DE GRUYTER OPEN Open Math. 2017; 15: 859–883

Open Mathematics Open Access

Research Article

Oleh Buhrii* and Nataliya Buhrii

Integro-differential systems with variable exponents of nonlinearity

DOI 10.1515/math-2017-0069
Received October 18, 2016; accepted May 10, 2017.

Abstract: Some nonlinear integro-differential equations of fourth order with variable exponents of the nonlinearity are considered. The initial-boundary value problem for these equations is investigated and the existence theorem for the problem is proved.

Keywords: Nonlinear parabolic equation, Integro-differential equation, Generalized Lebesgue space, Generalized Sobolev space, Variable exponents of nonlinearity

MSC: 47G20, 46E35, 35K52, 35K55

1 Introduction

Let $n, N \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$, $Q_{0,T} = \Omega \times (0,T)$, $S_{0,T} = \partial\Omega \times (0,T)$. We seek a weak solution $u = (u_1, \dots, u_N) : Q_{0,T} \to \mathbb{R}^N$ of the problem

$$u_{k,t} + \alpha\, \Delta^2 u_k - \sum_{i=1}^{n} \big(a_{ik}(x,t)|u_{x_i}|^{p(x)-2} u_{k,x_i}\big)_{x_i} + \Delta\big(b_k(x,t)|u|^{\gamma(x)-2}u_k\big) + (\mathcal{N}u)_k =$$

$$= \sum_{i,j=1}^{n} \big(f_{ijk}(x,t)\big)_{x_i x_j} - \sum_{i=1}^{n} \big(f_{ik}(x,t)\big)_{x_i} + f_{0k}(x,t), \quad (x,t) \in Q_{0,T}, \quad k = \overline{1,N}, \tag{1}$$

$$u|_{S_{0,T}} = 0, \qquad \Delta u|_{S_{0,T}} = 0, \qquad u|_{t=0} = u_0(x). \tag{2}$$

Here $\alpha > 0$ is a number, $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian, $\Delta^2 := \Delta(\Delta)$, $|u| := (|u_1|^2 + \ldots + |u_N|)^{1/2}$, $|u_{x_i}| := (|\frac{\partial u_1}{\partial x_i}|^2 + \ldots + |\frac{\partial u_N}{\partial x_i}|)^{1/2}$, $i = \overline{1,n}$,

$$(\mathcal{N}u)_k(x,t) := (Gu)_k(x,t) + (Bu)_k(x,t) + \phi_k((Eu)_k(x,t)), \quad (x,t) \in Q_{0,T}, \tag{3}$$

$$(Gu)_k(x,t) := g_k(x,t)|u(x,t)|^{q(x)-2}u_k(x,t), \quad (x,t) \in Q_{0,T}, \tag{4}$$

$$(Bu)_k(x,t) := -\beta_k(x,t)(u_k(x,t))^-, \quad (x,t) \in Q_{0,T}, \tag{5}$$

$$(Eu)_k(x,t) := \int_\Omega \epsilon_k(x,t,y)\big(\widetilde{u}_k(x+y,t) - \widetilde{u}_k(x,t)\big)\, dy, \quad (x,t) \in Q_{0,T}, \tag{6}$$

$a_{ik}, b_k, g_k, \beta_k, \phi_k, \epsilon_k, f_{ijk}, f_{ik}, f_{0k}, p, \gamma, q, u_0$ are some functions, $(u_k)^- := \max\{-u_k, 0\}$, and $\widetilde{u}_k$ is zero extension of u_k from $Q_{0,T}$ into $(\mathbb{R}^n \setminus \Omega) \times (0,T)$, where $i, j = \overline{1,n}$, $k = \overline{1,N}$.

Equation (1) describes, for example, the long-scale evolution of the thin liquid films. The function $u(x,t)$ is a height of the liquid films in the point x at the time t, the fourth-order terms describe capillary force of the liquid

***Corresponding Author: Oleh Buhrii:** Ivan Franko National University of Lviv, Lviv, Ukraine, E-mail: ol_buhrii@i.ua
Nataliya Buhrii: National University “Lviv Polytechnic”, Lviv, Ukraine, E-mail: n_buhrii@i.ua

surface tension, and the second-order terms describe the evaporation (condensation) process into the liquid (see [1–3] for more details). The investigation of the fourth-order degenerate parabolic equations of the thin liquid films was started in [4] by F. Bernis and A. Friedman (see also [2, 5–8], and the references given there). The Dirichlet problem for the Cahn–Hilliard equation (1) ($N = 1$, $\alpha > 0$, $a_{i1} = g_1 = \epsilon_1 = f_{ij1} = f_{i1} = f_{01} = 0$, where $i, j = \overline{1,n}$) was considered in [5] where $\gamma(x) \equiv 2m$, $m \in \mathbb{N}$, and $b_1 < 0$. The corresponding Neumann problem was studied in [6]. The Neumann problem for equation (1) ($N = 1$, $\alpha > 0$, $a_{i1} > 0$, $p(x) \equiv \text{const} > 2$, $g_1 = \epsilon_1 = f_{ij1} = f_{i1} = f_{01} = 0$, where $i, j = \overline{1,n}$) was considered in [2] if $\gamma(x) \equiv 2$ and $b_1 > 0$.

The initial-boundary value problems for the parabolic equations with variable exponents of the nonlinearity and without integral terms in equation were considered for instance in [9–15]. Integral terms (6) arise in many applications (see [16–18]). The second-order parabolic equations with variable exponents of the nonlinearity and integral term (6) were considered in [17, 19].

2 Notation and statement of theorem

Let $\|\cdot\|_B \equiv \|\cdot; B\|$ be a norm of some Banach space B, $B^N := B \times \ldots \times B$ (N times) be the Cartesian product of the B, B^* be a dual space for B, and $\langle \cdot, \cdot \rangle_B$ be a scalar product between B^* and B. We use the notation $X \circlearrowleft Y$ if the Banach space X is continuously embedded into Y; the notation $X \bar{\circlearrowleft} Y$ means the continuous and dense embedding; the notation $X \overset{K}{\subset} Y$ means the compact embedding.

If $w \in B$, $z = (z_1, \ldots, z_N) \in B^N$, and $v = (v_1, \ldots, v_N) \in B^N$, then we set

$$\langle v, w \rangle := \Big(\langle v_1, w \rangle_B, \ \ldots, \ \langle v_N, w \rangle_B \Big) \in \mathbb{R}^N, \qquad \langle v, z \rangle := \sum_{k=1}^{N} \langle v_k, z_k \rangle_B \in \mathbb{R}, \tag{7}$$

and $\|z; B^N\| := \|z_1; B\| + \ldots + \|z_N; B\|$.

Suppose that $m, d \in \mathbb{N}$, $p \in [1, \infty]$, X is the Banach space, Q is a measurable set in $\mathbb{R}^d$, $\mathcal{M}(\mathsf{Q})$ is a set of all measurable functions $v : \mathsf{Q} \to \mathbb{R}$ (see [20, p. 120]), Lip(Q) is a set of all Lipschitz-continuous functions $v : \mathsf{Q} \to \mathbb{R}$ (see [21, p. 29]), $C^m(\mathsf{Q})$ and $C_0^\infty(\mathsf{Q})$ are determined from [22, p. 9], $L^p(\mathsf{Q})$ is the Lebesgue space (see [22, p. 22, 24]), $W^{m,p}(\mathsf{Q})$ and $W_0^{m,p}(\mathsf{Q})$ are Sobolev spaces (see [22, p. 45]), $H^m(\mathsf{Q}) := W^{m,2}(\mathsf{Q})$, $H_0^m(\mathsf{Q}) := W_0^{m,2}(\mathsf{Q})$, $C([0,T]; X)$ and $C^m([0,T]; X)$ are determined from [23, p. 147], $L^p(0,T; X)$ is determined from [23, p. 155], $W^{m,p}(0,T; X)$ is determined from [24, p. 286], $H^m(0,T; X) := W^{m,2}(0,T; X)$, and

$$\mathcal{B}_+(\mathsf{Q}) := \{q \in L^\infty(\mathsf{Q}) \ \mid \ \operatorname*{ess\,inf}_{y \in \mathsf{Q}} q(y) > 0\}.$$

If $q \in \mathcal{B}_+(\mathsf{Q})$, then by definition, put

$$q_0 := \operatorname*{ess\,inf}_{y \in \mathsf{Q}} q(y), \quad q^0 := \operatorname*{ess\,sup}_{y \in \mathsf{Q}} q(y), \qquad S_q(s) := \max\{s^{q_0}, s^{q^0}\}, \quad s \geq 0, \tag{8}$$

$$q'(y) := \frac{q(y)}{q(y) - 1} \quad \text{for a.e.} \quad y \in \mathsf{Q} \quad \Big(\text{note that} \ \ \frac{1}{q(y)} + \frac{1}{q'(y)} = 1 \quad \text{and} \quad q' \in \mathcal{B}_+(\mathsf{Q})\Big), \tag{9}$$

$$\rho_q(v; \mathsf{Q}) := \int_{\mathsf{Q}} |v(y)|^{q(y)} \, dy, \quad v \in \mathcal{M}(\mathsf{Q}). \tag{10}$$

Assume that $q \in \mathcal{B}_+(\mathsf{Q})$, $q_0 > 1$, and $m \in \mathbb{N}$. The set

$$L^{q(y)}(\mathsf{Q}) := \{v \in \mathcal{M}(\mathsf{Q}) \ \mid \ \rho_q(v; \mathsf{Q}) < +\infty\}$$

is called a generalized Lebesgue space. It is well known that $L^{q(y)}(\mathsf{Q})$ is a Banach space which is reflexive and separable (see [25, p. 599, 600, 604]) with respect to the Luxemburg norm

$$\|v; L^{q(y)}(\mathsf{Q})\| := \inf\{\lambda > 0 \ \mid \ \rho_q(v/\lambda; \mathsf{Q}) \leq 1\}.$$

The set $W^{m,q(y)}(\mathsf{Q}) := \{v \in L^{q(y)}(\mathsf{Q}) \mid D^\alpha v \in L^{q(y)}(\mathsf{Q}),\ |\alpha| \le m\}$ is called a generalized Sobolev space. It is well known that $W^{m,q(y)}(\mathsf{Q})$ is a Banach space which is reflexive and separable (see [25, p. 604]) with respect to the norm

$$||v; W^{m,q(y)}(\mathsf{Q})|| := \sum_{|\alpha|\le m} ||D^\alpha v; L^{q(y)}(\mathsf{Q})||. \tag{11}$$

The closure of $C_0^\infty(\mathsf{Q})$ with respect to the norm (11) is called a generalized Sobolev space and is denoted by $W_0^{m,q(y)}(\mathsf{Q})$.

The generalized Lebesgue space was first introduced in [26]. The properties of the generalized Lebesgue and Sobolev spaces were widely studied in [25, 27–30].

Let us define the set $\Upsilon(\Omega) \subset \mathcal{M}(\Omega)$ as follows. For every $p \in \Upsilon(\Omega)$ there exist numbers $m \in \mathbb{N}$, $s_1, s_1^*, \ldots, s_m, s_m^* \in \mathbb{R}$, and open sets $\Omega_1, \ldots, \Omega_m \subset \Omega$ such that the following conditions hold:

1) $\Omega_1, \ldots, \Omega_m$ consist of the finite numbers of the components with the Lipschitz boundaries;

2) $\operatorname{mes}\Big(\Omega \setminus \bigcup_{j=1}^{m} \Omega_j\Big) = 0$;

3) $1 = s_1 < s_2 < s_1^* < s_3 < s_2^* < \ldots < s_{m-1} < s_{m-2}^* < n < s_m < s_{m-1}^* < s_m^* = +\infty$;

4) for every $j \in \{1, \ldots, m\}$ the inequality $s_j \le p(x) \le s_j^*$ holds a.e. for $x \in \Omega_j$;

5) for every $k \in \{1, \ldots, m-1\}$ the inequality $s_k^* < R(s_k)$ holds, where

$$\mathsf{R}(q) := \begin{cases} \frac{nq}{n-q} & \text{if } 1 \le q < n, \\ \text{arbitrary } s > 1 & \text{if } n \le q. \end{cases} \tag{12}$$

Note that $W^{1,q}(\Omega) \circlearrowleft L^{\mathsf{R}(q)}(\Omega)$, where $q \in [1, +\infty)$ (see [23, p. 47]).

Suppose that $\Delta^0 v := v$, $\Delta^1 v := \Delta v$, $\Delta^r v := \Delta(\Delta^{r-1} v)$,

$$H_\Delta^{2r}(\Omega) := \{v \in H^{2r}(\Omega) \mid v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = \ldots = \Delta^{r-1} v|_{\partial\Omega} = 0\}, \quad r \in \mathbb{N}. \tag{13}$$

By definition, put $Z := H_\Delta^2(\Omega)$, $X := W_0^{1,p(x)}(\Omega)$, $\mathcal{O} := L^{q(x)}(\Omega)$, $H := L^2(\Omega)$,

$$V := Z \cap X \cap \mathcal{O} \cap H, \tag{14}$$

$$U(Q_{0,T}) := \{u : (0,T) \to V^N \mid D^\alpha u \in [L^2(Q_{0,T})]^N,\ |\alpha| = 2,\ u_{x_1}, \ldots, u_{x_n} \in [L^{p(x)}(Q_{0,T})]^N,$$
$$u \in [L^{q(x)}(Q_{0,T})]^N \cap [L^2(Q_{0,T})]^N\}, \tag{15}$$

and

$$W(Q_{0,T}) := \{w \in U(Q_{0,T}) \mid w_t \in [U(Q_{0,T})]^*\}.$$

We will need the following assumptions:

(P): $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, and one of the following alternatives holds:

(i) $p \in \Upsilon(\Omega)$; (ii) $p^0 \le \mathsf{R}(p_0)$; (iii) $p \in C(\overline{\Omega})$;

(Γ): $\gamma \in \mathcal{B}_+(\Omega)$, $\gamma_0 > 1$;

(Q): $q \in \mathcal{B}_+(\Omega)$, $q_0 > 1$;

(Z): $\alpha > 0$, $\gamma^0 \le 2$; $s_0 := \min\{2, p_0, q_0\}$, $s^0 := \max\{2, p^0, q^0\}$, $r \in \mathbb{N}$, and
$$r \ge \tfrac{1}{2} \max\Big\{2,\ 1 + \tfrac{n(p^0-2)}{2p^0},\ \tfrac{n(q^0-2)}{2q^0}\Big\};$$

(A): $a_{ik} \in \mathcal{M}(Q_{0,T})$, $0 < a_0 \le a_{ik}(x,t) \le a^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$, where $i = \overline{1,n}$, $k = \overline{1,N}$;

(B): $b_k \in \mathcal{M}(Q_{0,T})$, $|b_k(x,t)| \le b^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$, where $k = \overline{1,N}$;

(G): $g_k \in \mathcal{M}(Q_{0,T})$, $0 < g_0 \le g_k(x,t) \le g^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$, where $k = \overline{1,N}$;

(BB): $\beta_1, \ldots, \beta_N \in \mathcal{B}_+(Q_{0,T})$;

(Φ): $\phi_k \in \operatorname{Lip}(\mathbb{R})$, $|\phi_k(\xi)| \le \phi^0|\xi|$ for every $\xi \in \mathbb{R}$, where $\phi^0 \in [0, +\infty)$, $k = \overline{1,N}$;

(E): $\epsilon_k \in \mathcal{M}(Q_{0,T} \times \Omega)$, $|\epsilon_k(x,t,y)| \le \epsilon^0 < +\infty$ for a.e. $(x,t,y) \in Q_{0,T} \times \Omega$, where $k = \overline{1,N}$;

(F): $f_{ijk} \in L^2(Q_{0,T})$, $f_{ik} \in L^{p'(x)}(Q_{0,T})$, $f_{0k} \in L^{q'(x)}(Q_{0,T})$, where $i, j = \overline{1,n}$, $k = \overline{1,N}$;

(U): $u_0 \in H^N$.

Let us introduce the following notation. If $t \in (0,T)$ and if $k \in \{1, \ldots, N\}$, then we set

$$\langle(\Lambda u)_k, w\rangle_Z := \int_\Omega \alpha\, \Delta u_k(x)\, \Delta w(x)\, dx, \quad u \in Z^N, \quad w \in Z, \tag{16}$$

$$\langle (A(t)u)_k, w\rangle_X := \int_\Omega \sum_{i=1}^n a_{ik}(x,t)|u_{x_i}(x)|^{p(x)-2} u_{k,x_i}(x) w_{x_i}(x)\,dx, \quad u \in X^N, \quad w \in X, \tag{17}$$

$$\langle (\Psi(t)u)_k, w\rangle_Z := \int_\Omega b_k(x,t)|u(x)|^{\gamma(x)-2} u_k(x)\,\Delta w(x)\,dx, \quad u \in Z^N, \quad w \in Z, \tag{18}$$

$$\langle (\mathcal{K}(t)u)_k, w\rangle_V := \langle (\Lambda u)_k, w\rangle_Z + \langle (A(t)u)_k, w\rangle_X + \langle (\Psi(t)u)_k, w\rangle_Z, \quad u \in V^N, \quad w \in V, \tag{19}$$

$$\langle F_k(t), w\rangle_V := \int_\Omega \Big[\sum_{i,j=1}^n f_{ijk}(x,t) w_{x_i x_j}(x) + \sum_{i=1}^n f_{ik}(x,t) w_{x_i}(x) + f_{0k}(x,t) w(x)\Big]\,dx, \; w \in V. \tag{20}$$

Using (16) and (7), we define the operator $\Lambda : Z^N \to [Z^N]^*$ by the rule

$$\Lambda u := \big((\Lambda u)_1, \ \ldots, \ (\Lambda u)_N\big), \qquad \langle \Lambda u, v\rangle_{Z^N} := \sum_{k=1}^N \langle (\Lambda u)_k, v_k\rangle_Z, \quad u \in Z^N, \quad v = (v_1, \ldots, v_N) \in Z^N.$$

Continuing in the same way, we define the operators $A(t) : X^N \to [X^N]^*$, $\Psi(t) : Z^N \to [Z^N]^*$, and $\mathcal{K}(t) : V^N \to [V^N]^*$, where $t \in [0,T]$. We write:

$$F(t) := (F_1(t), \ldots, F_N(t)), \quad t \in [0,T],$$

$$(\mathcal{N}w)(x,t) := ((\mathcal{N}w)_1(x,t), \ldots, (\mathcal{N}w)_N(x,t)), \quad (x,t) \in Q_{0,T},$$

$$(\mathcal{N}(t)w)(x) := (\mathcal{N}w)(x,t), \quad (x,t) \in Q_{0,T},$$

where $F_1, \ldots, F_N$ are defined in (20), $(\mathcal{N}w)_1, \ldots, (\mathcal{N}w)_N$ are defined in (3). Clearly,

$$F(t) \in [V^N]^*, \quad \mathcal{N}(t)(\mathcal{O}^N \cap H^N) \subset [\mathcal{O}^N \cap H^N]^*, \quad t \in [0,T].$$

Likewise we define the operators $G(t) : \mathcal{O}^N \to [\mathcal{O}^N]^*$, $B(t) : H^N \to H^N$, and $E(t) : H^N \to H^N$, where $t \in [0,T]$.

For the sake of convenience we have denoted $\phi(Eu) = (\phi_1((Eu)_1), \ldots, \phi_N((Eu)_N))$ and $\phi_k(Eu_k(t)) = \phi_k((Eu)_k(t))$, $k = \overline{1,N}$. By definition, put

$$(u,v)_\Omega := \begin{cases} \int_\Omega u(x)v(x)\,dx & \text{if } u : \Omega \to \mathbb{R}^N, \quad v : \Omega \to \mathbb{R}, \\ \int_\Omega (u(x), v(x))_{\mathbb{R}^N}\,dx & \text{if } u, v : \Omega \to \mathbb{R}^N, \end{cases} \tag{21}$$

Definition 2.1. *A real-valued function $u \in W(Q_{0,T}) \cap C([0,T]; H^N)$ is called a weak solution of problem ((1), (2)) if u satisfies (2) and for every $v \in U(Q_{0,T})$ we have*

$$\langle u_t, v\rangle_{U(Q_{0,T})} + \int_0^T \Big[\big\langle \mathcal{K}(t)u(t), v(t)\big\rangle_{V^N} + \big(\mathcal{N}(t)u(t), v(t)\big)_\Omega\Big]\,dt = \int_0^T \langle F(t), v(t)\rangle_{V^N}\,dt. \tag{22}$$

Theorem 2.2. *Suppose that conditions* **(P)-(U)** *and $\partial\Omega \in C^{2r}$ are satisfied. Then problem ((1), (2)) has a weak solution.*

3 Auxiliary facts

3.1 Properties of generalized Lebesgue and Sobolev spaces

The following Propositions are needed for the sequel.

Proposition 3.1 (see [31, p. 31]). *If $q \in \mathcal{B}_+(\mathsf{Q})$ and $q_0 > 1$, then for every $\eta > 0$ there exists a number $Y_q(\eta) > 0$ such that for every $a, b \geq 0$ and for a.e. $y \in \mathsf{Q}$ the generalized Young inequality*

$$ab \leq \eta\, a^{q(y)} + Y_q(\eta)\, b^{q'(y)} \tag{23}$$

holds. In addition, $Y_q(\eta)$ depends on q_0, q^0 and it is independent of y, $Y_2(\eta) = \frac{1}{4\eta}$, $Y_2(\frac{1}{2}) = \frac{1}{2}$, $Y_q(+0) = +\infty$, and $Y_q(+\infty) = 0$.

Proposition 3.2. *Assume that $q \in \mathcal{B}_+(\mathsf{Q})$ and $q_0 > 1$. Then the following statements are satisfied:*

(i) (see [25, p. 600]) if $q(y) \geq r(y) \geq 1$ for a.e. $y \in \mathsf{Q}$, then $L^{q(y)}(\mathsf{Q}) \circlearrowleft L^{r(y)}(\mathsf{Q})$ and

$$\|v; L^{r(y)}(\mathsf{Q})\| \leq (1 + \operatorname{mes} \mathsf{Q})\|v; L^{q(y)}(\mathsf{Q})\|, \quad v \in L^{q(y)}(\mathsf{Q});$$

(ii) (see [30, p. 431]) for every $u \in L^{q(y)}(\mathsf{Q})$ and $v \in L^{q'(y)}(\mathsf{Q})$ we get $uv \in L^1(\mathsf{Q})$ and the following generalized Hölder inequality is true

$$\int_{\mathsf{Q}} |u(y)v(y)|\, dy \leq 2\,\|u; L^{q(y)}(\mathsf{Q})\| \cdot \|v; L^{q'(y)}(\mathsf{Q})\|. \tag{24}$$

Proposition 3.3 (see [32, p. 168]). *Suppose that $q \in \mathcal{B}_+(\mathsf{Q})$, $q_0 \geq 1$, S_q is defined by (8), and ρ_q is defined by (10). Then for every $v \in \mathcal{M}(\mathsf{Q})$ the following statements are fulfilled:*

(i) $\|v; L^{q(y)}(\mathsf{Q})\| \leq S_{1/q}(\rho_q(v; \mathsf{Q}))$ if $\rho_q(v; \mathsf{Q}) < +\infty$;

(ii) $\rho_q(v; \mathsf{Q}) \leq S_q(\|v; L^{q(y)}(\mathsf{Q})\|)$ if $\|v; L^{q(y)}(\mathsf{Q})\| < +\infty$.

Proposition 3.4. *Suppose that $p \in \mathcal{B}_+(\Omega)$ and $p_0 > 1$. Then the following statements hold:*

(i) (see Theorem 3.10 [25, p. 610] and Theorem 2.7 [30, p. 443]) if either $p \in \Upsilon(\Omega)$ or $p \in C(\overline{\Omega})$, then

$$\|v; W_0^{1,p(x)}(\Omega)\| = \sum_{i=1}^{n} \|v_{x_i}; L^{p(x)}(\Omega)\|$$

is a equivalent norm of $W_0^{1,p(x)}(\Omega)$;

(ii) (see Lemma 5 [13, p. 48] and Theorem 3.1 [27, p. 76]) if $u_{x_1}, \ldots, u_{x_n} \in L^{p(x)}(\Omega)$ and either $p \in \Upsilon(\Omega)$ or $p^0 \leq \mathsf{R}(p_0)$ (see (12)), then $u \in L^{p(x)}(\Omega)$ and the generalized Poincaré inequality

$$\|u; L^{p(x)}(\Omega)\| \leq C_1\Big(\sum_{i=1}^{n} \|u_{x_i}; L^{p(x)}(\Omega)\| + \|u; L^1(\Omega)\|\Big),$$

holds, where $C_1 > 0$ is independent of u;

(iii) (see Lemma 2 [13, p. 46] and Theorem 3.2 [27, p. 77])

$$L^{p^0}(0, T; L^{p(x)}(\Omega)) \mathrel{\bar{\circlearrowleft}} L^{p(x)}(Q_{0,T}) \mathrel{\bar{\circlearrowleft}} L^{p_0}(0, T; L^{p(x)}(\Omega)). \tag{25}$$

3.2 Auxiliary functional spaces

Let $\mathcal{L}(X, Y)$ be a space of bounded linear operators from X into Y (see [33, p. 32]), $(\cdot, \cdot)_H$ be the Cartesian product in the Hilbert space H, and $H^{2r}_{\Delta}(\Omega)$ is defined in (13), where $r \in \mathbb{N}$. It is easy to verify that $H^{2r}_{\Delta}(\Omega)$ is the Hilbert space such that

$$H^{2r}_{\Delta}(\Omega) \circlearrowleft H^{2r}(\Omega), \qquad H^{2r}_{\Delta}(\Omega) \mathrel{\bar{\circlearrowleft}} L^2(\Omega) \mathrel{\bar{\circlearrowleft}} [H^{2r}_{\Delta}(\Omega)]^*. \tag{26}$$

If $\partial\Omega \subset C^1$, then the following integration by parts formula is true

$$\int_{\Omega} v\, \Delta^r u\, dx = \int_{\Omega} u\, \Delta^r v\, dx, \quad u, v \in H^{2r}_{\Delta}(\Omega). \tag{27}$$

Note that for every $r \in \mathbb{N}$ the space $H^{2r}_{\Delta}(\Omega)$ is reflexive.

Let $\{w^j\}_{j\in\mathbb{N}}$ be a set of all eigenfunctions of the problem

$$-\Delta w^j = \lambda_j w^j \quad \text{in} \quad \Omega, \qquad w^j|_{\partial\Omega} = 0, \quad j \in \mathbb{N}. \tag{28}$$

Here $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ is the set of the corresponding eigenvalues. Suppose that $\{w^j\}_{j\in\mathbb{N}}$ is an orthonormal set in $L^2(\Omega)$. It is easy to verify that solutions to problem (28) satisfy the equalities

$$(-1)^r \Delta^r w = \lambda^r w, \qquad w|_{\partial\Omega} = \Delta w|_{\partial\Omega} = \ldots = \Delta^{r-1} w|_{\partial\Omega} = 0. \tag{29}$$

The following propositions are needed for the sequel.

Proposition 3.5 (see Theorem 8 [34, p. 230]). *If $\partial\Omega \subset C^{2r}$, then the set $\{w^j\}_{j\in\mathbb{N}}$ of all eigenfunction of the problem (28) is a basis for the space $H^{2r}_{\Delta}(\Omega)$.*

Proposition 3.6 (see Lemma 3 [34, p. 229]). *If $\partial\Omega \subset C^{2r}$, then there exists a constant $C_2 > 0$ such that for all $v \in H^{2r}_{\Delta}(\Omega)$ we obtain*

$$||v; H^{2r}(\Omega)|| \le C_2 ||\Delta^r v; L^2(\Omega)||. \tag{30}$$

Define

$$\mathcal{W}_r := [H^{2r}_{\Delta}(\Omega)]^N, \qquad \mathcal{W}_r^* := [\mathcal{W}_r]^*, \tag{31}$$

where r is determined from condition **(Z)**. We consider the space V^N (see (14)) with respect to the norm

$$||v; V^N|| := ||\Delta v; H^N|| + \sum_{i=1}^{n} ||v_{x_i}; [L^{p(x)}(\Omega)]^N|| + ||v; \mathcal{O}^N|| + ||v; H^N||.$$

Since r satisfies **(Z)** and (14) holds, it is easy to verify that

$$\mathcal{W}_r \overset{\bar{}}{\circlearrowleft} V^N \overset{\bar{}}{\circlearrowleft} H^N \cong [H^N]^* \overset{\bar{}}{\circlearrowleft} [V^N]^* \overset{\bar{}}{\circlearrowleft} \mathcal{W}_r^*. \tag{32}$$

The following Lemma is needed for the sequel.

Lemma 3.7. $L^\infty(0,T; H^N) \cap C([0,T]; [V^N]^*) = C([0,T]; H^N)$.

The proof is omitted (see for comparison Lemma 8.1 [35, p. 307]).

We consider the space $U(Q_{0,T})$ (see (15)) with respect to the norm

$$||u; U(Q_{0,T})|| := \sum_{i,j=1}^{n} ||u_{x_i x_j}; [L^2(Q_{0,T})]^N|| + \sum_{i=1}^{n} ||u_{x_i}; [L^{p(x)}(Q_{0,T})]^N||$$
$$+ ||u; [L^{q(x)}(Q_{0,T})]^N|| + ||u; [L^2(Q_{0,T})]^N||.$$

It is easy to verify that the space $U(Q_{0,T})$ is reflexive. Taking into account the embedding of type (25) and inequality (30), we obtain

$$L^{s^0}(0,T; V^N) \overset{\bar{}}{\circlearrowleft} U(Q_{0,T}) \overset{\bar{}}{\circlearrowleft} L^{s_0}(0,T; V^N), \tag{33}$$

where s_0 and s^0 are determined from condition **(Z)**. Whence,

$$L^{\frac{s_0}{s_0-1}}(0,T; [V^N]^*) \overset{\bar{}}{\circlearrowleft} [U(Q_{0,T})]^* \overset{\bar{}}{\circlearrowleft} L^{\frac{s^0}{s^0-1}}(0,T; [V^N]^*). \tag{34}$$

Similarly, using (32) we obtain

$$L^{s^0}(0,T; \mathcal{W}_r) \overset{\bar{}}{\circlearrowleft} U(Q_{0,T}) \overset{\bar{}}{\circlearrowleft} [L^2(Q_{0,T})]^N \overset{\bar{}}{\circlearrowleft} [U(Q_{0,T})]^* \overset{\bar{}}{\circlearrowleft} L^{\frac{s^0}{s^0-1}}(0,T; \mathcal{W}_r^*). \tag{35}$$

Hence an arbitrary element of the spaces $[U(Q_{0,T})]^*$ or $U(Q_{0,T})$ belongs to $D^*((0,T); [V^N]^*)$. Therefore, we have distributional derivative of $u \in U(Q_{0,T}) \subset D^*((0,T); [V^N]^*)$. Together with (34), we conclude that an arbitrary element $w \in [U(Q_{0,T})]^*$ belongs to $L^{\frac{s^0}{s^0-1}}(0,T; [V^N]^*)$. Thus, if $u \in U(Q_{0,T})$ belongs to $L^{s^0}(0,T; V^N)$, then $\langle w, u\rangle_{U(Q_{0,T})} = \int_0^T \langle w(t), v(t)\rangle_{V^N}\, dt$. In particular, this equality is true if $u \in C([0,T]; V^N)$.

Lemma 3.8. *Suppose that conditions **(P)** and **(Q)** are satisfied, $u \in U(Q_{0,T})$, $\{w^{\mu}\}_{\mu\in\mathbb{N}}$ is a basis for the space V. Then for every $\varepsilon > 0$ there exist a number $m \in \mathbb{N}$ and functions $\{\varphi_{\mu k}\}_{\mu=1,k=1}^{m,\ N} \subset C^{\infty}([0,T])$ such that $||u - \psi_m; U(Q_{0,T})|| < \varepsilon$, where $\psi_m = (\psi_{m1},\dots,\psi_{mN})$ and $\psi_{mk}(x,t) = \sum\limits_{\mu=1}^{m} \varphi_{\mu k}(t) w^{\mu}(x)$, $(x,t) \in Q_{0,T}$, $k = \overline{1,N}$.*

The proof is omitted (see for comparison [36, p. 5] and [13, 27]).

3.3 Projection operator

Let $\mathcal{H}$ be the Hilbert space and $\mathcal{V}$ be the reflexive separable Banach space such that

$$\mathcal{V} \,\bar{\circlearrowleft}\, \mathcal{H} \cong \mathcal{H}^* \,\bar{\circlearrowleft}\, \mathcal{V}^*. \tag{36}$$

Notice that if $g \in \mathcal{V}^*$ and $g \in \mathcal{H}$, then

$$\langle g, v\rangle_{\mathcal{V}} = (g, v)_{\mathcal{H}}, \quad v \in \mathcal{V}. \tag{37}$$

Suppose $\{w^j\}_{j\in\mathbb{N}}$ is a orthonormal basis for the space $\mathcal{H}$, $m \in \mathbb{N}$ is a fixed number, $\mathfrak{M}$ is a set of all linear combinations of the elements from $\{w^1,\dots,w^m\}$, $\mathfrak{M}^{\perp}$ is a orthogonal complements of $\mathfrak{M}$ (see [37, p. 476]). Then (see [37, p. 526]) $\mathfrak{M}$ is a closed subset of $\mathcal{H}$ and $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^{\perp}$.

Define an unique orthogonal projection $P_m : \mathcal{H} \to \mathfrak{M}$ by the rule (see [37, p. 527])

$$P_m h := \sum_{j=1}^{m} (h, w^j)_{\mathcal{H}}\, w^j, \quad h \in \mathcal{H}. \tag{38}$$

This is a linear self-adjoint continuous operator (see Theorem 7.3.6 [37, p. 515]) such that

$$||P_m h||_{\mathcal{H}} \le ||h||_{\mathcal{H}}, \quad h \in \mathcal{H}. \tag{39}$$

If $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{V}$, then let us define an operator $\widehat{P}_m : \mathcal{V} \to \mathcal{V}$ (not necessarily self-adjoint) by the rule

$$\widehat{P}_m v := P_m v \quad \text{for every} \quad v \in \mathcal{V}. \tag{40}$$

We shall find a conjugate operator $\widehat{P}_m^* : \mathcal{V}^* \to \mathcal{V}^*$. Take elements $v \in \mathcal{V}$, $z \in \mathcal{V}^*$. Then

$$\langle z, P_m v\rangle_{\mathcal{V}} = \Big\langle z, \sum_{j=1}^{m} (v, w^j)_{\mathcal{H}}\, w^j \Big\rangle_{\mathcal{V}} = \sum_{j=1}^{m} (v, w^j)_{\mathcal{H}} \langle z, w^j\rangle_{\mathcal{V}} = \Big(v, \sum_{j=1}^{m} \langle z, w^j\rangle_{\mathcal{V}}\, w^j \Big)_{\mathcal{H}}.$$

Since $v, w^1, \dots, w^m \in \mathcal{V}$, (37) yields that

$$\Big(v, \sum_{j=1}^{m} \langle z, w^j\rangle_{\mathcal{V}}\, w^j \Big)_{\mathcal{H}} = \Big(\sum_{j=1}^{m} \langle z, w^j\rangle_{\mathcal{V}}\, w^j, v \Big)_{\mathcal{H}} = \Big\langle \sum_{j=1}^{m} \langle z, w^j\rangle_{\mathcal{V}}\, w^j, v \Big\rangle_{\mathcal{V}}.$$

Thus, $\langle z, P_m v\rangle_{\mathcal{V}} = \langle \widehat{P}_m^* z, v\rangle_{\mathcal{V}}$, where

$$\widehat{P}_m^* z = \sum_{j=1}^{m} \langle z, w^j\rangle_{\mathcal{V}}\, w^j, \quad z \in \mathcal{V}^*. \tag{41}$$

In addition, (41) implies that $\widehat{P}_m^*(\mathcal{V}^*) \subset \mathcal{V}$.

Lemma 3.9. *Assume that $\{w^j\}_{j\in\mathbb{N}}$ is a orthonormal basis for the space $\mathcal{H}$ such that $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{V}$, $\psi_1^m, \dots, \psi_m^m \in \mathbb{R}$ are some numbers, and $F \in \mathcal{V}^*$. Then $z^m = \sum\limits_{s=1}^{m} \psi_s^m w^s \in \mathcal{V}$ satisfies*

$$\begin{cases} \langle z^m, w^1\rangle_{\mathcal{V}} = \langle F, w^1\rangle_{\mathcal{V}}, \\ \quad \vdots \\ \langle z^m, w^m\rangle_{\mathcal{V}} = \langle F, w^m\rangle_{\mathcal{V}}, \end{cases} \tag{42}$$

iff the following equality holds

$$z^m = \widehat{P}^*_m F \quad in \quad \mathcal{V}^*. \tag{43}$$

Proof. Clearly, (43) implies (42). We shall prove that (42) implies (43). Take $v \in \mathcal{V}$. There exist numbers $\alpha_1^m, \dots, \alpha_m^m \in \mathbb{R}$ such that $P_m v = \widehat{P}_m v = \sum_{\mu=1}^{m} \alpha_\mu^m w^\mu$. Multiplying both sides of μ-th equality of (42) by α_μ^m and summing the obtained equalities, we get $\langle z^m, \widehat{P}_m v\rangle_{\mathcal{V}} = \langle F, \widehat{P}_m v\rangle_{\mathcal{V}}$. Hence, $\langle \widehat{P}^*_m z^m, v\rangle_{\mathcal{V}} = \langle \widehat{P}^*_m F, v\rangle_{\mathcal{V}}$ for every $v \in \mathcal{V}$. Thus,

$$\widehat{P}^*_m z^m = \widehat{P}^*_m F \quad \text{in} \quad \mathcal{V}^*. \tag{44}$$

Taking into account (37), the inclusions $z^m, w^1, \dots, w^m \in \mathcal{V}$, and the orthonormality condition for $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{H}$, from (41) we obtain

$$\widehat{P}^*_m z^m = \sum_{j=1}^{m} \langle z^m, w^j\rangle_{\mathcal{V}}\, w^j = \sum_{j=1}^{m}\Big(\sum_{s=1}^{m} \psi_s^m w^s, w^j\Big)_{\mathcal{H}} w^j = \sum_{s,j=1}^{m} \psi_s^m (w^s, w^j)_{\mathcal{H}}\, w^j = \sum_{s=1}^{m} \psi_s^m w^s = z^m.$$

Therefore, (42) yields (43). □

In the sequel, we only consider the case $\mathcal{H} = L^2(\Omega)$, $\mathcal{V} = H^{2r}_\Delta(\Omega)$ (see (13)), and $\{w^j\}_{j\in\mathbb{N}}$ is determined from problem (28). Then (38) implies that (see (21))

$$(P_m u)(x) = \sum_{j=1}^{m} (u, w^j)_\Omega\, w^j(x), \quad x \in \Omega, \quad u : \Omega \to \mathbb{R}. \tag{45}$$

This operator $P_m : L^2(\Omega) \to L^2(\Omega)$ is a linear self-adjoint continuous projection operator such that $||P_m||_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} = 1$.

To prove that $\widehat{P}_m$ belongs to $\mathcal{L}(H^{2r}_\Delta(\Omega), H^{2r}_\Delta(\Omega))$, we take $v \in H^{2r}_\Delta(\Omega)$. Then $\Delta^r \widehat{P}_m v \in L^2(\Omega)$ and Corollary 6.2.10 [38, p. 171] implies that there exists a function $h \in L^2(\Omega)$ such that $||h||_{L^2(\Omega)} = 1$ and $(h, \Delta^r \widehat{P}_m v)_{L^2(\Omega)} = ||\Delta^r \widehat{P}_m v||_{L^2(\Omega)}$. By (45), (40), (29), and (27) we obtain

$$||\widehat{P}_m v||_{H^{2r}_\Delta(\Omega)} = ||\Delta^r \widehat{P}_m v||_{L^2(\Omega)} = (h, \Delta^r \widehat{P}_m v)_{L^2(\Omega)} = \Big(h, \Delta^r \sum_{j=1}^{m} (v, w^j)_\Omega\, w^j\Big)_\Omega$$

$$= \Big(h, \sum_{j=1}^{m} (v, w^j)_\Omega \Delta^r w^j\Big)_\Omega = \Big(h, \sum_{j=1}^{m} (v, w^j)_\Omega\, (-1)^r \lambda_j^r w^j\Big)_\Omega = \Big(h, \sum_{j=1}^{m} (v, (-1)^r \lambda_j^r w^j)_\Omega\, w^j\Big)_\Omega$$

$$= \Big(h, \sum_{j=1}^{m} (v, \Delta^r w^j)_\Omega\, w^j\Big)_\Omega = \sum_{j=1}^{m} (v, \Delta^r w^j)_\Omega (h, w^j)_\Omega = \Big(v, \sum_{j=1}^{m} (h, w^j)_\Omega\, \Delta^r w^j\Big)_\Omega$$

$$= (v, \Delta^r \widehat{P}_m h)_\Omega = (\Delta^r v, \widehat{P}_m h)_\Omega = (\Delta^r v, P_m h)_\Omega.$$

Using Cauchy-Bunyakowski-Schwarz's inequality and estimating (39) with $\mathcal{H} = L^2(\Omega)$, we show that $|(\Delta^r v, P_m h)_\Omega| \le ||\Delta^r v||_{L^2(\Omega)} ||P_m h||_{L^2(\Omega)} \le ||\Delta^r v||_{L^2(\Omega)} ||h||_{L^2(\Omega)}$. Therefore,

$$||\widehat{P}_m v||_{H^{2r}_\Delta(\Omega)} \le ||v||_{H^{2r}_\Delta(\Omega)}, \quad v \in H^{2r}_\Delta(\Omega). \tag{46}$$

Suppose now that $f \in L^s(0,T;\mathcal{H})$, $s > 1$. If $P_m : \mathcal{H} \to \mathfrak{M}$ is determined from (38), then $P_m f(t) \in \mathcal{H}$ for every $t \in [0,T]$,

$$P_m f(t) = \sum_{j=1}^{m} (f(t), w^j)_{\mathcal{H}}\, w^j, \tag{47}$$

and from (39) we get $\int_0^T |P_m f(t)|^s_{\mathcal{H}}\, dt \le \int_0^T |f(t)|^s_{\mathcal{H}}\, dt$, i.e.

$$||P_m f; L^s(0,T;\mathcal{H})|| \le ||f; L^s(0,T;\mathcal{H})||, \quad f \in L^s(0,T;\mathcal{H}). \tag{48}$$

Finally assume that $\widehat{P}_m : \mathcal{V} \to \mathcal{V}$ is determined from (40), $\mathcal{H} = L^2(\Omega)$, and $\mathcal{V} = H^{2r}_\Delta(\Omega)$. Taking into account (46) and (48), we have that

$$||\widehat{P}_m u; L^s(0,T;H^{2r}_\Delta(\Omega))|| \le ||u; L^s(0,T;H^{2r}_\Delta(\Omega))||, \quad u \in L^s(0,T;H^{2r}_\Delta(\Omega)), \quad s \ge 1. \tag{49}$$

Clearly, we can prove (38)-(49) if we replace $L^2(\Omega)$, $H^{2r}_\Delta(\Omega)$ by $[L^2(\Omega)]^N$, $[H^{2r}_\Delta(\Omega)]^N$ respectively.

3.4 Differentiability of the nonlinear expressions

Take a function $\sigma \in \mathcal{M}(\Omega)$ and by definition, put

$$\psi_{\sigma(x)}(s) := \begin{cases} s^{\sigma(x)} & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases} \quad x \in \Omega. \tag{50}$$

Similarly to Theorem A.1 [39, p. 47], we obtain that if $v \in W^{1,p}(0,T; L^p(\Omega))$ $(1 \le p \le \infty)$, then $v^+ := \max\{u, 0\} \in W^{1,p}(0,T; L^p(\Omega))$ and $(v^+)_t = \widetilde{\chi}(v) v_t$ almost everywhere in $Q_{0,T}$, where

$$\widetilde{\chi}(s) := \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases} \tag{51}$$

The function $v^- := \max\{-u, 0\}$ has a similar property.

The following Propositions are needed for the sequel.

Proposition 3.10. *(see Theorem 2 [24, p. 286]). If X is a Banach space and $1 \le p \le \infty$, then $W^{1,p}(0,T; X) \circlearrowleft C([0,T]; X)$ and the following integration by parts formula holds:*

$$\int_s^\tau u_t(t)\, dt = u(\tau) - u(s), \qquad 0 \le s < \tau \le T, \quad u \in W^{1,p}(0,T; X). \tag{52}$$

Proposition 3.11. *(the Aubin theorem, see [40] and [41, p. 393]). If $s, h > 1$ are fixed numbers, $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, and $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

$$\{u \in L^s(0,T; \mathcal{W}) \mid u_t \in L^h(0,T; \mathcal{B})\} \overset{K}{\subset} L^s(0,T; \mathcal{L}) \cap C([0,T]; \mathcal{B}).$$

Lemma 3.12. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{0,1}$-domain. Then the integration by parts formula*

$$\int_{Q_{s,\tau}} w_t\, z\, dxdt = \int_{\Omega_t} w\, z\, dx \Big|_{t=s}^{t=\tau} - \int_{Q_{s,\tau}} w\, z_t\, dxdt, \quad 0 \le s < \tau \le T, \tag{53}$$

holds if one of the following alternatives hold:

(i) $w \in L^{q(x)}(Q_{0,T})$, where $q \in \mathcal{B}_+(\Omega)$ and $q_0 > 1$, $w_t \in L^1(Q_{0,T})$, $z \in L^\infty(Q_{0,T})$, $z_t \in L^{q'(x)}(Q_{0,T})$;

(ii) $w, w_t \in L^1(Q_{0,T})$, $z, z_t \in L^\infty(Q_{0,T})$.

Proof. (i). Take $W := \{w \in L^{q(x)}(Q_{0,T}) \mid w_t \in L^1(Q_{0,T})\}$, $Z := \{z \in L^\infty(Q_{0,T}) \mid z_t \in L^{q'(x)}(Q_{0,T})\}$. If $\varphi \in C^1([0,T])$ and $z \in Z$, then $\varphi z \in W^{1,1}(0,T; L^{\frac{q^0}{q^0-1}}(\Omega))$. Using (52) with $u = \varphi(t) z(x,t)$, we get

$$\int_s^\tau \varphi_t(t) z(x,t)\, dt = \varphi(\tau) z(x,\tau) - \varphi(s) z(x,s) - \int_s^\tau \varphi(t) z_t(x,t)\, dt, \quad x \in \Omega. \tag{54}$$

Take a function $v \in C^1(\overline{\Omega})$. By (54), we obtain that

$$\int_{Q_{s,\tau}} \varphi_t v\, z\, dxdt = \int_{\Omega_t} \varphi v\, z\, dx \Big|_{t=s}^{t=\tau} - \int_{Q_{s,\tau}} \varphi v\, z_t\, dxdt. \tag{55}$$

Clearly, $C^1([0,T]; C^1(\overline{\Omega})) \,\bar{\circlearrowleft}\, W \,\bar{\circlearrowleft}\, W^{1,1}(0,T; L^1(\Omega))$. Then the set

$$\Big\{ \sum_{i=1}^m \varphi_i(t) v_i(x) \;\Big|\; m \in \mathbb{N}, \quad \varphi_1, \ldots, \varphi_m \in C^1([0,T]), \quad v_1, \ldots, v_m \in C^1(\overline{\Omega}) \Big\}$$

is dense in W and (55) yields (53).

We shall omit the proof of *(ii)* because it is analogous to the previous one. □

Lemma 3.13. *Suppose that $\sigma \in \mathcal{B}_+(\mathsf{Q})$, $p, q \in \mathcal{B}_+(\mathsf{Q})$, $p_0, q_0 > 1$, $p(y) \geq \sigma(y)$ and $q(y) \leq \frac{p(y)}{\sigma(y)}$ for a.e. $y \in \mathsf{Q}$, and $\psi_{\sigma(y)}$ is determined from (50) if we replace $\sigma(x)$ by $\sigma(y)$. Then for every $u \in L^{p(y)}(\mathsf{Q})$ we have that $\psi_{\sigma(y)}(u) \in L^{\frac{p(y)}{\sigma(y)}}(\mathsf{Q})$,*

$$\rho_{p/\sigma}(\psi_{\sigma(y)}(u); \mathsf{Q}) \leq \rho_p(u; \mathsf{Q}), \tag{56}$$

$$\|\psi_{\sigma(y)}(u); L^{q(y)}(\mathsf{Q})\| \leq C_3 S_{\sigma/p}\big(\rho_p(u; \mathsf{Q})\big), \tag{57}$$

where $C_3 > 0$ is independent of u.

Proof. Clearly, $\frac{p(y)}{\sigma(y)} \geq 1$ for a.e. $y \in \mathsf{Q}$, $|\psi_{\sigma(y)}(u)|^{\frac{p(y)}{\sigma(y)}} = |u^+|^{p(y)} \leq |u|^{p(y)} \in L^1(\mathsf{Q})$. Then by [42, p. 297], we obtain $\psi_{\sigma(y)}(u) \in L^{\frac{p(y)}{\sigma(y)}}(\mathsf{Q})$. Moreover, (56) and

$$\|\psi_{\sigma(y)}(u); L^{q(y)}(\mathsf{Q})\| \leq C_4 \|\psi_{\sigma(y)}(u); L^{\frac{p(y)}{\sigma(y)}}(\mathsf{Q})\| \leq C_4 S_{\sigma/p}\Big(\rho_{p/\sigma}(\psi_{\sigma(y)}(u); \mathsf{Q})\Big)$$

hold. This inequality and (56) imply (57). □

Lemma 3.14. *Suppose that $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, $\theta \in \mathcal{M}(\Omega \times \mathbb{R})$, for a.e. $x \in \Omega$ the function $\mathbb{R} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is continuously differentiable, and there exists a number $M > 0$ such that*

$$|\theta(x, \zeta) - \theta(x, \eta)| \leq M|\zeta - \eta|, \qquad |\theta_\xi(x, \xi)| \leq M \tag{58}$$

for a.e. $x \in \Omega$ and for every $\zeta, \eta, \xi \in \mathbb{R}$. If $u, u_t \in L^{p(x)}(Q_{0,T})$, then $\theta(x, u)$, $\big(\theta(x, u)\big)_t \in L^{p(x)}(Q_{0,T})$ and

$$\big(\theta(x, u)\big)_t = \theta_\xi(x, u)\, u_t\,. \tag{59}$$

Proof. Since $u, u_t \in L^{p(x)}(Q_{0,T})$, there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset C^1(\overline{Q_{0,T}})$ such that $u^m \underset{m\to\infty}{\longrightarrow} u$ and $u_t^m \underset{m\to\infty}{\longrightarrow} u_t$ strongly in $L^{p(x)}(Q_{0,T})$ and almost everywhere in $Q_{0,T}$. Clearly,

$$\big(\theta(x, u^m(x,t))\big)_t = \lim_{h\to 0} \frac{\theta(x, u^m(x,t+h)) - \theta(x, u^m(x,t))}{u^m(x,t+h) - u^m(x,t)} \frac{u^m(x,t+h) - u^m(x,t)}{h} = \theta_\xi(x, u^m(x,t))\, u_t^m(x,t),$$

where $(x,t) \in Q_{0,T}$, $m \in \mathbb{N}$. In addition, $|\theta(x, u^m) - \theta(x, u)| \leq M|u^m - u|$. Hence, $\theta(x, u^m) \underset{m\to\infty}{\longrightarrow} \theta(x, u)$ strongly in $L^{p(x)}(Q_{0,T})$ and so $\theta(x, u) \in L^{p(x)}(Q_{0,T})$.

Clearly, $\theta_\xi(x, u^m)u_t^m - \theta_\xi(x, u)u_t = A_m + B_m$, where

$$A_m = \theta_\xi(x, u^m)(u_t^m - u_t), \quad B_m = (\theta_\xi(x, u^m) - \theta_\xi(x, u))u_t.$$

On the other hand, $|A_m|^{p(x)} \leq M^{p(x)}|u_t^m - u_t|^{p(x)} \underset{m\to\infty}{\longrightarrow} 0$ in $L^1(Q_{0,T})$. Then $A_m \underset{m\to\infty}{\longrightarrow} 0$ in $L^{p(x)}(Q_{0,T})$. Moreover, $|B_m|^{p(x)} \leq (2M|u_t|)^{p(x)} \in L^1(Q_{0,T})$, $B_m \underset{m\to\infty}{\longrightarrow} 0$ almost everywhere in $Q_{0,T}$, and $B_m \underset{m\to\infty}{\longrightarrow} 0$ in $L^{p(x)}(Q_{0,T})$. Therefore, $\theta_\xi(x, u^m)u_t^m \underset{m\to\infty}{\longrightarrow} \theta_\xi(x, u)u_t$ in $L^{p(x)}(Q_{0,T})$ and so $\theta_\xi(x, u)u_t \in L^{p(x)}(Q_{0,T})$.

Finally let us prove (59). Take a function $\varphi \in C_0^\infty(Q_{0,T})$. Then (59) holds because

$$\int\limits_{Q_{0,T}} \theta_\xi(x, u)u_t\, \varphi\, dxdt = \lim_{m\to\infty} \int\limits_{Q_{0,T}} \theta_\xi(x, u^m)u_t^m\, \varphi\, dxdt = \lim_{m\to\infty} \int\limits_{Q_{0,T}} \big(\theta(x, u^m)\big)_t\, \varphi\, dxdt$$

$$= -\lim_{m\to\infty} \int\limits_{Q_{0,T}} \theta(x, u^m)\, \varphi_t\, dxdt = -\int\limits_{Q_{0,T}} \theta(x, u)\, \varphi_t\, dxdt. \qquad \square$$

Notice that Lemma 3.14 generalizes the results of Lemma 3 [43, p. 18], where the case $\theta(x, u) = \theta(u)$ was considered.

Corollary 3.15. *Suppose that $-\infty < a < b < +\infty$ and one of the following alternatives holds: (i) $I = [a, b]$; (ii) $I = [a, +\infty)$; (iii) $I = (-\infty, b]$. Assume also that $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, $\theta \in \mathcal{M}(\Omega \times I)$, a.e. for $x \in \Omega$ the function $I \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is continuously differentiable, and there exists a number $M > 0$ such that a.e. or $x \in \Omega$ and for every $\zeta, \eta, \xi \in I$, (58) holds. If $u, u_t \in L^{p(x)}(Q_{0,T})$ and $u(x,t) \in I$ a.e. for $(x,t) \in Q_{0,T}$, then $\theta(x, u)$, $\big(\theta(x, u)\big)_t \in L^{p(x)}(Q_{0,T})$ and (59) holds.*

Proof. For the sake of convenience, only the case $I = (-\infty, b]$ is considered (see for comparison [44, p. 98]). Let us extend θ outside I as follows

$$\Theta(x,\xi) := \begin{cases} \theta(x,\xi) & \text{if } \xi \le b, \\ \theta_\xi(x,b)\xi + \theta(x,b) - \theta_\xi(x,b)b & \text{if } \xi > b, \end{cases} \quad x \in \Omega.$$

Then Θ satisfies the conditions of Lemma 3.14 and $\Theta(x,u(x,t)) = \theta(x,u(x,t))$ for a.e. $(x,t) \in Q_{0,T}$. This completes the proof. □

Lemma 3.16. *Suppose that $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, $\theta \in \mathcal{M}(\Omega\times\mathbb{R})$, for a.e. $x \in \Omega$ the function $\mathbb{R} \ni \xi \mapsto \theta(x,\xi) \in \mathbb{R}$ is continuous and the function $\mathbb{R} \setminus \{\xi_1,\dots,\xi_N\} \ni \xi \mapsto \theta(x,\xi) \in \mathbb{R}$ is differentiable, and (58) holds for a.e. $x \in \Omega$, where $\zeta, \eta \in \mathbb{R}$, $\xi \in \mathbb{R} \setminus \{\xi_1,\dots,\xi_N\}$. If $u, u_t \in L^{p(x)}(Q_{0,T})$, then $\theta(x,u)$, $\big(\theta(x,u)\big)_t \in L^{p(x)}(Q_{0,T})$ and (59) holds.*

Proof. For the sake of convenience, only the case $N = 1$ and $\xi_1 = 0$ is considered (see for comparison [44, p. 100]). It is easy to verify that

$$\theta(x,u) := \theta(x,u^+) + \theta(x,-u^-) - \theta(x,0). \tag{60}$$

Since $u, u_t \in L^{p(x)}(Q_{0,T}) \subset L^{p_0}(Q_{0,T})$, we have that $(u^\pm)_t \in L^{p_0}(Q_{0,T})$ and $(u^\pm)_t = \pm\widetilde{\chi}(u)u_t$, where $\widetilde{\chi}$ is determined from (51). Then by Corollary 3.15, we obtain the formulas of type (59) for every term in (60). Therefore, (59) holds. By (58) and (59), we get $\big(\theta(x,u)\big)_t \in L^{p(x)}(Q_{0,T})$. □

Lemma 3.17. *Suppose that $\beta \in \mathcal{B}_+(\Omega)$, $\psi_{\beta(x)}$ is determined from (50) if we replace σ by β, and*

$$\chi_k(s) := \begin{cases} 1 \ \textit{if} \ s > \dfrac{1}{k}, \\ 0 \ \textit{if} \ s \le \dfrac{1}{k}, \end{cases} \quad k \in \mathbb{N}. \tag{61}$$

If $u \in C^1(\overline{Q_{0,T}})$ and $v, v_t \in L^1(Q_{0,T})$, then

$$\lim_{k\to+\infty} \int\limits_{Q_{0,T}} \chi_k(u)\,\beta(x)\,\psi_{\beta(x)-1}(u)u_t\, v\,dxdt = \int\limits_{\Omega_t} \psi_{\beta(x)}(u)\,v\,dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \psi_{\beta(x)}(u)\,v_t\,dxdt. \tag{62}$$

Proof. By definition, set

$$\psi_{\beta(x),k}(s) := \begin{cases} k^{\beta(x)} & \text{if } s \ge k, \\ s^{\beta(x)} & \text{if } \dfrac{1}{k} < s < k, \\ \dfrac{1}{k^{\beta(x)}} & \text{if } s \le \dfrac{1}{k}, \end{cases} \qquad \widetilde{\xi}_{\beta(x),k}(s) := \begin{cases} \beta(x)\,s^{\beta(x)-1} & \text{if } \dfrac{1}{k} < s < k, \\ 0 & \text{if } s \le \dfrac{1}{k} \text{ and } s \ge k, \end{cases}$$

$k \in \mathbb{N}$, $k \ge 2$, $x \in \Omega$. Clearly, $\psi_{\beta(x),k}(s) \underset{k\to\infty}{\longrightarrow} \psi_{\beta(x)}(s)$, where $s \in \mathbb{R}$, $x \in \Omega$. In addition, for $k \in \mathbb{N}$ $(k \ge 2)$ and $x \in \Omega$ the function $s \mapsto \psi_{\beta(x),k}(s)$ has the Lipschitz property in $\mathbb{R}$ and it is not differentiable only in the point $s = \frac{1}{k}$ and $s = k$. Moreover, $\frac{\partial}{\partial s}\psi_{\beta(x),k}(s) = \widetilde{\xi}_{\beta(x),k}(s)$ if $s \ne \frac{1}{k}$ and $s \ne k$. Whence, by Lemma 3.16, we obtain

$$\big(\psi_{\beta(x),k}(u)\big)_t = \widetilde{\xi}_{\beta(x),k}(u)u_t \quad \text{almost everywhere in} \quad Q_{0,T}. \tag{63}$$

Thus, $\psi_{\beta(x),k}(u)$, $\big(\psi_{\beta(x),k}(u)\big)_t \in L^\infty(Q_{0,T})$. Using case *(ii)* of Lemma 3.12 with $z = \psi_{\beta(x),k}(u)$ and $w = v$, we get (53), i.e.

$$\int\limits_{Q_{0,T}} \big(\psi_{\beta(x),k}(u)\big)_t\, v\,dxdt = \int\limits_{\Omega} \psi_{\beta(x),k}(u)\,v\,dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \psi_{\beta(x),k}(u)\,v_t\,dxdt. \tag{64}$$

Let $M := \max_{(x,t)\in\overline{Q_{0,T}}} |u(x,t)|$, $k_0 \in \mathbb{N}$, $k_0 \geq \max\{2, M\}$. Since $|u| \leq M \leq k_0 \leq k$, from (63) we have

$$\big(\psi_{\beta(x),k}(u)\big)_t = \widetilde{\xi}_{\beta(x),k}(u)u_t = \chi_k(u)\,\beta(x)\,\psi_{\beta(x)-1}(u)u_t,$$

where $k \geq k_0$. By $|\psi_{\beta(x),k}(u(x,t))| \leq M^{\beta(x)}\ \forall\ (x,t) \in \overline{Q_{0,T}}$ and Lebesgue's Dominate Convergence Theorem (see [33, p. 90]), we obtain

$$\lim_{k\to+\infty} \int_{\Omega_t} \psi_{\beta(x),k}(u)\, v\, dx = \int_{\Omega_t} \psi_{\beta(x)}(u)\, v\, dx \quad \text{if} \quad t = 0 \quad \text{and} \quad t = T,$$

$$\lim_{k\to+\infty} \int_{Q_{0,T}} \psi_{\beta(x),k}(u)\, v_t\, dxdt = \int_{Q_{0,T}} \psi_{\beta(x)}(u)\, v_t\, dxdt.$$

Therefore, (62) follows from (64). □

Theorem 3.18. *Suppose that $\sigma \in \mathcal{B}_+(\Omega)$, $\sigma_0 > 1$, and the function $\psi_{\sigma(x)}$ is determined from (50). Then the following statements are satisfied:*
1) if $u \in C^1(\overline{Q_{0,T}})$, then $\psi_{\sigma(x)}(u), \big(\psi_{\sigma(x)}(u)\big)_t \in L^\infty(Q_{0,T})$ and

$$\big(\psi_{\sigma(x)}(u)\big)_t = \sigma(x)\,\psi_{\sigma(x)-1}(u)\,u_t; \tag{65}$$

2) if $u, u_t \in L^{p(x)}(Q_{0,T})$, where $p \in L^\infty_+(\Omega)$ and $p(x) \geq \sigma(x)$ for a.e. $x \in \Omega$, then $\psi_{\sigma(x)}(u), \big(\psi_{\sigma(x)}(u)\big)_t \in L^{\frac{p(x)}{\sigma(x)}}(Q_{0,T})$, equality (65) is true, and the estimate

$$\rho_{p/\sigma}\Big(\big(\psi_{\sigma(x)}(u)\big)_t; Q_{0,T}\Big) \leq C_5 S_{1/\sigma'}\Big(\rho_p(u; Q_{0,T})\Big) S_{1/\sigma}\Big(\rho_p(u_t; Q_{0,T})\Big) \tag{66}$$

holds, where $C_5 > 0$ is independent of u.

Proof. First let us prove Case 1. Take a function $u \in C^1(\overline{Q_{0,T}})$. If $v, v_t \in C(\overline{Q_{0,T}})$, χ_k is determined from (61), and $k \in \mathbb{N}$, then $|\chi_k(u)\,\sigma(x)\,\psi_{\sigma(x)-1}(u)u_t\, v| \leq C_6$, where $C_6 > 0$ is independent of k, x, t. Hence, Lebesgue's Dominate Convergence Theorem (see [33, p. 90]) yields that

$$\lim_{k\to+\infty} \int_{Q_{0,T}} \chi_k(u)\,\sigma(x)\,\psi_{\sigma(x)-1}(u)u_t\, v\, dxdt = \int_{Q_{0,T}} \sigma(x)\,\psi_{\sigma(x)-1}(u)u_t\, v\, dxdt.$$

Using (62) with $\beta = \sigma > 1$, we obtain

$$\int_{Q_{0,T}} \sigma(x)\,\psi_{\sigma(x)-1}(u)u_t\, v\, dxdt = \int_{\Omega_t} \psi_{\sigma(x)}(u)\, v\, dx\bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\sigma(x)}(u)\, v_t\, dxdt. \tag{67}$$

Taking in (67) the function $v \in C_0^\infty(Q_{0,T})$, we get

$$\int_{Q_{0,T}} \sigma(x)\psi_{\sigma(x)-1}(u)u_t\, v\, dxdt = -\int_{Q_{0,T}} \psi_{\sigma(x)}(u)\, v_t\, dxdt$$

(notice that $\sigma\,\psi_{\sigma(x)-1}(u)u_t \in L^\infty(Q_{0,T})$ because $\sigma_0 > 1$). Therefore, (65) holds.

Since $\sigma_0 > 1$, from (50) we have $\psi_{\sigma(x)} \in L^\infty(Q_{0,T})$ and from (65) we have $\big(\psi_{\sigma(x)}(u)\big)_t \in L^\infty(Q_{0,T})$.

Now let us prove Case 2. Suppose $u \in U$, where $U := \{u \in L^{p(x)}(Q_{0,T}) \mid u_t \in L^{p(x)}(Q_{0,T})\}$. Clearly, $C^1([0,T]; C^1(\overline{\Omega}))\ \bar{\circlearrowleft}\ W^{1,p^0}(0,T; L^{p(x)}(\Omega))\ \bar{\circlearrowleft}\ U\ \bar{\circlearrowleft}\ W^{1,p_0}(0,T; L^{p(x)}(\Omega))$. Then there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset C^1(\overline{Q_{0,T}})$ such that $u^m \underset{m\to\infty}{\longrightarrow} u$ and $u_t^m \underset{m\to\infty}{\longrightarrow} u_t$ strongly in $L^{p(x)}(Q_{0,T})$, $u^m \underset{m\to\infty}{\longrightarrow} u$ in $C([0,T]; L^{p(x)}(\Omega))$.

Assume that $v, v_t \in C(\overline{Q_{0,T}})$. By (67), for every $m \in \mathbb{N}$ we obtain

$$\int_{Q_{0,T}} \sigma(x)\,\psi_{\sigma(x)-1}(u^m)u_t^m\, v\, dxdt = \int_{\Omega} \psi_{\sigma(x)}(u^m)\, v\, dx\bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\sigma(x)}(u^m)\, v_t\, dxdt. \tag{68}$$

Since $1 < \sigma(x) \le p(x)$, we get $\frac{p(x)}{\sigma(x)-1} > 1$ for a.e. $x \in \Omega$. Therefore,

$$\psi_{\sigma(x)-1}(u^m) \underset{m\to\infty}{\longrightarrow} \psi_{\sigma(x)-1}(u) \quad \text{strongly in} \quad L^{\frac{p(x)}{\sigma(x)-1}}(Q_{0,T}).$$

Clearly, $[L^{\frac{p(x)}{\sigma(x)-1}}(Q_{0,T})]^* \cong L^{\frac{p(x)}{p(x)-(\sigma(x)-1)}}(Q_{0,T})$. Since $p(x) \ge (\sigma(x)-1)+1$, we have that $p(x) \ge \frac{p(x)}{p(x)-(\sigma(x)-1)}$ for a.e. $x \in \Omega$. Therefore,

$$u_t^m \underset{m\to\infty}{\longrightarrow} u_t \quad \text{strongly in} \quad L^{\frac{p(x)}{p(x)-(\sigma(x)-1)}}(Q_{0,T}).$$

By Lemma 5.2 [23, p. 19], we obtain

$$\int\limits_{Q_{0,T}} \sigma(x)\psi_{\sigma(x)-1}(u^m)u_t^m\, v\, dxdt \underset{m\to\infty}{\longrightarrow} \int\limits_{Q_{0,T}} \sigma(x)\psi_{\sigma(x)-1}(u)u_t\, v\, dxdt \tag{69}$$

and $\sigma\, \psi_{\sigma(x)-1}(u)u_t \in L^1(Q_{0,T})$. It is easy to verify that

$$\psi_{\sigma(x)}(u^m(t)) \underset{m\to\infty}{\longrightarrow} \psi_{\sigma(x)}(u(t)) \quad \text{strongly in} \quad L^{\frac{p(x)}{\sigma(x)}}(\Omega) \quad \text{for} \quad t=0 \quad \text{and} \quad t=T, \tag{70}$$

$$\psi_{\sigma(x)}(u^m) \underset{m\to\infty}{\longrightarrow} \psi_{\sigma(x)}(u) \quad \text{strongly in} \quad L^{\frac{p(x)}{\sigma(x)}}(Q_{0,T}). \tag{71}$$

Letting $m \to \infty$ in (68) and using (69)-(71), we get (67) and (65).

By Lemma 3.13, we get $\psi_{\sigma(x)}(u) \in L^{\frac{p(x)}{\sigma(x)}}(Q_{0,T})$. By (65) and generalized Young's inequality, we obtain

$$\left|\left(\psi_{\sigma(x)}(u)\right)_t\right|^{\frac{p(x)}{\sigma(x)}} \le \sigma(x)^{\frac{p(x)}{\sigma(x)}}|u|^{\frac{p(x)}{\sigma'(x)}}|u_t|^{\frac{p(x)}{\sigma(x)}} \le C_7(|u|^{p(x)} + |u_t|^{p(x)}) \in L^1(Q_{0,T}).$$

Thus, $\left(\psi_{\sigma(x)}(u)\right)_t \in L^{\frac{p(x)}{\sigma(x)}}(Q_{0,T})$.

By (65) and the generalized Hölder's inequality, we obtain that

$$\int\limits_{Q_{0,T}} \left|\left(\psi_{\sigma(x)}(u)\right)_t\right|^{\frac{p(x)}{\sigma(x)}} dxdt \le C_8 \| \, |u|^{\frac{p(x)}{\sigma'(x)}}; L^{\sigma'(x)}(Q_{0,T})\| \cdot \| \, |u_t|^{\frac{p(x)}{\sigma(x)}}; L^{\sigma(x)}(Q_{0,T})\|$$

$$\le C_9 S_{1/\sigma'}\Big(\int\limits_{Q_{0,T}} |u|^{p(x)}\, dxdt\Big) S_{1/\sigma}\Big(\int\limits_{Q_{0,T}} |u_t|^{p(x)}\, dxdt\Big).$$

This implies (66) and completes the proof of Theorem 3.18. □

Note that the case $\sigma(x) \equiv \sigma \in (0,1]$ is considered in [45].

Theorem 3.19. *Suppose that $r \in \mathcal{B}_+(\Omega)$. Then the following statements are satisfied:*
1) If $r_0 > 1$, then the equality

$$\left(|u|^{r(x)}\right)_t = r(x)|u|^{r(x)-2}u\, u_t \tag{72}$$

is true if one of the following alternatives holds:
(i) $u \in C^1(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)}, \left(|u|^{r(x)}\right)_t \in L^\infty(Q_{0,T})$);
(ii) $u, u_t \in L^{p(x)}(Q_{0,T})$ and $p(x) \ge r(x)$ for a.e. $x \in \Omega$ (here we have $|u|^{r(x)}, \left(|u|^{r(x)}\right)_t \in L^{\frac{p(x)}{r(x)}}(Q_{0,T})$).
2) If $r_0 > 2$, then the equality

$$\left(|u|^{r(x)-2}u\right)_t = (r(x)-1)|u|^{r(x)-2}u_t \tag{73}$$

is true if one of the following alternatives hold:
(i) $u \in C^1(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)-2}u, \left(|u|^{r(x)-2}u\right)_t \in L^\infty(Q_{0,T})$);
(ii) $u, u_t \in L^{p(x)}(Q_{0,T})$ and $p(x) \ge r(x)-1$ for a.e. $x \in \Omega$ (here $|u|^{r(x)-2}u, \left(|u|^{r(x)-2}u\right)_t \in L^{\frac{p(x)}{r(x)-1}}(Q_{0,T})$).

Proof. Suppose that $\psi_{r(x)-2}$ is determined from (50) if we replace σ by $r-2$. Then the proof follows from Theorem 3.18 since

$$|s|^{r(x)} = \psi_{r(x)}(s) + \psi_{r(x)}(-s), \quad |s|^{r(x)-2}s = \psi_{r(x)-1}(s) - \psi_{r(x)-1}(-s), \quad x \in \Omega, \quad s \in \mathbb{R}.$$ □

3.5 Cauchy's problem for system of ordinary differential equations

Take $Q = (0,T) \times \mathbb{R}^\ell$, where $\ell \in \mathbb{N}$. In this section, we seek a weak solution $\varphi : [0,T] \to \mathbb{R}^\ell$ of the problem

$$\varphi'(t) + L(t, \varphi(t)) = M(t), \quad t \in [0,T], \qquad \varphi(0) = \varphi^0, \tag{74}$$

where $M : [0,T] \to \mathbb{R}^\ell$, $L : Q \to \mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t,0) = 0$ for every $t \in [0,T]$) and $\varphi^0 = (\varphi^0_1, \dots, \varphi^0_\ell) \in \mathbb{R}^\ell$.

The following Definitions are needed for the sequel.

Definition 3.20. *A real-valued function $\varphi \in W^{1,1}(0,T;\mathbb{R}^k)$ is called a weak solution of problem (74) if u satisfies the initial value condition and satisfies the equation almost everywhere.*

Definition 3.21. *We shall say that a function $L : Q \to \mathbb{R}^\ell$ satisfies the Carathéodory condition if for every $\xi \in \mathbb{R}^\ell$ the function $(0,T) \ni t \mapsto L(t,\xi) \in \mathbb{R}^\ell$ is measurable and if for a.e. $t \in (0,T)$ the function $\mathbb{R}^\ell \ni \xi \mapsto L(t,\xi) \in \mathbb{R}^\ell$ is continuous.*

Definition 3.22 (see [46, p. 241])**.** *We shall say that a function $L : Q \to \mathbb{R}^\ell$ satisfies the L^p-Carathéodory condition if L satisfies the Carathéodory condition and for every $R > 0$ there exists a function $h_R \in L^p(0,T)$ such that*

$$|L(t,\xi)| \le h_R(t) \tag{75}$$

a.e. for $t \in (0,T)$ and for every $\xi \in \overline{D_R} := \{y \in \mathbb{R}^\ell \mid |y| \le R\}$.

Proposition 3.23 (Gronwall-Bellman's Lemma [47, p. 25])**.** *Suppose that $A, B \in L^1(0,T)$ and $y \in C([0,T])$ are nonnegative functions. If for every $\tau \in [0,T]$ we have*

$$y(\tau) \le C + \int_0^\tau \Big[A(t)y(t) + B(t)\Big]\,dt, \tag{76}$$

where C is a nonnegative number, then the following inequality is true

$$y(\tau) \le \Big(C + \int_0^\tau B(t)\,e^{-\int_0^t A(s)\,ds}\,dt\Big)\,e^{\int_0^\tau A(t)\,dt}, \quad \tau \in [0,T]. \tag{77}$$

We will need the following Theorem.

Theorem 3.24 (Carathéodory-LaSalle's Theorem)**.** *Suppose that $p \ge 2$, function $L : Q \to \mathbb{R}^\ell$ satisfies L^p-Carathéodory condition, $M \in L^p(0,T;\mathbb{R}^\ell)$, and $\varphi^0 \in \mathbb{R}^\ell$. If there exists a nonnegative functions $\alpha, \beta \in L^1(0,T)$ such that for every $\xi \in \mathbb{R}^\ell$ and for a.e. $t \in [0,T]$ the inequality*

$$(L(t,\xi),\xi)_{\mathbb{R}^\ell} \ge -\alpha(t)|\xi|^2 - \beta(t) \tag{78}$$

holds, then problem (74) has a global weak solution $\varphi \in W^{1,p}(0,T;\mathbb{R}^\ell)$.

Proof. We modify the method employed in the proof of Theorem 3 [48, p. 240]. According to the Carathéodory Theorem [49, p. 17], we have a local weak solution $\varphi \in W^{1,p}(0,b;\mathbb{R}^\ell)$ ($b \in (0,T]$) to the Cauchy problem (74) such that for every $\tau \in [0,b]$ the equality

$$\varphi(\tau) = \varphi^0 + \int_0^\tau M(t)\,dt - \int_0^\tau L(t,\varphi(t))\,dt \tag{79}$$

holds. If $b = T$, then Theorem 3.24 is proved. If $b < T$, then we take $\varphi^1 := \varphi(b)$ and consider the equation from (74) with new initial value condition $\varphi(b) = \varphi^1$. Using the Carathéodory Theorem and (79), we extend solution to problem (74) into $[b, b_1]$, where $b_1 \le T$ etc. Thus, similarly to [50, p. 22-24], we have one of the following possibility:

1) solution to problem (74) can be extended into $[0, T]$;
2) there exists a weak solution to problem (74) which is defined on right maximal interval of existence $[0, \overline{b})$, where $\overline{b} \le T$.

We shall prove that Case 2 is impossible. Assume the converse. Then for every $\tau \in (0, \overline{b})$ this local weak solution φ belongs to $W^{1,p}(0, \tau; \mathbb{R}^\ell)$. Define

$$R := \Big\{\Big(|\varphi^0|^2 + \int_0^T \Big[2\beta(t) + |M(t)|^2\Big]\, dt\Big) e^{\int_0^T [2\alpha(t)+1]\, dt}\Big\}^{1/2}, \tag{80}$$

where α and β are determined from (78). Since L satisfies the L^p-Carathéodory condition and R is determined from (80), there exists a function $h_R \in L^p(0, T)$ such that for a.e. $t \in (0, T)$ and for every $\xi \in \overline{D_R} := \{y \in \mathbb{R}^\ell \mid |y| \le R\}$ inequality (75) holds.

Taking into account (see (78)) the following inequalities

$$(L(t, \varphi(t)), \varphi(t))_{\mathbb{R}^\ell} \ge -\alpha(t)|\varphi(t)|^2 - \beta(t), (M(t), \varphi(t))_{\mathbb{R}^\ell} \le |M(t)| \cdot |\varphi(t)| \le \frac{1}{2}|M(t)|^2 + \frac{1}{2}|\varphi(t)|^2,$$

from (74) we get

$$(\varphi'(t), \varphi(t))_{\mathbb{R}^\ell} - \alpha(t)|\varphi(t)|^2 - \beta(t) \le \frac{1}{2}|M(t)|^2 + \frac{1}{2}|\varphi(t)|^2, \quad t \in [0, \overline{b}).$$

Hence,

$$\int_0^\tau (\varphi'(t), \varphi(t))_{\mathbb{R}^\ell}\, dt \le \int_0^\tau \Big[\Big(\alpha(t) + \frac{1}{2}\Big)|\varphi(t)|^2 + \beta(t) + \frac{1}{2}|M(t)|^2\Big]\, dt, \quad \tau \in (0, \overline{b}). \tag{81}$$

Since $\varphi \in W^{1,p}(0, \tau; \mathbb{R}^\ell)$ and $p \ge 2$, we obtain

$$|\varphi|^2 \in W^{1,\frac{p}{2}}(0, \tau), \quad (|\varphi(t)|^2)' = 2(\varphi'(t), \varphi(t))_{\mathbb{R}^\ell}, \quad t \in (0, \tau),$$

(see Case 1.ii of Theorem 3.19). Hence Proposition 3.10 implies that

$$\int_0^\tau (\varphi'(t), \varphi(t))_{\mathbb{R}^\ell}\, dt = \frac{1}{2}|\varphi(\tau)|^2 - \frac{1}{2}|\varphi(0)|^2.$$

Whence (81) has a form (76), where $C = |\varphi^0|^2$,

$$y(t) = |\varphi(t)|^2, \quad A(t) = 2\alpha(t) + 1, \quad B(t) = 2\beta(t) + |M(t)|^2, \quad t \in (0, \tau).$$

Therefore, from (77) we get

$$y(\tau) \le \Big(C + \int_0^\tau B(t)\, e^{-\int_0^t A(s)\, ds}\, dt\Big) e^{\int_0^\tau A(t)\, dt} \le \Big(C + \int_0^\tau B(t)\, dt\Big) e^{\int_0^\tau A(t)\, dt} \le R^2,$$

where R is determined from (80). Thus $|\varphi(\tau)| \le R$, $\tau \in (0, \overline{b})$, i.e. the point $\varphi(t)$ belongs to D_R, where $t \in (0, \overline{b})$. By (75), we have that $|L(t, \varphi(t))| \le h_R(t)$, where $t \in (0, \overline{b})$. Therefore, (79) yields that

$$|\varphi(t_2) - \varphi(t_1)| = \Big|\int_{t_1}^{t_2} L(t, \varphi(t))\, dt\Big| \le \Big|\int_{t_1}^{t_2} h_R(t)\, dt\Big| \underset{t_1, t_2 \to \overline{b}-0}{\longrightarrow} 0.$$

Finally we have an existence of the finite limit $\lim\limits_{t \to \overline{b}-0} \varphi(t)$. Then solution to problem (74) can be extended to $[0, \overline{b}]$ by the rule $\varphi(\overline{b}) := \lim\limits_{t \to \overline{b}-0} \varphi(t) < \infty$. This contradiction completes the proof Theorem 3.24. □

If L is slowly continuous with respect to the φ, then Theorem 3.24 follows from Theorem 3 [48, p. 240]. If $M \equiv 0$ and L is continuous, then Theorem 3.24 coincides with Lemma 4 [51, p. 67].

3.6 Some integral expressions

The following lemmas will be needed in the sequel.

Lemma 3.25 (see for comparison Lemma 2.3 [31, p. 26]). *Suppose that condition **(Q)** is satisfied,* $g \in L^\infty(Q_{0,T})$, $z \in L^{q(x)}(\Omega)$, $m \in \mathbb{N}$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $w^1, \dots, w^m \in L^{q(x)}(\Omega)$, *and* $w(x,\xi) = \sum_{l=1}^m \xi_l w^l(x)$. *Then the function*

$$I(t,\xi) := \int_\Omega g(x,t)|w(x,\xi)|^{q(x)-2} w(x,\xi) z(x)\, dx, \quad t \in (0,T), \quad \xi \in \mathbb{R}^m, \tag{82}$$

satisfies the L^∞*-Carathéodory condition.*

Proof. Step 1. The Fubini Theorem [33, p. 91] yields that $I(\cdot,\xi) \in L^1(0,T)$. Then the function $[0,T] \ni t \mapsto I(t,\xi) \in \mathbb{R}$ is measurable.

Step 2. We prove that the function $\mathbb{R} \ni \xi_1 \mapsto I(t,\xi_1,\dots,\xi_m) \in \mathbb{R}$ is continuous at the point $\xi_1^0 \in \mathbb{R}$. Take $\xi = (\xi_1, \xi_2, \dots, \xi_m)$, $\xi^0 = (\xi_1^0, \xi_2, \dots, \xi_m)$, where $|\xi - \xi^0| \le 1$.

By Theorem 2.1 [52, p. 2], we get

$$\big|\,|\eta_1|^{q(x)-2}\eta_1 - |\eta_2|^{q(x)-2}\eta_2\big| \le C_{10}(|\eta_1| + |\eta_2|)^{q(x)-1-\beta(x)}|\eta_1 - \eta_2|^{\beta(x)}, \tag{83}$$

where $0 < \beta(x) \le \min\{1, q(x)-1\}$, $\eta_1, \eta_2 \in \mathbb{R}$, $C_{10} > 0$ is independent of η_1, η_2, x. Hence,

$$|I(t,\xi) - I(t,\xi^0)| = \Big|\int_\Omega g\Big(|w(x,\xi)|^{q(x)-2}w(x,\xi) - |w(x,\xi^0)|^{q(x)-2}w(x,\xi^0)\Big) z\, dx\Big|$$

$$\le C_{11}\int_\Omega \big(|w(x,\xi)| + |w(x,\xi^0)|\big)^{q(x)-1-\beta(x)}|w(x,\xi) - w(x,\xi^0)|^{\beta(x)}|z|\, dx = C_{11}(I_1 + I_2), \tag{84}$$

where

$$I_1 = \int_{\Omega_1} h(x,\xi,\xi^0)\, dx, \qquad I_2 = \int_{\Omega_2} h(x,\xi,\xi^0)\, dx,$$

$\Omega_1 = \{x \in \Omega \mid q(x) \le 2\}$, $\Omega_2 = \{x \in \Omega \mid q(x) > 2\}$, and

$$h(x,\xi,\xi^0) = \big(|w(x,\xi)| + |w(x,\xi^0)|\big)^{q(x)-1-\beta(x)}|w(x,\xi) - w(x,\xi^0)|^{\beta(x)}|z(x)|, \quad x \in \Omega.$$

By taking $\beta(x) = q(x) - 1$, where $x \in \Omega_1$, we obtain

$$I_1 = \int_{\Omega_1} |w(x,\xi) - w(x,\xi^0)|^{q(x)-1}|z(x)|\, dx = \int_{\Omega_1} |\xi_1 - \xi_1^0|^{q(x)-1}|w^1(x)|^{q(x)-1}|z(x)|\, dx$$

$$\le |\xi_1 - \xi_1^0|^{q_0-1}\int_{\Omega_1} |w^1(x)|^{q(x)-1}|z(x)|\, dx = C_{12}|\xi_1 - \xi_1^0|^{q_0-1} \underset{\xi_1 \to \xi_1^0}{\longrightarrow} 0.$$

By taking $\beta(x) = 1$, where $x \in \Omega_2$, we obtain

$$I_2 = \int_{\Omega_2} \big(|w(x,\xi)| + |w(x,\xi^0)|\big)^{q(x)-2}|w(x,\xi) - w(x,\xi^0)| \cdot |z(x)|\, dx$$

$$= |\xi_1 - \xi_1^0|\int_{\Omega_2} \big(|w(x,\xi)| + |w(x,\xi^0)|\big)^{q(x)-2}|w^1(x)| \cdot |z(x)|\, dx \le C_{13}(\xi_1^0)|\xi_1 - \xi_1^0| \underset{\xi_1 \to \xi_1^0}{\longrightarrow} 0.$$

Therefore, by (84), we obtain that $|I(t,\xi) - I(t,\xi^0)| \underset{\xi_1 \to \xi_1^0}{\longrightarrow} 0$. Continuing in the same way, we see that I is continuous with respect to $\xi_2, \dots, \xi_m$.

Step 3. Taking into account the results of Step 1 and Step 2, we obtain that the function I satisfies the Carathéodory condition. Since $g \in L^\infty(Q_{0,T})$, the L^∞-Carathéodory condition holds. □

Lemma 3.26. *Suppose that condition* **(E)** *is satisfied,*

$$(Eu)(x,t) := \int\limits_{\Omega} \epsilon(x,t,y)\Big(\widetilde{u}(x+y,t) - \widetilde{u}(x,t)\Big)\,dy, \quad (x,t) \in Q_{0,T}, \tag{85}$$

where $u \in L^1(Q_{0,T})$, $\widetilde{u}$ is the zero extension of u from $Q_{0,T}$ into $(\mathbb{R}^n \setminus \Omega) \times (0,T)$. Then for every $s > 1$ the operator $E : L^s(Q_{0,T}) \to L^s(Q_{0,T})$ is linear bounded continuous and

$$||Eu; L^s(Q_{0,\tau})|| \le C_{14}||u; L^s(Q_{0,\tau})||, \quad u \in L^s(Q_{0,T}), \quad \tau \in (0,T], \tag{86}$$

where $C_{14} > 0$ is independent of u and τ.

The proof is trivial.

Lemma 3.27. *Suppose that $\phi \in \mathrm{Lip}\,(\mathbb{R})$, $\epsilon \in L^\infty(Q_{0,T} \times \Omega)$, $z \in L^2(\Omega)$, $m \in \mathbb{N}$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $w^1, \dots, w^m \in L^2(\Omega)$, $w(x,\xi) = \sum\limits_{l=1}^{m} \xi_l w^l(x)$, $x \in \Omega$, and the operator E is determined from (85). Then the function*

$$J(t,\xi) := \int\limits_{\Omega} \phi\Big((Ew(\cdot,\xi))(x,t)\Big) z(x)\,dx, \quad t \in (0,T), \quad \xi \in \mathbb{R}^m, \tag{87}$$

satisfies the L^∞-Carathéodory condition.

Proof. Step 1. Lemma 3.26 implies that $Ew \in L^2(Q_{0,T})$ if $\xi \in \mathbb{R}^m$. Hence $\phi(Ew) \in L^2(Q_{0,T}) \subset L^1(Q_{0,T})$. The Fubini Theorem [33, p. 91] yields that $J(\cdot,\xi) \in L^1(0,T)$. Then the function $[0,T] \ni t \mapsto J(t,\xi) \in \mathbb{R}$ is measurable.

Step 2. Take a point $t \in (0,T)$. We prove that the function $\mathbb{R} \ni \xi_1 \mapsto I(t,\xi_1,\dots,\xi_m) \in \mathbb{R}$ is continuous at the point $\xi_1^0 \in \mathbb{R}$. Take $\xi = (\xi_1, \xi_2, \dots, \xi_m)$, $\xi^0 = (\xi_1^0, \xi_2, \dots, \xi_m)$. Then

$$\begin{aligned}
|J(t,\xi) - J(t,\xi^0)| &\le \int\limits_{\Omega} \Big|\phi\big((Ew(\cdot,\xi))(x,t)\big) - \phi\big((Ew(\cdot,\xi^0))(x,t)\big)\Big| \cdot |z(x)|\,dx \\
&\le C_{15} \int\limits_{\Omega} \Big|(Ew(\cdot,\xi))(x,t) - (Ew(\cdot,\xi^0))(x,t)\Big| \cdot |z(x)|\,dx \\
= C_{15} \int\limits_{\Omega} \Big| \int\limits_{\Omega} &\epsilon(x,t,y)\Big((w(x+y,\xi) - w(x,\xi) - (w(x+y,\xi^0) - w(x,\xi^0))\Big)\,dy\Big| \cdot |z(x)|\,dx \\
&\le C_{16}|\xi_1 - \xi_1^0| \int\limits_{\Omega}\int\limits_{\Omega} \Big[|w^1(x+y)| + w^1(x)\Big] \cdot |z(x)|\,dxdy = C_{17}|\xi_1 - \xi_1^0| \underset{\xi_1 \to \xi_1^0}{\longrightarrow} 0.
\end{aligned}$$

Continuing in the same way, we see that J is continuous with respect to $\xi_2, \dots, \xi_m$.

Step 3. Taking into account the results of Step 1 and Step 2, we obtain that the function J satisfies the Carathéodory condition. Since $\epsilon \in L^\infty(Q_{0,T} \times \Omega)$, the L^∞-Carathéodory condition holds. □

Clearly, the operator $\Lambda(t) : Z^N \to [Z^N]^*$ (see (16)) is linear, bounded, continuous and monotone. Similarly as in Theorem 3.4 [53, p. 454], we prove that $A(t) : X^N \to [X^N]^*$ (see (17)) is bounded, semicontinuous and monotone if $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, and condition **(A)** is satisfied. The operator $G(t) : \mathcal{O}^N \to [\mathcal{O}^N]^*$ (see (4)) is bounded, semicontinuous and monotone. Similarly to (86), we get the estimate

$$||(Ew)(t); [L^s(\Omega)]^N|| \le C_{18}||w; [L^s(\Omega)]^N||, \quad w \in [L^s(\Omega)]^N, \quad t \in [0,T], \tag{88}$$

where $s > 1$ and $C_{18} > 0$ is independent of w and t. Using condition (Φ), we get that the operator $[L^s(Q_{0,T})]^N \ni u \mapsto \phi(Eu) \in [L^s(Q_{0,T})]^N$ is bounded and continuous.

Lemma 3.28. *Suppose that conditions* (Γ)*,* **(B)**, *and* **(Z)** *are satisfied, the operator* Ψ *is determined from (18). Then* $\Psi(t) : Z^N \to [Z^N]^*$ *is bounded and semicontinuous. Moreover,*

$$|\langle \Psi(t)u, v\rangle| \le C_{19} S_{1/\gamma'}\big(S_\gamma(\|u; H^N\|)\big)\|v; Z^N\|, \quad u, v \in Z^N, \quad t \in (0,T), \tag{89}$$

where $S_{1/\gamma'}$ *and* S_γ *are defined by (8),* $C_{19} > 0$ *is independent of* u, v *and* t.

Proof. Similar to [54, p. 159], we use the generalized Hölder inequality, Proposition 3.3 with $q = \gamma$, and notation (7). We get the estimate

$$|\langle \Psi(t)u, v\rangle| = \Big|\int_\Omega \sum_{k=1}^N b_k(x,t)|u|^{\gamma(x)-2}u_k\,\Delta v_k\,dx\Big| \le b^0 \int_\Omega |u|^{\gamma(x)-1}|\Delta v|\,dx$$

$$\le 2b^0\|\,|u|^{\gamma(x)-1}; L^{\gamma'(x)}(\Omega)\| \cdot \|\,|\Delta v|; L^{\gamma(x)}(\Omega)\| \le 2b^0 S_{1/\gamma'}\Big(\int_\Omega |u|^{(\gamma(x)-1)\gamma'(x)}\,dx\Big)$$

$$\times\|\,|\Delta v|; L^{\gamma(x)}(\Omega)\| \le C_{20} S_{1/\gamma'}\Big(S_\gamma\Big(\|u; [L^{\gamma(x)}(\Omega)]^N\|\Big)\Big) \cdot \|\Delta v; [L^{\gamma(x)}(\Omega)]^N\|.$$

Since $\gamma^0 \le 2$, we obtain that (89) holds and the operator Ψ is bounded. We omit the proof that Ψ is semicontinuous (it is similar to the proof of Lemma 3.25). □

Let us consider the Banach space $\mathcal{V}$ such that $\mathcal{V} \circlearrowleft Z^N$. Let us define the family of operators $\Psi_{\mathcal{V}}(t) : \mathcal{V} \to \mathcal{V}^*$ by the rule

$$\langle \Psi_{\mathcal{V}}(t)u, v\rangle_{\mathcal{V}} := \langle \Psi(t)u, v\rangle, \quad u, v \in \mathcal{V}, \quad t \in [0,T].$$

By (89), we obtain

$$|\langle \Psi_{\mathcal{V}}(t)u, v\rangle_{\mathcal{V}}| \le C_{21} S_{1/\gamma'}\big(S_\gamma(\|u; \mathcal{V}\|)\big)\|u; \mathcal{V}\|, \quad u, v \in \mathcal{V}, \quad t \in (0,T), \tag{90}$$

where $C_{21} > 0$ is independent of u, v and t. Then $\Psi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is bounded. We will replace this space $\mathcal{V}$ by V^N and $\mathcal{W}_r$. For the sake of convenience we have replaced Ψ_{V^N} and $\Psi_{\mathcal{W}_r}$ by Ψ and we have replaced $\langle\cdot,\cdot\rangle_{V^N}$ and $\langle\cdot,\cdot\rangle_{\mathcal{W}_r}$ by $\langle\cdot,\cdot\rangle$. The same notation we need for $\Lambda(t)$, $A(t)$, and $\mathcal{K}(t)$, $t \in (0,T)$. According to the above remarks, we have that the operator $\mathcal{K}(t)$ (see (19)) is bounded and semicontinuous from V^N into $[V^N]^*$ and is bounded from $\mathcal{W}_r$ into $\mathcal{W}_r^*$.

Lemma 3.29. *Suppose that* (Γ)*,* **(Q)**, **(A)-(E)**, *(7), and (21) hold. Assume also that* $\alpha > 0$, $p \in \mathcal{B}_+(\Omega)$, $p_0 > 1$, $\{w^j\}_{j\in\mathbb{N}} \subset V$, $m \in \mathbb{N}$, $L = (L_{11}, L_{21}, \ldots, L_{m1}, \ldots, L_{1N}, L_{2N}, \ldots, L_{mN})$*, where*

$$L_{\mu k}(t,\xi) = \langle (\mathcal{K}(t)z)_k, w^\mu\rangle + ((\mathcal{N}(t)z)_k, w^\mu)_\Omega, \ \ k = \overline{1,N}, \ \ \mu = \overline{1,m}, \ \ t \in (0,T), \tag{91}$$

$\xi = (\xi_{11}, \xi_{21}, \ldots, \xi_{m1}, \ldots, \xi_{1N}, \xi_{2N}, \ldots, \xi_{mN})$, $z = (z_1, \ldots, z_N)$*, and*

$$z_k(x) = \sum_{\ell=1}^m \xi_{\ell k} w^\ell(x), \quad x \in \Omega, \quad k = 1,N.$$

Then

$$(L(t,\xi),\xi)_{\mathbb{R}^{mN}} \ge \int_\Omega \Big[\frac{\alpha}{2}|\Delta z|^2 + a_0 \sum_{i=1}^n |z_{x_i}|^{p(x)} + g_0|z|^{q(x)} - C_{22}|z|^2\Big]\,dx - C_{23}, \quad t \in (0,T), \tag{92}$$

where $C_{22}, C_{23} > 0$ *are independent of* z, ξ *and* t.

Proof. Clearly,

$$(L(t,\xi),\xi)_{\mathbb{R}^{mN}} = \langle \mathcal{K}(t)z, z\rangle + (\mathcal{N}(t)z, z)_\Omega = \sum_{k=1}^N \int_\Omega \Big[\alpha|\Delta z_k|^2 + \sum_{i=1}^n a_{ik}(t)|z_{x_i}|^{p(x)-2}|z_{k,x_i}|^2$$

$$+b_k(t)|z|^{\gamma(x)-2}z_k\Delta z_k + g_k(t)|z|^{q(x)-2}|z_k|^2 + \beta_k(t)|(z_k)^-|^2\Big]\,dx + (\phi(Ez(t)), z)_\Omega. \tag{93}$$

Taking into account **(A)**, **(G)**, and **(BB)**, we obtain

$$\sum_{k=1}^{N}\Big[\alpha|\Delta z_k|^2 + \sum_{i=1}^{n} a_{ik}(t)|z_{x_i}|^{p(x)-2}|z_{k,x_i}|^2 + g_k(t)|z|^{q(x)-2}|z_k|^2 + \beta_k(t)|(z_k)^-|^2\Big]$$

$$\geq \alpha|\Delta z|^2 + a_0|z_{x_i}|^{p(x)} + g_0|z|^{q(x)}. \tag{94}$$

Using the generalized Young inequality, we get

$$\sum_{k=1}^{N}|b_k|z|^{\gamma(x)-2}z_k\Delta z_k| = b^0|z|^{\gamma(x)-1}|\Delta z| \leq C_{24}(\varkappa_1)|z|^{\gamma(x)} + \varkappa_1|\Delta z|^{\gamma(x)}$$

$$\leq \varkappa_1|\Delta z|^2 + C_{25}(\varkappa_1)(1+|z|^2), \tag{95}$$

where $\varkappa_1 > 0$, $C_{25}(\varkappa_1) > 0$ is independent of x, t, k and m.

Taking into account condition **(Φ)**, Cauchy-Bunyakowski-Schwarz's inequality, and (88), we obtain

$$\Big|(\phi(Ez(t)), z)_\Omega\Big| \leq \phi^0\int_\Omega |Ez(t)|\cdot|z|\,dx \leq C_{26}||Ez(t);[L^2(\Omega)]^N||\cdot||z;[L^2(\Omega)]^N||$$

$$\leq C_{18}||z;[L^2(\Omega)]^N||\cdot||z;[L^2(\Omega)]^N|| \leq C_{27}\int_\Omega |z|^2\,dx, \tag{96}$$

where $C_{27} > 0$ is independent of z, t and m.

Using (94)-(96) and choosing $\varkappa_1 = \frac{\alpha}{2}$ we can show that (93) yields (92). □

4 Proof of main Theorem

The solution will be constructed via Faedo-Galerkin's method.

Step 1. Let $\{w^j\}_{j\in\mathbb{N}}$ be a set of all eigenfunctions of the problem (28) which are an orthonormal in $L^2(\Omega)$,

$$\mathfrak{M}_m^N := \Big\{x \mapsto \Big(\sum_{\mu=1}^{m}\alpha_{\mu 1}^m w^\mu(x),\ \ldots,\ \sum_{\mu=1}^{m}\alpha_{\mu N}^m w^\mu(x)\Big)\ \Big|\ \alpha_{\mu k}^m \in \mathbb{R},\ k=\overline{1,N},\ \mu=\overline{1,m}\Big\},\quad m\in\mathbb{N},$$

r is determined from condition **(Z)**, $\mathcal{W}_r$ and $\mathcal{W}_r^*$ are defined by (31), and V is defined by (14). Taking into account Proposition 3.5 and (32), we obtain that $\mathfrak{M}^N := \bigcup_{m\in\mathbb{N}}\mathfrak{M}_m^N$ is dense in $\mathcal{W}_r$ and V^N.

Take $m\in\mathbb{N}$ and $u^m := (u_1^m, \ldots, u_N^m)$, where

$$u_k^m(x,t) := \sum_{\mu=1}^{m}\varphi_{\mu k}^m(t)w^\mu(x), \quad (x,t)\in Q_{0,T}, \quad k=\overline{1,N},$$

$\varphi^m := (\varphi_{11}^m, \varphi_{21}^m, \ldots, \varphi_{m1}^m, \ldots, \varphi_{1N}^m, \varphi_{2N}^m, \ldots, \varphi_{mN}^m)$ is a solution to the problem

$$\langle u_t^m(t), w^\mu\rangle + \langle\mathcal{K}(t)u^m(t), w^\mu\rangle + (\mathcal{N}(t)u^m(t), w^\mu)_\Omega = \langle F(t), w^\mu\rangle, \quad t\in(0,T), \tag{97}$$

$$\varphi_{\mu k}^m(0) = \beta_{\mu k}^m, \quad k=\overline{1,N}, \quad \mu=\overline{1,m} \tag{98}$$

(see (3), (19), and (20) for definition of the elements of $\mathcal{N}$, $\mathcal{K}$, and F), the functions $u_0^m := (u_{01}^m, \ldots, u_{0N}^m)$ satisfies the condition

$$u_0^m \underset{m\to\infty}{\longrightarrow} u_0 \quad \text{strongly in} \quad H^N,$$

and $u_{0k}^m(x) := \sum\limits_{\mu=1}^{m} \beta_{\mu k}^m \, w^\mu(x)$, $x \in \Omega$, $k = \overline{1,N}$. Clearly,

$$u^m(0) = \Big(\sum_{\mu=1}^{m} \varphi_{\mu 1}^m(0) w^\mu(x), \; \dots, \; \sum_{\mu=1}^{m} \varphi_{\mu N}^m(0) w^\mu(x) \Big) = u_0^m. \tag{99}$$

The problem ((97), (98)) coincides with (74) if $\ell = mN$,

$$\varphi^0 = (\beta_{11}^m, \beta_{21}^m, \dots, \beta_{m1}^m, \dots, \beta_{1N}^m, \beta_{2N}^m, \dots, \beta_{mN}^m),$$

$$M = (M_{11}, M_{21}, \dots, M_{m1}, \dots, M_{1N}, M_{2N}, \dots, M_{mN}), \qquad M_{\mu k}(t) = \langle F_k(t), w^\mu \rangle,$$

$$L = (L_{11}, L_{21}, \dots, L_{m1}, \dots, L_{1N}, L_{2N}, \dots, L_{mN}),$$

$$L_{\mu k}(t, \varphi^m) = \Big\langle \big(\mathcal{K}(t) u^m(t)\big)_k, w^\mu \Big\rangle + \Big(\big(\mathcal{N}(t) u^m(t)\big)_k, w^\mu \Big)_\Omega, \quad k = \overline{1,N}, \quad \mu = \overline{1,m}, \quad t \in (0,T). \tag{100}$$

By **(F)**, we have $M \in L^2(0,T;\mathbb{R}^{mN})$. Taking into account the lemmas such as Lemmas 3.27 and 3.25, we see that L satisfies the L^∞-Carathéodory condition. From (92) we obtain

$$(L(t,\varphi^m), \varphi^m)_{\mathbb{R}^{mN}} \geq -C_{28} \int\limits_\Omega |u^m|^2 \, dx - C_{29}$$

$$\geq -C_{30}(m) \int\limits_\Omega \sum_{k=1}^{N} \sum_{\mu=1}^{m} |\varphi_{\mu k}^m|^2 |w^\mu(x)|^2 \, dx - C_{29} = -C_{31}(m)\, |\varphi^m|^2 - C_{29}, \tag{101}$$

where $C_{29}, C_{31} > 0$ are independent of t, φ^m. Then Carathéodory-LaSalle's Theorem 3.24 implies that there exists a solution $\varphi^m \in H^1(0,T;\mathbb{R}^{mN})$ to problem (97), (98). If we combine the condition $\partial\Omega \in C^{2r}$ with Proposition 3.5 and embedding (26), we get $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{W}_r \subset [H^{2r}(\Omega)]^N$. Thus,

$$u^m \in H^1(0,T;\mathcal{W}_r) \subset H^1(0,T;[H^{2r}(\Omega)]^N) \subset [H^1(Q_{0,T})]^N. \tag{102}$$

Step 2. Multiplying both sides of the corresponding equality (97) by $\varphi_{\mu k}^m(t)$, summing the obtained equalities, and integrating in $t \in (0,\tau) \subset (0,T)$, we get

$$\int\limits_{Q_{0,\tau}} (u_t^m, u^m) \, dxdt + \int\limits_0^\tau (L(t,\varphi^m(t)), \varphi^m(t))_{\mathbb{R}^{mN}} \, dt$$

$$= \int\limits_{Q_{0,\tau}} \Big[\sum_{i,j=1}^{n} (f_{ij}, u_{x_i x_j}^m) + \sum_{i=1}^{n} (f_i, u_{x_i}^m) + (f_0, u^m) \Big] \, dxdt, \quad \tau \in (0,T]. \tag{103}$$

By (102), similar to Case 1.ii of Theorem 3.19 (with $p(x) = r(x) \equiv 2$), we obtain

$$|u^m|^2 \in W^{1,1}(0,T;L^1(\Omega)), \qquad (|u^m|^2)_t = 2(u_t^m, u^m).$$

Then, the integration by parts formula and (99) yield that

$$\int\limits_{Q_{0,\tau}} (u_t^m, u^m) \, dxdt = \frac{1}{2} \int\limits_\Omega |u^m(x,\tau)|^2 \, dx - \frac{1}{2} \int\limits_\Omega |u_0^m(x)|^2 \, dx.$$

By (92), we get

$$\int\limits_0^\tau (L(t,\varphi^m(t)), \varphi^m(t))_{\mathbb{R}^{mN}} \, dt \geq \int\limits_{Q_{0,\tau}} \Big[\frac{\alpha}{2} |\Delta u^m|^2 + a_0 \sum_{i=1}^{n} |u_{x_i}^m|^{p(x)} + g_0 |u^m|^{q(x)} - C_{32} |u^m|^2 \Big] \, dx - C_{33},$$

where $C_{32}, C_{33} > 0$ are independent of m and τ. In addition, Young's inequality, the condition $\partial\Omega \in C^2$, and estimate (30) yield that

$$\Big|\int\limits_{Q_{0,\tau}} \sum_{i,j=1}^{n} (f_{ij}, u^m_{x_i x_j})\, dxdt\Big| \le \int\limits_{Q_{0,\tau}} \sum_{i,j=1}^{n} \Big[\varkappa_1 |u^m_{x_i x_j}|^2 + \frac{1}{4\varkappa_1}|f_{ij}|^2\Big]\, dxdt$$

$$\le \int\limits_{Q_{0,\tau}} \Big[\varkappa_1 C_{34}|\Delta u^m|^2 + \frac{1}{4\varkappa_1}\sum_{i,j=1}^{n} |f_{ij}|^2\Big]\, dxdt,$$

where $\varkappa_1 > 0$, the constant $C_{34} > 0$ is independent of m and $\varkappa_1$. By (23), we get

$$\Big|\sum_{i=1}^{n}(f_i, u^m_{x_i}) + (f_0, u^m)\Big| \le \Big[\varkappa_2 \sum_{i=1}^{n} |u^m_{x_i}|^{p(x)} + Y_p(\varkappa_2)\sum_{i=1}^{n} |f_i|^{p'(x)} + \varkappa_3 |u^m|^{q(x)} + Y_q(\varkappa_3)|f_0|^{q'(x)}\Big].$$

According to the above remarks, from (103) we have the following inequality

$$\frac{1}{2}\int\limits_{\Omega_\tau} |u^m|^2\, dx + \int\limits_{Q_{0,\tau}} \Big[\Big(\frac{\alpha}{2} - \varkappa_1 C_{34}\Big)|\Delta u^m|^2 + (a_0 - \varkappa_2)\sum_{i=1}^{n} |u^m_{x_i}|^{p(x)} + (g_0 - \varkappa_3)|u^m|^{q(x)}\Big]\, dxdt$$

$$\le \frac{1}{2}\int\limits_{\Omega} |u^m_0|^2\, dx + C_{35}(\varkappa_1, \varkappa_2, \varkappa_3)\Big(1 + \int\limits_{Q_{0,\tau}} \Big[\sum_{i,j=1}^{n} |f_{ij}|^2 + \sum_{i=1}^{n} |f_i|^{p'(x)} + |f_0|^{q'(x)}\Big]\, dxdt$$

$$+ \int\limits_{Q_{0,\tau}} |u^m|^2\, dxdt\Big), \quad \tau \in (0, T], \tag{104}$$

where $C_{35} > 0$ is independent of m and τ.

Let $y(t) := \int_\Omega |u^m(x,t)|^2\, dx$, $t \in [0, T]$. Choosing $\varkappa_1, \varkappa_2, \varkappa_3 > 0$ sufficiently small, from (104) we can obtain that $y(\tau) \le C_{36} + C_{37}\int_0^\tau y(t)\, dt$, $\tau \in (0, T]$. Then the Gronwall-Bellman Lemma yields that

$$\int\limits_{\Omega} |u^m(x,\tau)|^2\, dx \le C_{38}, \quad \tau \in (0, T], \tag{105}$$

and so

$$\int\limits_{Q_{0,\tau}} |u^m|^2\, dxdt \le C_{38}T, \quad \tau \in (0, T]. \tag{106}$$

Using (104), (106), and choosing $\varkappa_1, \varkappa_2, \varkappa_3 > 0$ sufficiently small, we get

$$\int\limits_{Q_{0,\tau}} \Big[|\Delta u^m|^2 + \sum_{i=1}^{n} |u^m_{x_i}|^{p(x)} + |u^m|^{q(x)}\Big]\, dxdt \le C_{39}, \quad \tau \in (0, T]. \tag{107}$$

Here $C_{38}, C_{39} > 0$ are independent of m and τ.

By (105)-(107), we have that there exists a sequence $\{u^{m_j}\}_{j\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad *-\text{weakly in} \quad L^\infty(0, T; H^N) \quad \text{and weakly in} \quad U(Q_{0,T}). \tag{108}$$

Step 3. We define the element $\mathcal{F} \in [U(Q_{0,T})]^*$ and the operator $\mathcal{A} : U(Q_{0,T}) \to [U(Q_{0,T})]^*$ by the rules

$$\langle \mathcal{F}, v\rangle_{U(Q_{0,T})} := \int\limits_0^T \langle F(t), v(t)\rangle\, dt, \quad v \in U(Q_{0,T}), \tag{109}$$

$$\langle \mathcal{A}u, v\rangle_{U(Q_{0,T})} := \int\limits_0^T \Big[\langle \mathcal{K}(t)u(t), v(t)\rangle + (\mathcal{N}(t)u(t), v(t))_\Omega\Big]\, dt, \quad u, v \in U(Q_{0,T}). \tag{110}$$

Using (24), (86), (106) and (107), we get

$$\langle \mathcal{A}u^m, v\rangle_{U(Q_{0,T})} = \int\limits_{Q_{0,T}} \sum_{k=1}^{N}\Big[\alpha\, \Delta u_k^m\, \Delta v_k + \sum_{i=1}^{n} a_{ik}|u_{x_i}^m|^{p(x)-2}u_{k,x_i}^m v_{k,x_i} + b_k|u^m|^{\gamma(x)-2}u_k^m\,\Delta v_k$$

$$+g_k|u^m|^{q(x)-2}u_k^m\, v_k - \beta_k(u_k^m)^- v_k + \phi_k(Eu_k^m)v_k\Big]\,dxdt \le C_{40}\int\limits_{Q_{0,T}}\Big[|\Delta u^m|\cdot|\Delta v|$$

$$+\sum_{i=1}^{n}|u_{x_i}^m|^{p(x)-1}|v_{x_i}| + |u^m|^{\gamma(x)-1}\,|\Delta v| + |u^m|^{q(x)-1}\,|v| + |u^m|\cdot|v| + |Eu^m|\cdot|v|\Big]\,dxdt$$

$$\le C_{40}\Big(||\,|\Delta u^m|; L^2(Q_{0,T})||\cdot||\,|\Delta v|; L^2(Q_{0,T})|| + 2\sum_{i=1}^{n}||\,|u_{x_i}^m|^{p(x)-1}; L^{p'(x)}(Q_{0,T})||$$

$$\times||\,|v_{x_i}|; L^{p(x)}(Q_{0,T})|| + 2||\,|u^m|^{\gamma(x)-1}; L^{\gamma'(x)}(Q_{0,T})||\cdot||\,|\Delta v|; L^{\gamma(x)}(Q_{0,T})||$$

$$+2||\,|u^m|^{q(x)-1}; L^{q'(x)}(Q_{0,T})||\cdot||\,|v|; L^{q(x)}(Q_{0,T})|| + ||\,|u^m|; L^2(Q_{0,T})||\cdot||\,|v|; L^2(Q_{0,T})||$$

$$+||\,|Eu^m|; L^2(Q_{0,T})||\cdot||\,|v|; L^2(Q_{0,T})||\Big) \le C_{41}||v; U(Q_{0,T})||,$$

where $C_{41} > 0$ is independent of m, v. Then

$$||\mathcal{A}u^m; [U(Q_{0,T})]^*|| \le C_{41} \tag{111}$$

and so

$$\mathcal{A}u^{m_j} \underset{j\to\infty}{\longrightarrow} \chi \quad \text{weakly in} \quad [U(Q_{0,T})]^*. \tag{112}$$

Step 3. Suppose that the numbers r and s^0 are determined from condition **(Z)**, the spaces $\mathcal{W}_r$ and $\mathcal{W}_r^*$ are defined by (31), $P_m : H^N \to H^N$ is the projection operator from (45) (see also (21)), $\widehat{P}_m$ is defined by (40), where $\mathcal{H} = H^N$ and $\mathcal{V} = \mathcal{W}_r$. Similarly to [54, p. 77] and [55, p. 62-63], using Lemma 3.9, notation (109) and (110), we rewrite (97) as

$$u_t^m = \widehat{P}_m^*\big(\mathcal{F} - \mathcal{A}u^m\big). \tag{113}$$

By (49), we get

$$||\widehat{P}_m f; L^{s^0}(0,T;\mathcal{W}_r)|| \le ||f; L^{s^0}(0,T;\mathcal{W}_r)||, \quad f \in L^{s^0}(0,T;\mathcal{W}_r). \tag{114}$$

Since $||D^*||_{\mathcal{L}(B^*,A^*)} = ||D||_{\mathcal{L}(A,B)}$ for every $D \in \mathcal{L}(A,B)$ (see [42, p. 231]), using (114), we have

$$||\widehat{P}_m^* h; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le ||h; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)||, \quad h \in L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*). \tag{115}$$

Taking into account (115), (35), and (109), we obtain

$$||\widehat{P}_m^*\mathcal{F}; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le ||\mathcal{F}; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le C_{42}||\mathcal{F}; [U(Q_{0,T})]^*|| \le C_{43}. \tag{116}$$

By (115), (110), (111), and (35), we get

$$||\widehat{P}_m^*\mathcal{A}u^m; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le ||\mathcal{A}u^m; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le C_{44}||\mathcal{A}u^m; [U(Q_{0,T})]^*|| \le C_{45}. \tag{117}$$

Using (113), (116), and (117) (see for comparison [54, 55]), we obtain

$$||u_t^m; L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*)|| \le C_{46}. \tag{118}$$

Here $C_{43},\dots,C_{46} > 0$ are independent of m. Therefore,

$$u_t^{m_j} \underset{j\to\infty}{\longrightarrow} u_t \quad \text{weakly in} \quad L^{\frac{s^0}{s^0-1}}(0,T;\mathcal{W}_r^*). \tag{119}$$

Step 4. Suppose the numbers r and s_0 are determined from condition **(Z)**. Then (32) implies that $V^N \overset{K}{\subset} H^N \circlearrowleft \mathcal{W}_r^*$. By (33), (106), and (107), we get

$$||u^m; L^{s_0}(0,T;V^N)|| \le C_{47}||u^m; U(Q_{0,T})|| \le C_{48}, \tag{120}$$

where $C_{48} > 0$ is independent of m.

Taking into account (120), (118), the Aubin theorem (see Proposition 3.11), and Lemma 1.18 [23, p. 39], we obtain

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{strongly in} \quad L^2(0,T;H^N) \quad \text{and in} \quad C([0,T];\mathcal{W}_r^*), \tag{121}$$

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{almost everywhere in} \quad Q_{0,T}. \tag{122}$$

Clearly, $V^N \overset{K}{\subset} [H_0^1(\Omega)]^N \circlearrowleft \mathcal{W}_r^*$. Then (120), (118), and the Aubin theorem yield that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{strongly in} \quad L^2(0,T;[H_0^1(\Omega)]^N). \tag{123}$$

Hence for every $i \in \{1,\dots,n\}$ we have

$$\int\limits_{Q_{0,T}} |u_{x_i}^{m_j} - u_{x_i}|^2\,dxdt \le ||u^{m_j} - u; L^2(0,T;[H_0^1(\Omega)]^N)||^2 \underset{j\to\infty}{\longrightarrow} 0.$$

Thus $u_{x_i}^{m_j} \underset{j\to\infty}{\longrightarrow} u_{x_i}$ strongly in $[L^2(Q_{0,T})]^N$ and so Lemma 1.18 [23, p. 39] implies that

$$u_{x_i}^{m_j} \underset{j\to\infty}{\longrightarrow} u_{x_i} \quad \text{almost everywhere in} \quad Q_{0,T}, \quad i = \overline{1,n}. \tag{124}$$

By (122) and (124), we obtain the equality $\chi = \mathcal{A}u$.

Step 5. Using (97) and (102), we obtain

$$-\int\limits_0^T (u^{m_j}(t), w)_\Omega\, \varphi'(t)\,dt + \langle \mathcal{A}u^{m_j}, w\varphi\rangle_{U(Q_{0,T})} = \langle \mathcal{F}, w\varphi\rangle_{U(Q_{0,T})}, \tag{125}$$

where $\varphi \in C_0^\infty((0,T))$, $w \in \mathfrak{M}_k^N$, $k \in \mathbb{N}$, $k \le m_j$, $j \in \mathbb{N}$. Letting $j \to +\infty$ and using Lemma 3.8, we get the equality $u_t + \mathcal{A}u = \mathcal{F}$. Whence, $u_t = \mathcal{F} - \mathcal{A}u \in [U(Q_{0,T})]^*$, $u \in W(Q_{0,T})$, and (22) holds. Moreover, we obtain the inclusion $u_t \in L^{\frac{s^0}{s^0-1}}(0,T;[V^N]^*)$ because (34) is true. Hence, $u \in C([0,T];[V^N]^*)$. By (108), we have that $u \in L^\infty(0,T;H^N)$. Thus, Lemma 3.7 yields that $u \in C([0,T];H^N)$ and so u is a weak solution to initial-boundary value problem (1), (2). □

References

[1] Borcia O. D., Borcia R., Bestehorn M., Long wave instabilities in binary mixture thin liquid film, J. of Optoelectronics and Advanced Materials., 2006, 8 (3), 1033-1036

[2] King B. B., Stein O., Winkler M., A fourth-order parabolic equation modeling epitaxial thin film growth, J. Math. Anal. Appl., 2003, 286, 459–490

[3] Oron A., Davis S. H., Bankoff S. G., Long-scale evolution of thin liquid films, Reviews of Modern Physics., 1997, 69 (3), 931-980

[4] Bernis F., Friedman A., Higher order nonlinear degenerate parabolic equations, J. Diff. Equ., 1990, 83, 179-206

[5] Bai F., Elliott C. M., Gardiner A., Spence A., and Stuart A. M., The viscous Cahn-Hillard equation. Part I.: Computations. Stanford University (Preprint SCCM-93-12), 1993, 23 p.

[6] Elliott C. M., French D. A., and Milner F. A., A second order splitting method for the Cahn-Hillard equation, Numer. Math., 1989, 54, 575-590

[7] Shyshkov A. E., Taranets R. M., On a flow thin-film equation with nonlinear convection in multidimensional domains, Ukr. Math. Bull., 2004, 1 (3), 402-444 (in Russian)

[8] Taranets R. M., Propagation of perturbations in thin capillary film equations with nonlinear diffusion and convection, Siberian Math. J., 2006, 47 (4), 751–766

[9] Alkhutov Y., Antontsev S., Zhikov V., Parabolic equations with variable order of nonlinearity, in: Collection of Works of Inst. of Math. NAS of Ukraine., 2009, 6 (1), 23-50

[10] Antontsev S., Shmarev S., Evolution PDEs with nonstandard growth conditions. Existence, uniqueness, localization, blow-up (Atlantis Press, Paris, 2015)

[11] Bokalo M. M., Buhrii O. M., Mashiyev R. A., Unique solvability of initial-boundary-value problems for anisotropic elliptic-parabolic equations with variable exponents of nonlinearity, J. of Nonlinear Evolution Equat. Appl. 2014, 2013 (6), 67-87

[12] Bokalo M. M., Sikorsky B. M., On properties of solution to problem without initial conditions for equations og polytropic filtration type, Visn. Lviv. Univ. (Herald of Lviv Univ.). Series Mech. Math., 1998, 51, 85-98 (in Ukrainian)

[13] Buhrii O., Domans'ka G., Protsakh N., Initial boundary value problem for nonlinear differential equation of the third order in generalized Sobolev spaces, Visn. Lviv. Univ. (Herald of Lviv Univ.). Series Mech. Math., 2005, 64, 44-61 (in Ukrainian)

[14] Kholyavka O., Buhrii O., Bokalo M., and Ayazoglu (Mashiyev) R., Initial-boundary-value problem for third order equations of Kirchhoff type with variable exponents of nonlinearity, Advances in Math. Sciences and Appl., 2013, 23 (2), 509-528

[15] Kováčik O., Parabolic equations in generalized Sobolev spaces $W^{k,p(x)}$, Fasciculi Mathematici., 1995, 25, 87-94

[16] Averina T. A., Rybakov K. A., New methods of influence of Poisson's delta-impulse in radioelectronic's problems J. of Radioelectron., 2013, 1, 1-20 (in Russian)

[17] Buhrii O., Buhrii M., On existence in generalized Sobolev spaces solutions of the initial-boundary value problems for nonlinear integro-differential equations arising from theory of European option, Visn. Lviv Univ. (Herald of Lviv University). Ser. Mech.-Math., 2016, 81, 61-84 (in Ukrainian)

[18] Merton R. C., Option pricing when underlying stock returns are discontinuous, J. of Financial Economics., 1976, 3, 125-144

[19] Buhrii O. M., Initial-boundary value problem for doubly nonlinear integro-differential equations with variable exponents of nonlinearity, Dopovidi NANU., 2017, 2, 3-9 (in Ukrainian)

[20] Dunford N., Schwartz J. T., Linear operators. Part 1: General theory (Izd. Inostran Lit., Moscow, 1962) (translated from: Interscience Publ., New York, London, 1958) (in Russian)

[21] Ladyzhenskaya O. A., Ural'tseva N. N., Linear and quasilinear elliptic equations, 2th edition (Nauka, Moscow, 1973) (in Russian)

[22] Adams R. A., Sobolev spaces (Academic Press, New York, San Francisco, London, 1975)

[23] Gajewski H., Groger K., Zacharias K., Nonlinear operator equations and operator differential equations (Mir, Moscow, 1978) (translated from: Akademie-Verlag, Berlin, 1974) (in Russian)

[24] Evans L. C., Partial differential equations. Graduate Studies in Mathematics. (Amer. Math. Soc., Providence, RI, 1998)

[25] Kováčik O., Rákosník J., On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J., 1991, 41 (116), 592-618

[26] Orlicz W. Uber Konjugierte Exponentenfolgen. Studia Mathematica (Lviv), 1931, 3, 200-211

[27] Buhrii O. M., Parabolic variational inequalities without initial conditions, Ph. D. thesis, Ivan Franko National University of Lviv (Lviv, Ukraine, 2001) (in Ukrainian)

[28] Buhrii O. M., Mashiyev R. A., Uniqueness of solutions of the parabolic variational inequality with variable exponent of nonlinearity, Nonlinear Analysis: Theory, Methods and Appl., 2009, 70 (6), 2325-2331

[29] Diening L., Harjulehto P., Hasto P., Ruzicka M., Lebesgue and Sobolev spaces with variable exponents (Springer, Heidelberg, 2011)

[30] Fan X.-L., Zhao D., On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 2001, 263, 424-446

[31] Panat O. T., Problems for hyperbolic equations and hyperbolic-parabolic systems in generalized Sobolev spaces, Ph.D. thesis, Ivan Franko National University of Lviv (Lviv, Ukraine, 2010) (in Ukrainian)

[32] Buhrii O. M., Finiteness of the stabilization time of solution of a nonlinear parabolic variational inequality with variable degree of nonlinearity, Matem. Studii., 2005, 24 (2), 167-172 (in Ukrainian)

[33] Brezis H., Functional Analysis, Sobolev Spaces and Partial Differential Equations (Springer, New York, Dordrecht, Heidelberg, London, 2011)

[34] Mikhailov V. P., Partial differential equations (Nauka, Moscow, 1976) (in Russian)

[35] Lions J.-L., Magenes E., Nonhomogeneous boundary value problems and its applications (Mir, Moscow, 1971) (translated from: Dunod, Paris, 1968) (in Russian)

[36] Antontsev S., Shmarev S., Energy solution of evolution equations with nonstandard growth conditions. (CMAF, University of Lisboa, Portugal. Preprint-2011-008)

[37] Suhubi E., Functional analysis (Kluwer Acad. Publ., Dordrecht, Boston, London, 2003)

[38] Hutson V. C. L., Pym J. S., Applications of functional analysis and operator theory (Mir, Moscow, 1983) (translated from: Academic Press, London, New York, Toronto, Sydney, San Francisco, 1980) (in Russian)

[39] Kinderlehrer D., Stampacchia G., Introduction to variational inequalities and its applications (Mir, Moscow, 1983) (translated from: Academic Press, New York, London, Toronto, Sydney, San Francisco, 1980) (in Russian)

[40] Aubin J.-P., Un theoreme de compacite, Comptes rendus hebdomadaires des seances de l'academie des sciences., 1963, 256 (24), 5042-5044

[41] Bernis F., Existence results for doubly nonlinear higher order parabolic equations on unbounded domains, Math. Ann. 1988, 279, 373-394

[42] Kolmogorov A. N., Fomin S. V., Elements of theory of functions and functional analysis (Nauka, Moscow, 1972) (in Russian)

[43] Bokalo T., Buhrii O., Some integrating by parts formulas in variable indices of nonlinearity function spaces, Visn. Lviv. Univ. (Herald of Lviv Univ.). Series Mech. Math. 2009, 71, 13-26 (in Ukrainian)
[44] Mikhailov V. P., Gushchin A. K., Additional chapters of the course “Equations of mathematical physics” (Moscow, 2007) (in Russian)
[45] Buhrii O. M., On formulaes of integration by parts for special type of power functions, Matem. Studii., 2016, 45 (2), 118–131 (in Ukrainian)
[46] Lee J. W., O’Regan D., Existence results for differential equations in Banach spaces, Comment. Math. Univ. Carolin., 1993, 34 (2), 239-251
[47] Roubicek T., Nonlinear partial differential equations with applications (Birkhauser Verlag, Basel, Boston, Berlin, 2005)
[48] Kato S., On existence and uniqueness conditions for nonlinear ordinary differential equations in Banach spaces, Funkcialaj Ekvacioj., 1976, 19, 239-245
[49] O’Regan D., Existence theory for nonlinear ordinary differential equations (Kluwer, Dordrecht, 1997)
[50] Coddington E. A., Levinson N., Theory of ordinary differential equations, (Mir, Moscow, 1983) (translated from: McGraw-Hill Book Company Inc., New York, Toronto, London, 1955) (in Russian)
[51] Dubinsky Yu. A., Quasilinear elliptic and parabolic equations of an arbitrary order, Uspehi Mat. Nauk., 1968, 23, 45-90 (in Russian)
[52] Byström J., Sharp constants for some inequalities connected to the p-Laplace operator, J. of Ineq. in Pure and Appl. Math., 2005, 6 (2): Article 56
[53] Mashiyev R. A., Buhrii O. M., Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity, J. Math. Anal. Appl., 2011, 377, 450-463
[54] Lions J.-L., Some methods of solving of nonlinear boundary value problems (Mir, Moscow, 1972) (translated from: Dunod, Gauthier-Villars, Paris, 1969) (in Russian)
[55] Lions J.-L., Strauss W. A., Some non-linear evolution equations, Bulletin de la S. M. F., 1965, 93, 43-96