y)\}$.

Let X and Y be two graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f from X to itself is called an *endomorphism* of X. A endomorphism f is said to be *half-strong* if $\{f(a), f(b)\} \in E(X)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. Denote by $hEnd(X)$ the set of all half-strong endomorphisms of X.

A *retraction* of a graph X is an endomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ to $V(Y)$ is the identity mapping on $V(Y)$. It is well known that the idempotents of $End(X)$ are retractions of X. Denote by $Idpt(X)$ the set of all idempotents of $End(X)$. Let $f \in End(X)$. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f.

We use the standard terminology and notation of semigroup theory as in [2, 15] and of graph theory as in [16]. We list some known results which are used in this paper.

Lemma 1.1 ([17]). *Let X be a graph and $f \in End(X)$. Then*
(1) *$f \in hEnd(X)$ if and only if I_f is an induced subgraph of X.*
(2) *If f is regular, then $f \in hEnd(X)$.*

Lemma 1.2 ([18]). *Let G be a graph and $f \in End(G)$. Then f is completely regular if and only if there exists $g \in Idpt(G)$ such that $\rho_g = \rho_f$ and $I_g = I_f$.*

Lemma 1.3 ([6]). *Let $X + Y$ be a join of split graphs and $f \in End(X + Y)$. Then the following statements are equivalent.*
(1) *There exists $h \in Idpt(X + Y)$ such that $I_h = I_f$.*
(2) *I_f is an induced subgraph of $X + Y$ and $\{x, y\} \notin E(X + Y)$ for any $x \in K_i \backslash I_f$ and $y \in S_i \cap I_f$ (where $i = 1, 2$).*

Lemma 1.4 ([6]). *Let $X + Y$ be a join of split graphs and $f \in End(X + Y)$. Then there exists $g \in Idpt(X + Y)$ such that $\rho_g = \rho_f$ if and only if there exists $b \in [a]_{\rho_f}$ such that $N(b) = \cup_{x \in [a]_{\rho_f}} N(x)$ for any $a \in V(X + Y)$.*

2 Main results

Let X be a split graph with $V(X) = K_1 \cup S_1$, where $K_1 = \{k_1, k_2, \cdots, k_n\}$ is a maximal complete set and $S_1 = \{x_1, x_2, \cdots, x_s\}$ is an independent set. Let Y be another split graph with $V(Y) = K_2 \cup S_2$, where $K_2 = \{r_1, r_2, \cdots, r_m\}$ is a maximal complete set and $S_2 = \{y_1, y_2, \cdots, y_t\}$ is an independent set. It is easy to see that the vertex set $V(X + Y)$ of $X + Y$ can be partitioned into three parts K, S_1 and S_2, i.e., $V(X + Y) = K \cup S_1 \cup S_2$, where $K = K_1 \cup K_2$ is a complete set, S_1 and S_2 are independent sets. Obviously, the subgraph of $X+Y$ induced by K is complete and the subgraph of $X+Y$ induced by $S = S_1 \cup S_2$ is complete bipartite. Hence in graph $X+Y$, $N(x_i) = N_X(x_i) \cup V(Y)$ for $x_i \in S_1$ and $N(y_i) = N_Y(y_i) \cup V(X)$ for $y_i \in S_2$. Clearly, $X+Y$ is a split graph extended by the edge set $\{\{x_i, y_j\} \mid x_i \in S_1, y_j \in S_2\}$. In this section, we investigate the completely regular endomorphisms of $X+Y$ and give the conditions under which the completely regular endomorphisms of $X+Y$ form a monoid.

First, we give a characterization of the completely regular endomorphisms for $X+Y$.

Theorem 2.1. *Let $X+Y$ be a join of split graphs and $f \in End(X+Y)$. Then f is completely regular if and only if the following conditions hold:*

(1) *I_f is an induced subgraph of $X+Y$ and $\{x,y\} \notin E(X+Y)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$.*

(2) *$N(b) = \cup_{x \in [b]_{\rho_f}} N(x)$ for any $b \in V(I_f)$ with $[b]_{\rho_f} \subseteq S$.*

(3) *$f(a) \neq f(b)$ for any $a, b \in V(I_f)$ with $a \neq b$.*

Proof. Necessity. Since f is completely regular, by Lemma 1.2, there exists $g \in Idpt(X+Y)$ such that $\rho_g = \rho_f$ and $I_g = I_f$. By Lemma 1.3, I_f is an induced subgraph of $X+Y$ and $\{x,y\} \notin E(X+Y)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$. Hence (1) holds. Let $a \in V(I_f)$. Note that $g \in Idpt(X+Y)$ and $I_g = I_f$, then $g(a) = a$. It follows from $\rho_g = \rho_f$ that $g(x) = a$ for any $x \in [a]_{\rho_f}$. Let $b \in V(I_f)$ be such that $[b]_{\rho_f} \subseteq S$. Then $b \in S$ and $N(b) \cap K \subseteq V(I_f)$ by (1). Thus $g(x) = x$ for any $x \in N(b) \cap K$ and so $N(b) \cap K \subseteq g(N(b))$. We claim that $N(b) = \cup_{x \in [b]_{\rho_f}} N(x)$. Otherwise, there exists $y \in [b]_{\rho_f}$ such that $N(y) \nsubseteq N(b)$. Then there exists $k \in K$ such that $k \in N(y)$ and $k \notin N(b)$. Note that $\{k,t\} \in E$ for any $t \in N(b)$. Then $\{g(k), g(t)\} \in E$ and so $g(k) \notin g(N(b))$. In particular, $g(k) \notin N(b) \cap K$. If $g(k) \in N(b) \cap S$, then $g^2(k) = g(k)$ since g is idempotent. Since $\{k, g(k)\} \in E(X+Y)$, $g(k)$ forms a loop in $X+Y$. This is a contradiction. Hence $g(k) \notin N(b)$. Now we get that $\{g(y), g(k)\} = \{b, g(k)\} \notin E(X+Y)$. This contradicts $\{y,k\} \in E$. Hence (2) holds. If $f(a) = f(b)$ for some $a, b \in V(I_f)$, then $[a]_{\rho_f} = [b]_{\rho_f}$. Note that $g \in Idpt(X+Y)$ and $\rho_g = \rho_f$, then $g(a) = g(b)$. This means that $\{a,b\} \nsubseteq V(I_g)$ and so $I_g \neq I_f$, which yields a contradiction. Hence (3) holds.

Sufficiency. Let $X+Y$ be a join of split graphs and $f \in End(X+Y)$ be such that (1), (2) and (3). Note that $f(a) \neq f(b)$ for any $a, b \in V(I_f)$. Then for any $a \in V(X+Y)$, there exists only one vertex in $[a]_{\rho_f} \cap V(I_f)$. Denote it by $\overline{[a]_{\rho_f}}$. Define a mapping g from $V(X+Y)$ to itself by

$$g(a) = \overline{[a]_{\rho_f}} \quad for\ all\ a \in V(X+Y).$$

Then g is well-defined. In the following, we show that $g \in End(X+Y)$. Let $\{a,b\} \in E(X+Y)$ for some $a, b \in V(X+Y)$. Since K is a maximum complete set of $X+Y$, $f(K)$ is a clique of size $n+m$. We have $K \subseteq I_f$, $|K \cap I_f| = n+m-1$ or $|K \cap I_f| = n+m-2$.

Assume that $K \subseteq I_f$. Then $g(k) = k$ for any $k \in K$. If $a, b \in V(I_f)$, then $g(a) = a$ and $g(b) = b$. Thus $\{g(a), g(b)\} = \{a,b\} \in E(X+Y)$. If $a \in V(I_f)$ and $b \notin V(I_f)$, then $b \in S$. If $[b]_{\rho_f} \subseteq S$, then $g(b) = \overline{[b]_{\rho_f}}$. Note that $N(\overline{[b]_{\rho_f}}) = \cup_{x \in [b]_{\rho_f}} N(x)$. Then $a \in N(\overline{[b]_{\rho_f}})$. Hence $\{g(a), g(b)\} = \{a, \overline{[b]_{\rho_f}}\} \in E(X+Y)$; If $[b]_{\rho_f} \nsubseteq S$, then there exists $k \in K \cap [b]_{\rho_f}$. Obviously, $\{a,k\} \in E(X+Y)$. Hence $\{g(a), g(b)\} = \{a,k\} \in E(X+Y)$. If $a \notin V(I_f)$ and $b \notin V(I_f)$, then $a, b \in S$. Without loss of generality, we may assume that $a \in S_1$ and $b \in S_2$. If $g(a) \in S$, then $g(a) = \overline{[a]_{\rho_f}}$ for some $\overline{[a]_{\rho_f}} \in S_1$; If $g(a) \in K$, then $g(a) = k_1$ for some $k_1 \in K_1$. Similarly, if $g(b) \in S$, then $g(a) = \overline{[b]_{\rho_f}}$ for some $\overline{[b]_{\rho_f}} \in S_2$; If $g(b) \in K$, then $g(b) = k_2$ for some $k_2 \in K_2$. It is a routine manner to check that $\{g(a), g(b)\} \in E(X+Y)$ for every case. Consequently, $g \in End(X+Y)$.

Assume that $|K \cap I_f| = n+m-1$. Then there exists $x_1 \in K \setminus I_f$. Since any endomorphism f maps a clique to a clique of the same size, $f(K)$ is a clique of size $n+m$ in $X+Y$. Thus there exist $y_1 \in S \cap I_f$ such that y_1 is adjacent to every vertex of $K \backslash \{x_1\}$. Now $g(x_1) = y_1$. If $a, b \in V(I_f)$, then $g(a) = a$ and $g(b) = b$. Thus $\{g(a), g(b)\} = \{a,b\} \in E(X+Y)$. If $a = x_1$ and $b \in K \backslash \{x_1\}$, then $g(b) = b$. Thus $\{g(a), g(b)\} = \{y_1, b\} \in E(X+Y)$. If $a = x_1$ and $b \in S$, then $b \notin V(I_f)$ by (1). Now $[b]_{\rho_f} \nsubseteq S$. Otherwise, there exists $\overline{[b]_{\rho_f}} \in V(I_f) \cap S$ such that $N(\overline{[b]_{\rho_f}}) = \cup_{x \in [b]_{\rho_f}} N(x)$. Since $\{x_1, b\} \in E(X+Y)$, $\{x_1, \overline{[b]_{\rho_f}}\} \in E(X)$. Note that $g(\overline{[b]_{\rho_f}}) = \overline{[b]_{\rho_f}} \in I_g = I_f$. This contradicts (1). Then there exists $k_1 \in K \backslash \{x_1\}$ such that $k_1 \in [b]_{\rho_f}$. Thus $\{g(a), g(b)\} = \{y_1, k_1\} \in E(X+Y)$. If $a \in K \backslash \{x_1\}$ and $b \in S \backslash I_f$, then $g(a) = a$. If $[b]_{\rho_f} \subseteq S$, then $g(b) = \overline{[b]_{\rho_f}}$. Note that $N(\overline{[b]_{\rho_f}}) = \cup_{x \in [b]_{\rho_f}} N(x)$. Then $a \in N(\overline{[b]_{\rho_f}})$. Thus $\{g(a), g(b)\} = \{a, \overline{[b]_{\rho_f}}\} \in E(X+Y)$;

If $[b]_{\rho_f} \not\subseteq S$, then there exists $k \in K \cap [b]_{\rho_f}$. Thus $\{g(a), g(b)\} = \{a, g(k)\}$. If $k \in K\backslash\{x_1\}$, then $g(k) = k$ and so $\{a, g(k)\} = \{a, k\} \in E(X+Y)$; If $k = x_1$, then $g(k) = y_1$ and so $\{a, g(k)\} = \{a, y_1\} \in E(X+Y)$. If $a \in S\backslash V(I_f)$ and $b \in S\backslash V(I_f)$, without loss of generality, we may assume that $a \in S_1$ and $b \in S_2$. If $g(a) \in S$, then $g(a) = \overline{[a]_{\rho_f}}$ for some $\overline{[a]_{\rho_f}} \in S_1$; If $g(a) \in K$, then $g(a) = k_1$ for some $k_1 \in K_1$. Similarly, if $g(b) \in S$, then $g(a) = \overline{[b]_{\rho_f}}$ for some $\overline{[b]_{\rho_f}} \in S_2$; If $g(b) \in K$, then $g(b) = k_2$ for some $k_2 \in K_2$. It is a routine manner to check that $\{g(a), g(b)\} \in E(X+Y)$ for each cases. Consequently, $g \in End(X+Y)$.

Assume that $|K \cap I_f| = n + m - 2$. Then there exist $x_1 \in K_1 \setminus I_f$ and $x_2 \in K_2 \setminus I_f$. Note that $f(K)$ is a clique of size $n + m$ in $X + Y$. Then there exist $y_1, y_2 \in S \cap I_f$ such that y_1 is adjacent to every vertex of $K\backslash\{x_1\}$ and y_2 is adjacent to every vertex of $K\backslash\{x_2\}$. Obviously, $g(x_1) = y_1$ and $g(x_2) = y_2$. If $a, b \in V(I_f)$, then $g(a) = a$ and $g(b) = b$. Thus $\{g(a), g(b)\} = \{a, b\} \in E(X+Y)$. If $a \in \{x_1, x_2\}$ and $b \in K\backslash\{x_1, x_2\}$, then $g(b) = b$. Without loss of generality, we may assume that $a = x_1$. Thus $\{g(a), g(b)\} = \{y_1, b\} \in E(X+Y)$. If $a \in \{x_1, x_2\}$ and $b \in S$, then $b \notin V(I_f)$ by (1). Without loss of generality, we may assume that $a = x_1$. We claim that $[b]_{\rho_f} \not\subseteq S$. Otherwise, there exists $\overline{[b]_{\rho_f}} \in V(I_f) \cap S$ such that $N(\overline{[b]_{\rho_f}}) = \cup_{x \in [b]_{\rho_f}} N(x)$. Since $\{x_1, b\} \in E(X+Y)$, $\{x_1, \overline{[b]_{\rho_f}}\} \in E(X+Y)$. Note that $g(\overline{[b]_{\rho_f}}) = \overline{[b]_{\rho_f}} \in I_g = I_f$. This contradicts (1). Then there exists $k_1 \in K\backslash\{x_1\}$ such that $k_1 \in [b]_{\rho_f}$. If $k_1 \neq x_2$, then $\{g(a), g(b)\} = \{y_1, k_1\} \in E(X+Y)$; If $k_1 = x_2$, then $\{g(a), g(b)\} = \{y_1, y_2\} \in E(X+Y)$. If $a \in K\backslash\{x_1, x_2\}$ and $b \in S\backslash I_f$, then $g(a) = a$. If $[b]_{\rho_f} \subseteq S$, then $g(b) = \overline{[b]_{\rho_f}}$. Note that $N(\overline{[b]_{\rho_f}}) = \cup_{x \in [b]_{\rho_f}} N(x)$. Then $a \in N(\overline{[b]_{\rho_f}})$. Thus $\{g(a), g(b)\} = \{a, \overline{[b]_{\rho_f}}\} \in E(X+Y)$; If $[b]_{\rho_f} \not\subseteq S$, then there exists $k \in K \cap [b]_{\rho_f}$. Thus $\{g(a), g(b)\} = \{a, g(k)\}$. If $k \in K\backslash\{x_1, x_2\}$, then $g(k) = k$ and so $\{a, g(k)\} = \{a, k\} \in E(X+Y)$; If $k \in \{x_1, x_2\}$, then $g(k) \in \{y_1, y_2\}$ and so $\{a, g(k)\} = \{a, y_i\}$ (where $i = 1$ or 2). Note that $a \notin \{x_1, x_2\}$, then $\{a, y_i\} \in E(X+Y)$. If $a = x_1$ and $b = x_2$, then $\{g(a), g(b)\} = \{y_1, y_2\} \in E(X+Y)$. If $a \in S\backslash V(I_f)$ and $b \in S\backslash V(I_f)$, without loss of generality, we may assume that $a \in S_1$ and $b \in S_2$. If $g(a) \in S$, then $g(a) = \overline{[a]_{\rho_f}}$ for some $\overline{[a]_{\rho_f}} \in S_1$; If $g(a) \in K$, then $g(a) = k_1$ for some $k_1 \in K_1$. Similarly, if $g(b) \in S$, then $g(a) = \overline{[b]_{\rho_f}}$ for some $\overline{[b]_{\rho_f}} \in S_2$; If $g(b) \in K$, then $g(b) = k_2$ for some $k_2 \in K_2$. It is routine manner to check that $\{g(a), g(b)\} \in E(X+Y)$ for each case. Consequently, $g \in End(X+Y)$.

It is easy to check that $g \in Idpt(X+Y)$, $\rho_g = \rho_f$ and $I_g = I_f$. By Lemma 1.2, f is completely regular. □

Next, we start to seek the conditions for $X + Y$ under which $cEnd(X+Y)$ forms a monoid.

Lemma 2.2. *If there exist $y_i, y_j \in S$ such that $N(y_i) \subset N(y_j)$, then $cEnd(X+Y)$ does not form a monoid.*

Proof. Suppose that there exist $y_i, y_j \in S$ such that $N(y_i) \subset N(y_j)$. Since K is a maximum complete set of $X + Y$, for any $x \in S$, there exists $k_x \in K$ such that $\{x, k_x\} \notin E(X+Y)$. Let

$$f(x) = \begin{cases} y_j, & x = y_i, \\ x, & \text{otherwise.} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} k_x, & x = y_j, \\ x, & \text{otherwise.} \end{cases}$$

Then f and g are idempotent endomorphisms of $X + Y$ and so they are completely regular. It is easy to see that $y_j = (fg)(y_i) \in I_{fg}$ and $(fg)^{-1}(y_j) = \{y_i\}$. Since $| N(y_i) \cap K | < | N(y_j) \cap K |$, I_{fg} is not an induced subgraph of $X + Y$. By Theorem 2.1, fg is not completely regular. Therefore $cEnd(X+Y)$ does not form a monoid. □

Lemma 2.3. *If there exist $y_i, y_j \in S$ with $i \neq j$ such that $| N(y_i) \cap K | = | N(y_j) \cap K |$, then $cEnd(X+Y)$ does not form a monoid.*

Proof. Suppose that there exist $y_i, y_j \in S$ with $i \neq j$ such that $| N(y_i) \cap K | = | N(y_j) \cap K |$. Let p be a bijection of K such that $p(N(y_i)) = N(y_j)$ and $p(N(y_j)) = N(y_i)$. Since K is a maximum complete set of $X + Y$, for any $x \in S$, there exists $k_x \in K$ such that $\{x, k_x\} \notin E(X)$. Let

$$f(x) = \begin{cases} p(x), & x \in K, \\ y_j, & x = y_i, \\ y_i, & x = y_j, \\ k_x, & \text{otherwise.} \end{cases}$$

It is easy to check that f is completely regular. Let

$$g(x) = \begin{cases} x, & x \in K, \\ y_i, & x = y_i, \\ k_x, & \text{otherwise.} \end{cases}$$

Then g is an idempotent endomorphism of $X + Y$ and so it is completely regular. It is easy to see that $y_j = (fg)(y_i) \in I_{fg}$, $(fg)(K) = K$ and $(fg)(y_j) \in K$. Thus there exists $k \in K$ such that $(fg)(y_j) = f(k)$. By Theorem 2.1 fg is not completely regular. Therefore $cEnd(X + Y)$ does not form a monoid. □

Up to now, we have obtained the following necessary conditions for $cEnd(X + Y)$ being a monoid:
(A) $N(y_i) \not\subset N(y_j)$ for any $y_i, y_j \in S$.
(B) $\mid N(y_i) \cap K \mid \neq \mid N(y_j) \cap K \mid$ for any $y_i, y_j \in S$ with $i \neq j$.

To show that (A) and (B) are also sufficient for $cEnd(X + Y)$ being a monoid, we need the following characterization of completely regular endomorphisms of $X + Y$ satisfying (A) and (B).

Lemma 2.4. *Let $X + Y$ be a join of split graphs satisfying (A) and (B), and let $f \in cEnd(X + Y)$. If there exists $y \in S$ such that $[y]_{\rho_f} \subseteq S$, then $[y]_{\rho_f} = \{y\}$.*

Proof. Let $f \in cEnd(X + Y)$. By Theorem 2.1 (3), $f(a) \neq f(b)$ for any $a, b \in V(I_f)$. Thus there exists $y_0 \in V(I_f) \cap [y]_{\rho_f}$. It follows from Theorem 2.1 (2) that $N(y_0) = \cup_{x \in [y]_{\rho_f}} N(x)$. Thus $N(b) \subseteq N(y_0)$ for any $b \in [y]_{\rho_f}$. It follows from (A) that $N(b) = N(y_0)$. In particular, $N(b) \cap K = N(y_0) \cap K$. It follows from (B) that $b = y_0 = y$. Hence $[y]_{\rho_f} = \{y\}$. □

Theorem 2.5. *Let $X + Y$ be a join of split graphs satisfying (A) and (B), and let $f \in End(X + Y)$. Then $f \in cEnd(X + Y)$ if and only if one of the following conditions hold:*
(1) *For $x \in K$, $f(x) \in K$; for $y \in S$, either $f(y) = y$, or $f(y) \in K$.*
(2) *$f(K) \neq K$ and $I_f \cong K$.*
(3) *There exist $x_1 \in K$, $y_1 \in S$ with $N(y_1) = K \backslash \{x_1\}$ such that $f(x_1) = y_1$ and $f(y_1) = x_1$; $f(K \backslash \{x_1\}) = K \backslash \{x_1\}$; for $y \in S$ with $\{y, y_1\} \notin E(X + Y)$, $f(y) \in K \backslash \{x_1\}$; for $y \in S$ with $\{y, y_1\} \in E(X + Y)$, either $f(y) = y$, or $f(y) \in K$.*

Proof. Necessity. Let $X + Y$ be a join of split graphs satisfying (A) and (B) and let $f \in cEnd(X + Y)$. We divide it into two cases to discuss:

Case 1. Assume that $f(K) = K$. For any $y \in S$, if $f(y) \notin K$, then $f(y) \in S$. Since $f(K) = K$, $[y]_{\rho_f} \subseteq S$. By Lemma 2.4, $[y]_{\rho_f} = \{y\}$. Since $f(K) = K$, $|N(y) \cap K| = |N(f(y)) \cap K|$. It follows from (B) that $f(y) = y$. Hence f is an endomorphism of $X + Y$ satisfying (1).

Case 2. Assume that $f(K) \neq K$. Then there exist $x_i \in K$, $y_1 \in S$ such that $f(x_i) = y_1$ and $|N(y_1) \cap K| = n + m - 1$. Now $N(y_1) \cap K = K \setminus \{x_1\}$ for some $x_1 \in K$. By (B), y_1 is the only vertex in S such that $|N(y_1) \cap K| = n + m - 1$. Thus there are exactly two cliques of order $n + m$ in $X + Y$. They are induced by K and $f(K) = (K \backslash \{x_1\}) \cup \{y_1\}$, respectively. Hence $f(K \backslash \{x_i\}) = K \backslash \{x_1\}$. There are two cases:

(i) $x_i \neq x_1$. Then $x_1 \notin I_f$. Otherwise, since $f(K \backslash \{x_i\}) = K \backslash \{x_1\}$, $f^{-1}(x_1) \subseteq S$. Let $y \in f^{-1}(x_1)$. By Lemma 2.4, $[y]_{\rho_f} = \{y\}$. It follows from $\{x_1, y_1\} \notin E(X + Y)$ that $\{x_i, y\} \notin E(X + Y)$. Thus $y \neq y_1$. It follows from (B) that $|N(y) \cap K| \leq n + m - 2$. Thus there exists $k_0 \in K \backslash \{x_i\}$ such that $\{k_0, y\} \notin E(X + Y)$. Obviously, $\{f(y), f(k_0)\} = \{x_1, f(k_0)\} \in E(X + Y)$. But $\{y, t\} \notin E(X + Y)$ for any $t \in [k_0]_{\rho_f}$. Hence f is not half-strong. Note that $f \in hEnd(X + Y)$ if and only if I_f is an induced subgraph of $X + Y$. This contradicts Theorem 2.1 (1). It follows from Theorem 2.1 (1) that $y \notin I_f$ for any $y \in S$ with $\{y, x_1\} \in E(X + Y)$. In particular, $y \notin I_f$ for any $y \in S$ with $\{y, y_1\} \in E(X + Y)$. Let $y \in S$ with $\{x_i, y\} \in E(X + Y)$. Then $\{f(x_i), f(y)\} = \{y_1, f(y)\} \in E(X + Y)$. Hence $f(y) \in N(y_1) \cap K = K \setminus \{x_1\}$.

Let $y \in S$ with $\{x_i, y\} \notin E(X + Y)$. Then $N(y) \cap K \subseteq K \backslash \{x_i\}$. If $f(y) \in S$, then $\{x_1, f(y)\} \notin E(X + Y)$ since $x_1 \notin I_f$. Thus $N(f(y)) \subseteq N(y_1)$. It follows from (A) that $N(f(y)) = N(y_1)$. By (B), we have $f(y) = y_1$. If $f(y) \in K$, then $f(y) \neq x_1$. Otherwise, $[y]_{\rho_f} \subseteq S$. Since $f \in cEnd(X + Y)$, I_f is an induced subgraph of

$X+Y$ and $[y]_{\rho_f}=\{y\}$. Note that $\{x,x_1\}\in E(X+Y)$ for any $x\in K\backslash\{x_1\}$. Then $N(y)\cap K=K\backslash\{x_i\}$. Thus we have $|N(y)\cap K|=|N(y_1)\cap K|=n+m-1$. This contradicts (B). Hence I_f is a subgraph of $X+Y$ induced by $(K\setminus\{x_1\})\cup\{y_1\}$ and so $I_f\cong K$.

(ii) $x_i=x_1$. Then $f(x_1)=y_1$ and $f(K\backslash\{x_1\})=K\backslash\{x_1\}$. Since $\{y_1,k\}\in E(X+Y)$ for any $k\in K\backslash\{x_1\}$, $f(y_1)$ is adjacent to every vertex of $f(K\backslash\{x_1\})=K\backslash\{x_1\}$. Thus $f(y_1)\in\{x_1,y_1\}$.

If $f(y_1)=y_1$, then $x_1\notin I_f$ by Theorem 2.1 (3). It follows from Theorem 2.1 (1) that $y\notin I_f$ for any $y\in S$ with $\{x_1,y\}\in E(X+Y)$. Note that y_1 is the only vertex in S, which is not adjacent to x_1. Then I_f is a subgraph of $X+Y$ induced by $(K\backslash\{x_1\})\cup\{y_1\}$ and so $I_f\cong K$.

If $f(y_1)=x_1$, then $x_1\in I_f$. Let $y\in S\setminus\{y_1\}$ be such that $\{y,y_1\}\notin E(X+Y)$. It follows from (A) and (B) that $\{x_1,y\}\in E(X+Y)$. Thus $\{f(x_1),f(y)\}=\{y_1,f(y)\}\in E(X+Y)$. If $f(y)\in S$, $f(y)$ and y_1 lie in the different S_i (where $i\in\{1,2\}$). Note that $[y]_{\rho_f}\subseteq S$, then $[y]_{\rho_f}=\{y\}$. Since f is half-strong, $f((N(y)\cap K)\backslash\{x_1\})=(N(f(y))\cap K)\backslash\{x_1\}$. Hence $|N(y)\cap K|=|N(f(y))\cap K|$. This contradicts (B). Hence $f(y)\in K\setminus\{x_1\}$ for any $y\in S\setminus\{y_1\}$ with $\{y,y_1\}\notin E(X+Y)$. Let $y\in S$ be such that $\{y,y_1\}\in E(X+Y)$. If $f(y)\notin K$, then $f(y)\in S$. Since $\{x_1,y\}\in E(X+Y)$, $\{f(x_1),f(y)\}=\{y_1,f(y)\}\in E(X+Y)$. Thus $f(y)\neq y_1$. Then $[y]_{\rho_f}\subseteq S$ and so $[y]_{\rho_f}=\{y\}$ by Lemma 2.4. Note that I_f is an induced subgraph of $X+Y$. Then $|N(y)\cap K|=|N(f(y))\cap K|$. It follows from (B) that $f(y)=y$. Hence f is an endomorphism of $X+Y$ satisfying (3).

Sufficiency. This follows directly from Theorem 2.1. □

Corollary 2.6. *Let $X+Y$ be a join of split graphs satisfying (A) and (B), and let $f\in End(X+Y)$. If $I_f\cong K$, then $f\in cEnd(X+Y)$.*

Proof. This follows directly from Theorem 2.5. □

Now we prove that $cEnd(X+Y)$ forms a monoid for $X+Y$ satisfying (A) and (B).

Theorem 2.7. *Let $X+Y$ be a join of split graphs satisfying (A) and (B). Then $cEnd(X+Y)$ forms a monoid.*

Proof. Let $X+Y$ be a join of split graphs satisfying (A) and (B). We only need to show that the composition of any two completely regular endomorphisms of $X+Y$ is also completely regular in every case. Let f be an arbitrary completely regular endomorphism of $X+Y$. Then by Theorem 2.5, f acts in one of the following ways:

(1) For $x\in K$, $f(x)\in K$; for $y\in S$, either $f(y)=y$, or $f(y)\in K$.
(2) $f(K)\neq K$ and $I_f\cong K$.
(3) There exist $x_1\in K$, $y_1\in S$ with $N(y_1)=K\backslash\{x_1\}$ such that $f(x_1)=y_1$ and $f(y_1)=x_1$; $f(K\backslash\{x_1\})=K\backslash\{x_1\}$; for $y\in S$ with $\{y,y_1\}\notin E(X+Y)$, $f(y)\in K\backslash\{x_1\}$; for $y\in S$ with $\{y,y_1\}\in E(X+Y)$, either $f(y)=y$, or $f(y)\in K$.

It is straightforward to see that the composition of any such two completely regular endomorphisms is still a completely regular endomorphism of $X+Y$. The proof is complete. □

Up to now we have

Theorem 2.8. *Let $X+Y$ be a join of split graphs. Then $cEnd(X+Y)$ forms a monoid if and only if*
(A) *$N(y_i)\not\subset N(y_j)$ for any $y_i,y_j\in S$, and*
(B) *$|N(y_i)\cap K|\neq|N(y_j)\cap K|$ for any $y_i,y_j\in S$ with $i\neq j$.*

Proof. Necessity follows directly from Lemmas 2.2 and 2.3.

Sufficiency follows directly from Theorem 2.7. □

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