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Linear dynamics of semigroups generated by differential operators

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Abstract: During the last years, several notions have been introduced for describing the dynamical behavior of linear operators on infinite-dimensional spaces, such as hypercyclicity, chaos in the sense of Devaney, chaos in the sense of Li-Yorke, subchaos, mixing and weakly mixing properties, and frequent hypercyclicity, among others. These notions have been extended, as far as possible, to the setting of C_0-semigroups of linear and continuous operators.

We will review some of these notions and we will discuss basic properties of the dynamics of C_0-semigroups. We will also study in detail the dynamics of the translation C_0-semigroup on weighted spaces of integrable functions and of continuous functions vanishing at infinity. Using the comparison lemma, these results can be transferred to the solution C_0-semigroups of some partial differential equations. Additionally, we will also visit the chaos for infinite systems of ordinary differential equations, that can be of interest for representing birth-and-death process or car-following traffic models.

Keywords: Hypercyclicity, Topological transitivity, Topologically mixing property, Devaney chaos, C_0-semigroups

MSC: 47A16

Linear dynamics has attracted the interest of researchers during the last three decades, after the seminal works of Kitai [1], Gethner and Shapiro [2] and Godefroy and Shapiro [3]. On the one hand, it is connected with the still unsolved Invariant Subspace Problem on Hilbert spaces, which asks for the existence of an operator with no non-trivial closed invariant subset. The answer was positive in Banach spaces, as it was seen by Read [4] in ℓ_1. On the other hand, there is a number of different connections of linear dynamics with different areas such as algebra, topology, real and complex analysis, functional analysis, approximation theory, number theory, and probability.

The advances in the area were first compiled by Grosse-Erdmann [5, 6]. The monographs of Bayart and Matheron [7] and Grosse-Erdmann and Peris [8] represent a good source of the state of the art in the area. The study of the size and the algebraic structure of the set of vectors with *wild behaviour*, provides an interesting field for continuing with the quest of surprising examples of sets of pathological elements that, however, are preserved by the elementary algebraic operators. In this line, a selection of topics was recently revisited by Aron et al. in [9, Ch. 4], see also [10].

Given a family $\{T_i\}_{i\in I}$ of linear and continuous operators on an infinite-dimensional separable Banach space X, we say that it is *universal* if there exists some element $x \in X$ such that, its orbit $\{T_i x : i \in I\}$ is dense in X.

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One of the core notions of linear dynamics is *hypercyclicity*, which corresponds to the particular case in which the family of operators is given by the powers of a single operator, i.e. $T_i = T_0^i$ for some $T_0 \in L(X)$ and $i \in \mathbb{N}$. Another particular case is the case of a C_0-*semigroup of operators* $\{T_t\}_{t\geq 0}$ in $L(X)$, or simply a *semigroup*.

Definition 0.1. *A family of operators $\{T_t\}_{t\geq 0}$ in $L(X)$ is said to be a semigroup if the following conditions hold:*
(i) $T_0 = I$,
(ii) $T_t T_s = T_{t+s}$ *for all* $t, s \geq 0$ *(semigroup law); and*
(iii) $\lim_{t\to s} T_t x = T_s x$ *for all* $x \in X, s \geq 0$.

The last condition expresses the pointwise convergence of the operators. Many results in the study of the linear dynamics of C_0-semigroups yield from the semigroup law and from condition (iii) in the definition, since it permits to obtain the local equicontinuity of the family of operators $\{T_t\}_{t\geq 0}$.

Replacing in the last condition the pointwise convergence by the convergence over the bounded sets of X, we say that the semigroup is *uniformly continuous*. These semigroups can be easily represented using the Taylor series of the exponential function. More precisely, a semigroup $\{T_t\}_{t\geq 0}$ is *uniformly continuous* if there exists $A \in L(X)$ such that $T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, t \geq 0$. In fact, such a semigroup corresponds to the solution semigroup of the following Abstract Cauchy Problem (ACP)

$$\left\{ \begin{array}{ll} u_t(t,x) = & Au(t,x) \\ u(0,x) = & \varphi(x) \end{array} \right\}, \tag{1}$$

where A is a linear operator defined on X. If A is defined on the whole space X, then the unique solution of this ACP is given by

$$u(t,x) = e^{tA}\varphi(x) \tag{2}$$

where $\varphi(x) \in X$. In that sense, $u(t,x)$ is called a *classical solution* of the abstract Cauchy problem (1) and the semigroup $\{T_t\}_{t\geq 0} = \{e^{tA}\}_{t\geq 0}$ is called the *solution semigroup* of (1). The operator A is considered to be the *infinitesimal generator* of the C_0-semigroup. This notion can be extended to C_0-semigroups that are not uniformly continuous. In this case, the *infinitesimal generator* is computed as

$$Ax := \lim_{h\to 0^+} \frac{T_h x - x}{h}. \tag{3}$$

This is a closed and densely defined linear operator that determines the semigroup uniquely. For more information about the properties of the infinitesimal generator of C_0-semigroups we refer to [11–13]. For the case of Fréchet spaces different from ω we refer to [14], and for ω we refer to [15, 16]. The corresponding study in locally convex spaces was performed in [17].

The study of the distinguished translation semigroup $\{\tau_t\}_{t\in\mathbb{C}}$ started with the work [18], where Birkhoff proved its transitivity (hypercyclicity) on $\mathcal{H}(\mathbb{C})$. After that, Rolewicz showed an analogous result in the space of weighted spaces of p-integrable functions $L^p_\rho(\mathbb{R})$, $1 \leq p < \infty$, $\rho(s) := a^{-s}, a > 1$ [19]. The first systematic study on hypercyclic semigroups was initiated by Desch, Schappacher and Webb in [20]. However, some examples and previous results were already obtained by that time, see for instance [21–25]. In that work, Desch et al introduced some computable conditions for hypercyclicity and Devaney chaos. One of them is stated in terms of the abundance of eigenvalues of the infinitesimal generator of the semigroup, see Theorem 2.7. This criterion allowed to extend several known examples whose solutions present a wild behaviour.

In this paper we review some results concerning the linear dynamics of C_0-semigroups of operators. In Section 1 we present several notions that have been considered in the study of linear dynamics of C_0-semigroups. For the clarity of the exposition, the formulations of these notions for single linear operators are omitted. The existence of computable criteria for hypercyclicity and other dynamical properties is considered in Section 2. One of the most well-known criteria in the area is the Desch, Schappacher, and Webb criterion in any of their formulations. We revisit different versions of the criterion, and some examples for which the criteria were applied.

In the same way as the (weighted) shifts are considered as a model for understanding the dynamics of linear operators on sequence spaces [26–28], the translation semigroup works in the same manner for the dynamics of semigroups [20, Sec. 4]. This has been fostered part because of the use of the comparison lemma for extending

the results to other semigroups. The characterization of the dynamics of the translation semigroup is considered in Section 3.

A special case is provided by semigroups generated by partial differential equations in spaces of analytic functions with some growth control, which is analyzed in Section 4. The revision of solution semigroups of partial differential equations defined on a special Banach sequence space, the Herzog space, is provided in Section 5. Finally, Section 6 deals with some open problems for future study on the dynamics of semigroups.

1 Preliminaries on the dynamics of semigroups

In this section, we gather the most significant notions that have been considered in the study of linear dynamics of semigroups in the last years.

1.1 Transitivity and hypercyclicity

First of all, we will recall the basic notions on linear dynamics. The study of hypercyclicity started with the work of Kitai [1, 29]. Beauzamy coined the notion of hypercyclicity for linear operators, see [30, 31]. Further information on the origins of this notion can be found in [5].

This concept is stronger than the ones of supercyclicity or cyclicity, in which one considers the multiples of the elements of the orbit or their linear combinations, respectively, instead of the orbit itself. In this work we will restrict ourselves to hypercyclicity. It consists on the existence of elements with dense orbit. The generalization to semigroups is natural, replacing the discrete orbit of an element by a continuous one.

Definition 1.1. *A semigroup $\{T_t\}_{t\geq 0}$ is said to be* hypercyclic *if there exists some $x \in X$ such that*

$$\mathrm{Orb}(\{T_t\}_{t\geq 0}, x) := \{T_t x \ : \ t \geq 0\} \text{ is dense in } X. \tag{4}$$

From hypercyclicity it directly follows that the space X must be separable and infinite-dimensional. A surprising result is that the orbit of a vector under a semigroup is dense as soon as it is somewhere dense (see [32], based on a result of Bourdon and Feldman [33]). This notion is equivalent to *transitivity*.

Definition 1.2. $\{T_t\}_{t\geq 0}$ *is* transitive *if for every pair of non-empty open sets $U, V \neq \emptyset$ there exists some $t > 0$ such that $T_t(U) \cap V \neq \emptyset$.*

Both notions are equivalent since Birkhoff transitivity theorem [34] also holds for C_0-semigroups, which relies on an application of Baire's Category Theorem [5, Th. 1], or [3, Th. 1.2] in the operator case. Hypercyclicity is a quite unstable property, since by small perturbations of the infinitesimal generator we can destroy it [35].

The separability of X also yields that the hypercyclicity of $\{T_t\}_{t\geq 0}$ is equivalent to the hypercyclicity of a sequence of operators $\{T_{t_n}\}_n$ for some discretization of the semigroup given by the sequence $(t_n)_n$ with $\lim_n t_n = \infty$, see for instance [36, Prop. 2.1] and [8, Page 186]. A detailed study of the equivalences between the dynamics of semigroups and the dynamics of their discretizations can be found in [36]. The particular case of autonomous discretizations, that is the existence of hypercyclic operators in the semigroup, requires a special attention. Oxtoby and Ulam obtained, from a clever use of the Baire's Category Theorem, that there is a G_δ set $A \subset (0, +\infty)$ such that T_{t_0} is hypercyclic for every $t_0 \in A$. Costakis and Sambarino proved that such a set can be shared by all the non-trivial operators of the translation semigroup on the complex plane [37]. Those results were later improved in [38], where it is shown that in a hypercylic semigroup $\{T_t\}_{t\geq 0}$ all the operators T_{t_0}, $t_0 > 0$ are hypercyclic and they share the set of hypercyclic vectors. This also holds for supercyclic semigroups and their non-trivial operators [39].

However, these results cannot be extended to the frame of semigroups whose index set is a sector, or the whole, complex plane. The translation semigroup considered on a suitable L^p-space may result to be hypercylic, or even mixing, but with no single operator satisfying such property [36, Ex. 4.14 and 4.15]. The situation does not improve if we restrict ourselves to holomorphic semigroups [40].

After the aforementioned works of Birkhoff and Rolewicz, the study of the wild dynamics of semigroups was taken on by Lasota, who considered the existence of turbulent orbits in [21]. In fluid dynamics, turbulence is associated with low momentum diffusion, high momentum convection, and rapid variation of pressure and flow velocity in space and time. The model in which he shows his results was given by the solution C_0-semigroup of

$$\begin{aligned} u_t(x,t) &= -xu_x(x,t) + \lambda u(x,t) \\ u(x,0) &= f(x) \text{ for } x \geq 0, \quad f \in \mathcal{C}(\mathbb{R}_0^+) \text{ with } f(0) = 0 \end{aligned} \tag{5}$$

where λ denotes the Reynolds number. On the one hand, if $\lambda < 1$ the solutions converge to the laminar solution $u \equiv 0$. On the other hand, if $\lambda \geq 2$ there are infinitely many turbulent solutions. A solution is *strictly turbulent* if its closure is a compact non-empty set and it does not contain periodic points. Roughly speaking, they are complicated and irregular. This was quite surprising since turbulence seemed to be tied to very strongly nonlinear partial differential equations of higher order.

Following this line, in [41], Lasota also showed an example of a solution semigroup associated to the following Abstract Cauchy Problem that describes the growth of a population of cells which constantly differentiate (change their properties) in time.

$$\begin{aligned} u_t(x,t) &= -c(t,x)u_x(x,t) + f(t,x,u) \text{ for some } (t,x) \in [0,\infty) \times [0,1] \\ u(0,x) &= v(x) \qquad \text{for } x \in [0,1]. \end{aligned}$$

Here t represents the time, x the degree of differentiation which goes from $x = 0$ (undifferentiated cells) to $x = L$ (mature cells), and $c(x,t)$ the velocity of cell differentiation [42–44]. The usual setting for posing these problems are the spaces of continuous or integrable functions on the interval.

The existence of semigroups is not limited to partial differential operators in function spaces. It is known that every separable infinite-dimensional Banach space admits a hypercyclic semigroup [45]. The proof relies on a construction based on a result on biorthogonal sequences on Banach spaces by Ovsepian and Pelczynski [46] and on an analogous result for single operators [47]. An alternative shorter proof was given in [48]. The generalization to Fréchet spaces different from ω was presented in [49]. In this case, the role of the result of Ovsepian and Pelczynski is played by a more general result on the setting of Fréchet spaces [50].

1.2 Devaney chaos

Godefroy and Shapiro introduced the notion of chaos in the sense of Devaney [51] for linear operators [3]. We recall that $x \in X$ is a *periodic point* for a C_0-semigroup $\{T_t\}_{t\geq 0}$ if there is some $t_0 > 0$ such that $T_{t_0}x = x$.

Definition 1.3. *A semigroup $\{T_t\}_{t\geq 0}$ is* chaotic in the sense of Devaney *if it is hypercyclic (transitive), it has a dense set of periodic points, and the sensitive dependence to initial conditions (SDIC) holds.*

In our setting, the last condition can be directly obtained from hypercyclicity. In the particular setting of linear operators the SDIC can be deduced from the mere hypercyclity [52].

In [53], following an ergodic-theoretical approach, Rudnicki showed the existence of invariant measures having strong analytic and mixing properties of the solution semigroup of equation (5). He also showed the existence of Devaney chaos, see also [54–56]. An updated revision of these results can be found in [57, 58].

The dynamics associated to this equation has been also considered in other spaces: in Hölder spaces of continuous functions [59–61] and Orlicz spaces [62], and in Sobolev spaces of type $W^{1,p}$, $1 \leq p < \infty$ [63]. The reader can find more information about the study of the dynamics of semigroups induced by semiflows in [64].

1.3 Mixing & weakly mixing properties

The notion of transitivity can be strengthened in some ways. The weak mixing property was considered in the setting of linear dynamics by Herrero [65].

Definition 1.4. *A semigroup $\{T_t\}_{t\geq 0}$ is said to be weakly mixing if $\{T_t \oplus T_t\}_{t\geq 0}$ is hypercyclic (transitive).*

He also posed there the question whether every hypercyclic operator was weakly mixing. This question was shown to be negative by De la Rosa and Read [66], see also [67]. However, the weakly mixing property was seen to be equivalent to the Hypercyclicity Criterion [68]. This link can be shown in the following result:

Theorem 1.5 ([36, Th. 2.4])**.** *Let $\{T_t\}_{t\geq 0}$ be a semigroup in L(X). The following are equivalent:*
1. *$\{T_t\}_{t\geq 0}$ is weakly mixing.*
2. *There exists a weakly mixing discretization.*
3. *All autonomous discretizations are weakly mixing.*

As a matter of fact, the weak mixing property is obtained whenever we have Devaney chaos [68].

The (topologically) mixing property was first analyzed for single operators in [69], and later studied for semigroups in [70].

Definition 1.6. *$\{T_t\}_{t\geq 0}$ is said to be* mixing *if for every pair of non-empty open sets $U, V \neq \emptyset$ there exists some $t_0 > 0$ such that*

$$T_t(U) \cap V \neq \emptyset \text{ for all } t \geq t_0. \tag{6}$$

This notion is clearly stronger than transitivity and weakly mixing. However, when all the discretizations of a semigroup satisfy either transitive, weakly mixing, or the mixing property, the other properties also hold.

Theorem 1.7 ([36, Th. 2.3])**.** *Let $\{T_t\}_{t\geq 0}$ be a semigroup in $L(X)$. The following are equivalent:*
1. *$\{T_t\}_{t\geq 0}$ is mixing.*
2. *Every discretization of $\{T_t\}_{t\geq 0}$ is mixing.*
3. *Every discretization of $\{T_t\}_{t\geq 0}$ is weakly mixing.*
4. *Every discretization of $\{T_t\}_{t\geq 0}$ is transitive.*

Other recurrent properties for hypercyclic semigroups were considered in [71] (see also [72] for a recent thorough study in the case of a single operator).

1.4 Frequent hypercyclicity

With the goal of studying linear transformations from the point of view of ergodic theory, Bayart and Grivaux introduced the notion of frequent hypercyclicity in order to quantify the frequency with which an orbit meets open sets [73]. This notion was extended to semigroups in [74]. For this purpose we recall the notion of lower density of a set of real numbers: Let $B \subseteq \mathbb{R}_0^+$, we define the lower density of B as $\underline{\text{Dens}}(B) = \liminf_{t\to\infty} \frac{\mu(B\cap[0,t])}{t}$ and the *upper density* as $\overline{\text{Dens}}(B) = \limsup_{t\to\infty} \frac{\mu(B\cap[0,t])}{t}$, where μ denotes the Lebesgue measure on $\mathbb{R}_0^+$.

Definition 1.8. *$\{T_t\}_{t\geq 0}$ is* frequently hypercyclic *if there exits some $x \in X$ such that for every $\emptyset \neq U \subset X$, we have*

$$\underline{\text{Dens}}(\{t > 0 : T_t(x) \cap U\}) > 0. \tag{7}$$

It is well-known that when a semigroup is frequent hypercyclic, all their non-trivial operators share the set of frequently hypercyclic vectors too [38, Th. 3.2]. The set of frequent hypercyclic vectors for an operator is meager [75]. However, if one considers the use of the upper density in the definition of frequent hypercyclicity, the set of upper frequent hypercyclic vectors is residual [75].

1.5 Li-Yorke chaos

In the celebrated paper of Li and Yorke [76], they introduced the concept of scrambled set. In this flavor, this notion was studied in linear dynamics in [77] and [78].

Definition 1.9. $\{T_t\}_{t\geq 0}$ *is said to be* Li-Yorke chaotic *if there exists an uncountable subset* $\Gamma \subset X$*, called the* scrambled set*, such that for every pair* $x, y \in \Gamma$ *of distinct points, we have*

$$\liminf_{t\to\infty} ||T_t x - T_t y|| = 0, \qquad \liminf_{t\to\infty} ||T_t x - T_t y|| > 0. \tag{8}$$

Clearly, every hypercyclic semigroup is Li-Yorke chaotic: we just have to fix a hypercyclic vector $x \in X$ and consider $\Gamma := \{\lambda x; |\lambda| \leq 1\}$ as the scrambled set.

Irregular vectors were introduced by Beauzamy [79], and their existence turned out to be equivalent to Li-Yorke chaos [77]. For a C_0-semigroup $\{T_t\}_{t\geq 0}$, a vector x is said to be *irregular* if $\liminf_{t\to\infty} ||T_t x|| = 0$ and $\liminf_{t\to\infty} ||T_t x|| = \infty$.

1.6 Distributional chaos

The notion of distributional chaos was inspired by the work of Schweizer and Smítal [80]. It was incorporated to the setting of linear dynamics in [81, 82], and thoroughly studied in [83]. The corresponding version for C_0-semigroups was given in [84]. In this notion we require a scrambled set S such that the orbits of any couple of distinct points are closed enough or at least at some distance, measured in terms of upper densities. We say that a semigroup $\{T_t\}_{t\geq 0}$ is *distributionally chaotic* if there is an uncountable set $\Gamma \subseteq X$, $\delta > 0$, so that for each $\varepsilon > 0$ and pair $x, y \in \Gamma$ of distinct points

$$\overline{\text{Dens}}(\{s \geq 0 : ||T_s x - T_s y|| \geq \delta\}) = 1, \quad \overline{\text{Dens}}(\{s \geq 0 : ||T_s x - T_s y|| < \varepsilon\}) = 1. \tag{9}$$

See also [85, 86] where distributionally chaotic semigroups defined on Fréchet spaces are characterized.

Inspired by the concept of irregular vectors, a new notion of distributionally irregular vector was given in [77] for single operators, later generalized for C_0-semigroups:

Definition 1.10 ([84])**.** *A vector* $x \in X$ *is said to be* distributionally irregular *for* $\{T_t\}_{t\geq 0}$ *if for every* $\delta > 0$ *we have:*

$$\overline{\text{Dens}}(\{s \geq 0 : ||T_s x|| > \delta\}) = 1, \quad \& \quad \overline{\text{Dens}}(\{s \geq 0 : ||T_s x|| < \delta\}) = 1. \tag{10}$$

It was shown in [83] that the existence of distributionally irregular vectors was equivalent to distributional chaos for single operators, and later generalized for C_0-semigroups in [84].

1.7 The specification property

The specification property was introduced by Bowen in [87] in order to indicate that there exist periodic points that can approximate prescribed parts of the orbits of certain elements. It is a very strong property, and it was stated for compact metric spaces. In the context of linear dynamics, the corresponding concept was given in [88], and thoroughly studied in [89] for single operators. A recent adaptation to C_0-semigroups was provided:

Definition 1.11 ([90])**.** *A* C_0*-semigroup* $\{T_t\}_{t\geq 0}$ *on a separable Banach space* X *has the* Semigroup Specification Property *(SgSP) if there exists an increasing sequence* $(K_n)_n$ *of* T*-invariant sets with* $0 \in K_1$ *and* $\overline{\cup_{n\in\mathbb{N}} K_n} = X$ *and there exists a* $t_0 > 0$*, such that for each* $n \in \mathbb{N}$ *and for any* $\delta > 0$ *there is a positive real number* $M_{\delta,n} \in \mathbb{R}^+$ *such that for any integer* $s \geq 2$*, any set* $\{y_1, \ldots, y_s\} \subset K_n$ *and any real numbers:* $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ *satisfying* $b_s + M_{\delta,n} \in \mathbb{N} \cdot t_0$ *and* $a_{r+1} - b_r \geq M_{\delta,n}$ *for* $r = 1, \ldots, s-1$*, there is a point* $x \in K_n$ *such that, for each* $t_r \in [a_r, b_r]$*,* $r = 1, 2, ..., s$*, the following conditions hold:*

$$||T_{t_r}(x) - T_{t_r}(y_r)|| < \delta,$$

$$T_t(x) = x, \quad \text{where } t = M_{\delta,n} + b_s.$$

The SgSP implies the mixing property, Devaney chaos, frequent hypercyclicity, and distributional chaos. Moreover, it holds for the semigroup if, and only if, it holds for some of its nontrivial operators [90].

2 A review of dynamical criteria

The existence of computable criteria for hypercyclicity and for stronger notions allowed a wide range of examples with wild behaviour. In the operator case, the first result in this line was the Hypercyclicity Criterion, that was first stated by Kitai [1], and later rediscovered by Gethner and Shapiro [2]. The following version is inspired in a reformulation by Bès and the fourth author [68].

Criterion 2.1 (Hypercyclicity Criterion (HC)). *Let $\{T_t\}_{t\geq 0}$ be a semigroup on X. Let us also consider $Y, Z \subseteq X$ dense subsets of X, an increasing sequence of real positive numbers $(t_k)_k$ tending to ∞, and a sequence of mappings $S_{t_k} : Z \to X$, $k \in \mathbb{N}$ such that*
(a) $\lim_{k\to\infty} T_{t_k} y = 0$ *for all* $y \in Y$,
(b) $\lim_{k\to\infty} S_{t_k} z = 0$ *for all* $z \in Z$, *and*
(c) $\lim_{k\to\infty} T_{t_k} S_{t_k} z = z$ *for all* $z \in Z$.
Then, the semigroup $\{T_t\}_{t\geq 0}$ is hypercyclic.

There are alternative formulations of this criterion that result to be equivalent:

Theorem 2.2 ([91]). *Let $\{T_t\}_{t\geq 0}$ be a semigroup. The following formulations are equivalent to the Hypercyclicity Criterion:*

1. *There exist $(t_k)_k \subset \mathbb{R}_+$ strictly increasing and tending to infinity, dense subspaces $Y, Z \subset X$, satisfying*
   (i) *For all $y \in Y$, $\lim_{k\to\infty} T_{t_k} y = 0$.*
   (ii) *Every $z \in Z$ admits a backward orbit $\{z_t : t \geq 0\}$ such that $\lim_k z_{t_k} = 0$.*
2. *There exist $(t_k)_k \subset \mathbb{R}_+$ strictly increasing and tending to infinity, dense subspaces $Y, Z \subset X$, satisfying*
   (i) *For all $y \in Y$, $\lim_{k\to\infty} T_{t_k} y = 0$.*
   (ii) *For all $z \in Z$ there is $(z_k)_k$ in X with $\lim_{k\to\infty} z_k = 0$ and $\lim_{k\to\infty} T_{t_k} z_k = z$.*

In the same way as for linear operators, $\{T_t\}_{t\geq 0}$ is weakly mixing if, and only if, there exists a discretization $\{T_{t_k}\}_k$ satisfying the HC [36, 91], which is also equivalent to the existence of a single weakly mixing operator T_{t_0}. Semigroups satisfying the HC were also characterized by El Mourchid [92].

The HC can be strengthened in order to get the mixing property [70].

Criterion 2.3 (Mixing Criterion (MC)). *Let $\{T_t\}_{t\geq 0}$ be a semigroup on X. Let us also consider $Y, Z \subseteq X$ dense subsets of X and a family of mappings $S_t : Z \to X$ such that*
(a) $\lim_{t\to\infty} T_t y = 0$ *for all* $y \in Y$,
(b) $\lim_{t\to\infty} S_t z = 0$ *for all* $z \in Z$, *and*
(c) $\lim_{t\to\infty} T_t S_t z = z$ *for all* $z \in Z$.
Then, the semigroup $\{T_t\}_{t\geq 0}$ is mixing.

Theorem 2.4 ([36, Th. 2.4]). *Let $\{T_t\}_{t\geq 0}$ be a semigroup in $L(X)$. The following are equivalent:*

1. *$\{T_t\}_{t\geq 0}$ is weakly mixing.*
2. *There exists a weakly mixing discretization.*
3. *All autonomous discretizations are weakly mixing,*
4. *There exists a discretization verifying the HC for sequences of operators.*
5. *All autonomous discretizations verify the HC for sequences of operators.*

6. *For every increasing sequence* $\{t_k\}_k$ *tending to* ∞ *with* $\sup_{k\in\mathbb{N}} |t_{k+1} - t_k| < \infty$, *the discretization* $\{T_{t_k}\}_k$ *is hypercyclic.*

Condition 6 above is expressed in terms of syndetic sequences in $\mathbb{R}^+$, and was inspired by the corresponding result for single operators [93].

In [94], Mangino and Peris obtained a sufficient condition for frequent hypercyclicity. This frequent hypercyclicity criterion is based on the Pettis integral.

Criterion 2.5 (Frequently Hypercyclicity Criterion)**.** *Let* $\{T_t\}_{t\geq 0}$ *be a semigroup in* $L(X)$. *If there exist* $X_0 \subset X$ *dense in* X *and maps* $S_t : X_0 \to X_0$, $t > 0$, *such that*

(i) $T_t S_t x = x, T_t S_r x = S_{r-t} x, t > 0, r > t > 0$ *for all* $x \in X_0$,

(ii) $t \to T_t x$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

(iii) $t \to S_t x$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

then $\{T_t\}_{t\geq 0}$ *is frequently hypercyclic.*

Moreover, in [95] it was shown that this criterion suffices for the existence of invariant Borel probability measures on X that are strongly mixing and have full support. In the same work, the authors analyzed the existence of these measures for the solution semigroup of the Black-Scholes equation completing the previous work performed in [20] where the authors studied its chaotic behaviour.

2.1 The role of the infinitesimal generator

Godefroy and Shapiro reformulated the HC in terms of the abundance of vectors of an operator [3], also known as the Eigenvalue Criterion, see also [96, Th. 3.7] and [97]. For hypercyclicity, it was necessary to have "plenty of" eigenvalues of modulus greater than 1 and smaller than 1 (and for Devaney chaos to have additionally many of them with modulus equal to 1).

For C_0-semigroups, especially for those associated to solutions of linear PDEs and infinite systems of linear ODEs, it turns out to be necessary to have at our disposal dynamical criteria that can be expressed in terms of the infinitesimal generator. In this subsection X will be a complex Banach space.

Criterion 2.6. *Let* $\{T_t\}_{t\geq 0}$ *be a semigroup whose infinitesimal generator is* A. *Suppose that the subspaces*

$$X_0 := \text{span}\{x \in X\,;\, Ax = \lambda x \text{ for some } \lambda \in \mathbb{K} \text{ with } \Re(\lambda) < 1\},$$
$$Y_0 := \text{span}\{x \in X\,;\, Ax = \lambda x \text{ for some } \lambda \in \mathbb{K} \text{ with } \Re(\lambda) > 1\},$$

are dense in X. *Then* $\{T_t\}_{t\geq 0}$ *is mixing (in particular hypercyclic).*

If, moreover, X *is a complex space and also the subspace*

$$Z_0 := \text{span}\{x \in X\,;\, Ax = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \Re(\lambda) = 0 \quad \text{and} \quad Im(\lambda) \in 2\pi i\mathbb{Q}\}.$$

is dense in X, *then* $\{T_t\}_{t\geq 0}$ *is Devaney chaotic.*

The abundance of eigenvectors of the infinitesimal generator can be obtained by applying the Hahn-Banach theorem.

Criterion 2.7 (Desch-Schappacher-Webb Criterion [20])**.** *Let* X *be a complex Banach space, and let* $(A, D(A))$ *be the generator of the semigroup* $\{T_t\}_{t\geq 0}$. *Assume that there exist an open connected subset* $U \subset \mathbb{C}$ *and a weakly holomorphic function* $f : U \to X$ *such that*

(i) $U \cap i\mathbb{R} \neq \emptyset$.

(ii) $f(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U$,

(iii) *If for some* $\varphi \in X^*$ *the function* $\langle f(\cdot), \varphi\rangle$ *is identically zero on* U, *then* $\varphi = 0$.

Then the semigroup $\{T_t\}_{t\geq 0}$ *is Devaney chaotic and mixing.*

Kalmes showed that all the nontrivial operators of such semigroups are in fact Devaney chaotic [98, Th. 2.1]. This criterion has presented several different formulations along the time, in order to relax the hypothesis, as far as possible [8, 94, 99–102]. Banasiak and Moszynski [103] reformulated it in order to find subspaces where hypercyclicity and chaos hold (sub-hypercyclicity and sub-chaoticity). A fractional version of criterion 2.7 for semigroups was provided in [104]. As an illustration of its usefulness, we recall the following result. The Desch-Schappacher-Webb Criterion is very strong since it also implies the Semigroup Specification Property [90], which is the strongest dynamical property mentioned in this paper.

Example 2.8 ([20])**.** *Consider the following PDE in* $L^2(\mathbb{R}^+, \mathbb{C})$*:*

$$\begin{aligned} u_t(x,t) &= au_{x,x}(x,t) + bu_x(x,t) + cu(x,t), \\ u(0,t) &= 0 \ \textit{for}\ t \geq 0, \\ u(x,0) &= f(x) \ \textit{for}\ x \geq 0, \ \textit{with some}\ f \in X. \end{aligned}$$

The solution semigroup is generated by the operator $Au := au_{x,x}(x,t) + bu_x(x,t) + cu(x,t)$*. When* $a, b, c > 0$ *and* $c < b^2/2a < 1$ *then the solution semigroup generated by A is Devaney chaotic.*

The next example is given by a perturbation of an Ornstein-Uhlenbeck operator [100]. A slight modification in the formulation of condition (ii) and (iii) is needed.

Example 2.9. *Consider the following PDE in* $L^2(\mathbb{R}^+, \mathbb{C})$*:*

$$\begin{aligned} u_t(x,t) &= u_{x,x}(x,t) + bxu_x(x,t) + cu(x,t), \\ u(0,t) &= 0 \ \textit{for}\ t \geq 0, \\ u(x,0) &= f(x) \ \textit{for}\ x \geq 0, \ \textit{with some}\ f \in X. \end{aligned}$$

The solution semigroup generated by the operator $Au := u_{x,x}(x,t) + bxu_x(x,t) + cu(x,t)$*, with* $b > 0$ *with* $c > b/2$ *is Devaney chaotic.*

Concerning distributional chaos, some criteria for C_0-semigroups were provided in [84], inspired by the ones given for single operators in [83]. We recall the following useful one:

Theorem 2.10 ([84])**.** *If there exist a dense subset* $X_0 \subset X$ *with* $\lim_{t\to\infty} T_t x = 0$ *for each* $x \in X_0$*, and a measurable subset* $B \subset \mathbb{R}_0^+$ *with* $\underline{\text{Dens}}(B) = 1$*. If either*

1. $\int_B \frac{1}{\|T_t\|} < \infty$ *or*
2. $\int_B \frac{1}{\|T_t\|^2} < \infty$ *when X is a complex Hilbert space,*

then $\{T_t\}_{t\geq 0}$ *has a dense manifold whose nonzero vectors are distributionally irregular vectors (i.e., a dense distributionally irregular manifold).*

This theorem can be rephrased in terms of the infinitesimal generator of the semigroup, in order to try to use directly some of its spectral properties, see [77, Cor. 31] for the discrete case and [105, Th. 3.7] for the continuous version.

Criterion 2.11. *Let* $\{T_t\}_{t\geq 0}$ *be a semigroup with infinitesimal generator A. If the following conditions hold:*

1. *there is a dense subset* $X_0 \subseteq X$ *with* $\lim_{t\to\infty} T_t x = 0$*, for each* $x \in X_0$*, and*
2. *there is some* $\lambda \in \sigma_p(A)$ *with* $\Re(\lambda) > 0$*,*

then $\{T_t\}_{t\geq 0}$ *has a dense distributionally irregular manifold. In particular,* $\{T_t\}_{t\geq 0}$ *is distributionally chaotic.*

Thus, whenever we obtain Devaney chaos due to an application of Desch-Schappacher-Webb criterion we, in fact, get the existence of a dense distributionally irregular manifold. For facilitating the lecture, we will omit this part in the sequel.

3 Dynamics of the translation semigroup and applications

The translation semigroup $\{\tau_t\}_{t\geq 0}$ has become one of the most clear test case to study and analyze different dynamical properties. Birkhoff proved the transtivity of the operator τ_1 on $\mathcal{H}(\mathbb{C})$, the space of entire functions endowed with the compact open topology [18], and this yields the transitivity of the whole translation semigroup $\{\tau_t\}_{t\in\mathbb{C}}$ on $\mathcal{H}(\mathbb{C})$. A revision of this result can be also found in [106]. Chan and Shapiro also considered the dynamics of the translation semigroup on Hilbert spaces of entire functions [107] (see also [108] for an analogous result on translations of harmonic functions).

Other important settings where the translation semigroup is considered are the weighted spaces of continuous and integrable functions vanishing at ∞. In this way, we define the following function spaces

$$L^p_\rho(I) := \left\{ f : I \to \mathbb{R} \ : f \text{ measurable with } \int_I |f(t)|^p \rho(t) dt \right\} \tag{11}$$

endowed with the norm $||f||_{p,\rho} := \left(\int_I |f(t)|^p \rho(t) dt\right)^{1/p}$ and

$$C_{0,\rho}(I) := \left\{ f : I \to \mathbb{R} \ : f \text{ continuous, with } \lim_{t\to\infty} f(t)\rho(t) = 0 \right\} \tag{12}$$

endowed with the norm $||f||_{p,\infty} := \sup_{t\in I} |f(t)|\rho(t)$ where $I = \mathbb{R}$ or $I = \mathbb{R}_0^+$.

The weight function $\rho : I \to \mathbb{R}^+$ is said to be an *admissible weight function* if the following property holds: There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(s) \leq Me^{wt}\rho(t+s)$ for all $s \in I$ and $t > 0$. These conditions yield that the translation semigroup defined as $\tau_t f(x) = f(x+t)$, for $t \geq 0, x \in \mathbb{R}, f \in X$, where X is any of the spaces above, is strongly continuous [8].

3.1 The translation semigroup of L^p_ρ, $1 \leq p < \infty$

Rolewicz analyzed the dynamics of the translation semigroup in the setting of weighted spaces of p-integrable functions $L^p_\rho(\mathbb{R})$, $1 \leq p < \infty$, with $\rho(s) := a^{-s}, a > 1$ [19]. As we will see, the linear dynamics of the translation semigroup has permitted to express the dynamics in terms of the behaviour of the weight function ρ.

Desch et al. showed that the translation semigroup $\{\tau_t\}_{t\geq 0}$ is hypercyclic in $L^p_\rho(\mathbb{R})$ if, and only if,

$$\liminf_{t\to\pm\infty} \rho(t) = 0. \tag{13}$$

Devaney chaos can also be characterized, but in this case the condition is stated in terms of the integrability of ρ. In [109] it is proved that $\{\tau_t\}_{t\geq 0}$ is Devaney chaotic on $L^p_\rho(\mathbb{R})$ if, and only if,

$$\int_{-\infty}^{+\infty} \rho(t) < \infty. \tag{14}$$

Moreover, this condition results to be equivalent to the existence of a single periodic point.

The mixing property for the translation semigroup on L^p_ρ is, somehow, in the middle of hypercyclicity and Devaney chaos. This can be seen in the corresponding characterization provided by [70]. $\{\tau_t\}_{t\geq 0}$ is mixing on $L^p_\rho(\mathbb{R})$ if, and only if

$$\lim_{t\to\pm\infty} \rho(t) = 0. \tag{15}$$

The translation semigroup is frequently hypercyclic on $L^p_\rho(\mathbb{R})$ if and only if $\int_{-\infty}^{+\infty} \rho(t)dt < \infty$, as it was completely characterized in [94], see also [110] for a generalization of this result.

Barrachina and Peris [111] showed that $\{\tau_t\}_{t\geq 0}$ is (densely) distributionally chaotic on $L^p_\rho(\mathbb{R}_0^+)$ if we can find a measurable subset $A \subseteq \mathbb{R}_0^+$ such that $\overline{\text{Dens}}(A) = 1$ and $\int_A \rho(s)ds < \infty$. Sufficient conditions for Li-Yorke chaos were provided in [112]. The interrelations between Devaney chaos and distributional chaos for this semigroup were

studied in [105]: There are examples of a hypercyclic and distributionally chaotic translations that are not Devaney chaotic. This fact can be compared with the one provided in [111, Ex. 2] of a distributionally chaotic translation semigroup that is neither hypercyclic nor Devaney chaotic. In [90] we get that the translation semigroup has the Semigroup Specification Property on $L^p_\rho(\mathbb{R})$ if, and only if, $\int_{-\infty}^{+\infty} \rho(s)ds < \infty$, which is in fact the condition for Devaney chaos.

3.2 The water hammer equations

A direct application of some of the aforementioned results can be seen when studying the solution of the water hammer equations [113]. A water hammer is a pressure wave that occurs, accidentally or intentionally, in a filled liquid pipeline when a tap is suddenly closed, or a pump starts or stops, or when a valve closes or opens. A water hammer wave propagates through pipes reflecting on features and boundaries. This phenomenon is governed by a pair of coupled quasi-linear partial differential equations of first order, the dynamic and continuity equations. A detailed derivation of the governing water hammer equations can be found in Chaudry [114]. Here we provide a representation of the classical solution of the linear water hammer equations with the help of the translation semigroup [113, Cor. 4.2].

$$Q_t + g\mathcal{A}H_x + \frac{f}{2\mathcal{D}\mathcal{A}}Q|Q| = 0, \text{ (Dynamic equation)} \tag{16}$$

$$\frac{v^2}{g\mathcal{A}}Q_x + H_t = 0, \text{ (Continuity equation)} \tag{17}$$

where $Q(x,t)$ represents the *discharge*, $H(x,t)$ denotes the *piezometric head* at the centerline of the conduit above the specified datum, f is the *friction factor* (which is assumed to be constant), g is the acceleration due to *gravity*, v is the *fluid wave velocity*, and $\mathcal{A}$ and $\mathcal{D}$ are the the *cross-sectional area* and the *diameter of a conduit*, respectively. The parameters $\mathcal{A}$ and $\mathcal{D}$, are characteristics of the conduit system and are time invariant, but may be functions of x.

This strategy is similar to the case in which the dynamics of the cosine family is analyzed [115, 116] to describe the orbit in terms of a forward and backward orbit. Here, we present how the discharge and the piezometric head evolve from a given initial condition $Q(x,0) = \varphi(x)$ and $H(x,0) = \phi(x)$.

$$\begin{aligned} Q(x,t) &= \frac{1}{2}\varphi(x-vt) + \frac{g\mathcal{A}}{2v}\phi(x-vt) + \frac{1}{2}\varphi(x+vt) + \frac{g\mathcal{A}}{2v}\varphi(x+vt) \\ H(x,t) &= \frac{v}{2g\mathcal{A}}\varphi(x-vt) + \frac{1}{2}\phi(x-vt) - \frac{v}{2g\mathcal{A}}\varphi(x+vt) + \frac{1}{2}\phi(x+vt) \end{aligned}$$

for every $x \in \mathbb{R}, t \geq 0$ and initial conditions $(Q(0), H(0)) = (\phi, \varphi) \in X \times X$.

Kalmes studied the semigroup induced by semiflows in [117]. He characterized hypercyclicity and mixing property for cosine operator functions generated by second order partial differential operators on space of integrable functions and continuous functions. Similar results for the case of cosine operator functions generated from shifts have been given by Chang and Chen in [118]. In addition, Chen also considered chaos in the sense of Devaney [116], giving a characterization of chaotic cosine operator functions generated by weighted translations on the Lebesgue space $L^p(G)$, where G is a locally compact group.

3.3 The quasi-linear Lasota equation

Hung has recently studied the hypercyclicity and Devaney chaos for the quasi-linear Lasota equation by reducing their dynamics to the one of the translation semigroup [119]. He considered the equation

$$\begin{aligned} u_t(x,t) &= k(x)u - c(x)u_x(x,t) \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq 1, \quad f \in \mathcal{C}([0,1]) \text{ with } f(0) = 0 \end{aligned}$$

where $k(x)$ is a bounded function on the interval and $c(x)$ satisfies

$$c(0) = 0, c(x) > 0, \quad \text{for} \quad x \in (0,1], \text{ and } \int_0^1 \frac{dx}{c(x)} = \infty. \tag{18}$$

The particular case $c(x) = -1$ and $k(x) = 0$ yields the translation semigroup.

3.4 The conjugation lemma

The dynamics of the translation semigroup has many applications, since the study of the behaviour of the solution semigroups associated to many partial differential equations and infinite systems of linear ordinary differential equations, can be reduced to the analysis of the translation semigroup in certain spaces. This is due to an application of the conjugation lemma, which can be formulated as follows: Given $\{S_t\}_{t\geq 0}$ a C_0-semigroup on a Banach space Y and $\{T_t\}_{t\geq 0}$ on a Banach space X. Then $\{T_t\}_{t\geq 0}$ is called *conjugate* to $\{S_t\}_{t\geq 0}$ if there exists a homeomorphism $\Phi : Y \to X$ such that $T_t \circ \Phi = \Phi \circ S_t$ for all $t \geq 0$. Dynamical properties such as frequent hypercyclicity, hypercyclicity, mixing, weak mixing and Devaney chaos for a C_0-semigroup are preserved under conjugacy.

Using the conjugation lemma we are going to revisit the dynamics of solution semigroups to some Lasota type equation. Further information on these examples can be found in [54, 105].

Example 3.1. *Let us consider*

$$\begin{aligned} u_t(x,t) &= k(x)u - c(x)u_x(x,t) \\ u(x,0) &= f(x) \ \textit{for } x \in \mathbb{R}_0^+, \quad f \in \mathcal{C}(\mathbb{R}_0^+) \end{aligned}$$

with $k(x), c(x)$ bounded and continuous functions on $\mathbb{R}_0^+$. If $c(x) = 1$, the solution semigroup $\{T_t\}_{t\geq 0}$ is given by

$$T_t f(x) = \exp\left(\int_x^{x+t} h(s)ds\right) f(x+t), \textit{ for } f \in X. \tag{19}$$

Its dynamics can be reduced to the one of the translation semigroup on certain L^p spaces. If we define $\rho(x) = \exp(-\int_0^x k(s)ds)$ and $\phi(f)(x) = (\rho(x))^{1/p} f(x)$, we have:

$$\begin{array}{ccc} L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{\tau_t} & L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) \\ \phi \downarrow & & \downarrow \phi \\ L^p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{T_t} & L^p(\mathbb{R}_0^+, \mathbb{C}) \end{array} \tag{20}$$

Clearly, if $k(x)$ is constant and equal to 1 then Devaney and distributional chaos hold. However, if $k(x) = -1$ for $x \in [n^2, n^2+1[$ for some $n \in \mathbb{N}$, and $k(x) = 1$ elsewhere, we have distributional but not Devaney chaos.

Example 3.2. *Let now consider*

$$\begin{aligned} u_t(x,t) &= k(x)u - c(x)u_x(x,t) \\ u(x,0) &= f(x) \ \textit{for } x \in \mathbb{R}_0^+, \quad f \in \mathcal{C}(\mathbb{R}_0^+). \end{aligned}$$

Take $c(x) = \gamma x$, $\gamma < 0$ and k continuous. If there is $\delta > 0$ such that $\Re(k(x)) \geq 0$ for $0 \leq x \leq \delta$, then the solution semigroup $\{T_t\}_{t\geq 0}$ is given by

$$T_t f(x) = \exp\left(\int_0^t k(e^{\gamma(t-r)}x) f(e^{\gamma t}x)\right) \textit{ for } t \geq 0. \tag{21}$$

Define $\rho(x) = \exp\{(1/\gamma)\int_x^1 (h(s)/s)ds\}$ *and set* $\phi(f)(x) = (\rho(x))^{1/p} f(x)$*:*

$$\begin{array}{ccc} L^p_\rho([0,1],\mathbb{C}) & \xrightarrow{S_t} & L^p_\rho([0,1],\mathbb{C}) \\ \phi\downarrow & & \downarrow\phi \\ L^p([0,1],\mathbb{C}) & \xrightarrow{T_t} & L^p([0,1],\mathbb{C}) \end{array} \tag{22}$$

then we have distributional chaos.

If we consider $c(x) = 1$, $k(x) = \dfrac{\alpha x}{1+x^\alpha}$*, then the solution is given by*

$$T_t f(x) = \frac{1+(x+t)^\alpha}{1+x^\alpha} f(x+t) \text{ for } x, \geq 0, t \geq 0. \tag{23}$$

If we define $\rho(x) = \dfrac{1}{1+x^\alpha}$ *and set* $\phi(f)(x) = (\rho(x))^{1/p} f(x)$*, we have:*

$$\begin{array}{ccc} L^p_\rho(\mathbb{R}^+_0,\mathbb{C}) & \xrightarrow{\tau_t} & L^p_\rho(\mathbb{R}^+_0,\mathbb{C}) \\ \phi\downarrow & & \downarrow\phi \\ L^p(\mathbb{R}^+_0,\mathbb{C}) & \xrightarrow{T_t} & L^p(\mathbb{R}^+_0,\mathbb{C}) \end{array} \tag{24}$$

This semigroup was known to be Devaney chaotic by El Mourchid [101], since $\int_0^\infty \rho(x)dx < \infty$*.*

4 Chaos on cell growth models

As mentioned before, Lasota considered in [41] a model that describes the growth of a population of cells which constantly differentiate (change their properties) in time. This type of process has been later studied either by single partial differential equations that described the evolution within a structured population, or by coupled infinite system of ordinary differential equations that represent "birth-and-death" process. These models are inspired in kinetic theory, and they have been also used to describe the chaotic behaviour of car-following models in traffic [120–122].

4.1 Cell structured models

Howard [123] studied the linear dynamics of the solution semigroup of a size structured model for describing cell growth that can be found in [124, 125].

$$\begin{aligned} u_t(x,t) + (xu(x,t))_x &= \nu u(x,t) - \mu(x)u(x,t) - \beta(x)u(x,t) + 4\beta(2x)u(2x,t), \text{ for } 0 \leq x \leq 1, t \geq 0. \\ u(x,0) &= \varphi(x), \quad 0 \leq x \leq 1. \end{aligned} \tag{25}$$

In particular, the idea of cells of size zero is considered for describing an abnormality in the division process within our population resulting in a accumulation of cells of various size including a population of non-functional "dwarf" cells. The presence of such "dwarf" cells is seen in the blood disorder Alpha-thalassemia, a genetic disease associated with sickle cell anemia [126].

If $\eta(x) := \nu(x) - \mu(x) - \beta(x) - 1$ and there exists some $\eta_0 \in \mathbb{R}$ such that $-1 < \eta_0 \leq \eta(x)$ for all $0 \leq x \leq 1$, then the solution semigroup of (25) is hypercyclic [123, Prop. 5.3]. This model was also considered by El Mourchid et al. [127, 128].

4.2 Birth-and-death models

Protopopescu and Azmy introduced kinetic models that describe "death" process in the study of linear dynamics [23].

$$(f_n)_t = -\alpha f_n + \beta f_{n+1}, \quad n \in \mathbb{N}_0. \tag{26}$$

Such a model can be used for describing chemical reactions, biological process, or a generalized automaton rule. From the biological point of view, this represents a population of neoplastic cells divided into subpopulations characterized by different levels of cellular resistance to antineoplastic drugs [129]. Here, particles at level n are absorbed at rate $\alpha > 0$ and particles of level $n+1$ are re-emited at rate $\beta > 0$ to level n. Particles with internal energy $n = 0$ are absorbed and cease to exist. The solution semigroup generated by the associated linear operator is Devaney chaotic provided that $\beta > \alpha \geq 0$ [23] and [130, Ex. 2.1].

The study of the dynamics of the general case in which the rates α and β are different for every stage was started in [131] and continued in [132, 133]:

$$(f_n)_t = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \in \mathbb{N}_0 \tag{27}$$

The associated Abstract Cauchy Problem can be restated as

$$\mathbf{f}_t(t) = A\mathbf{f}(t), \mathbf{f}(0) = A\mathbf{f}_0 \tag{28}$$

with

$$A = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ 0 & \alpha_2 & \beta_3 & & \\ & 0 & \alpha_3 & \beta_4 & \\ & & 0 & \alpha_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{29}$$

The natural setting for considering this problem is the space of summable sequences ℓ_1. The most general result that ensures its chaotic dynamics is the following [8, Ch. 7].

Theorem 4.1. *Let $\alpha_n > 0, \beta_n \in \mathbb{R}$ for every $n \in \mathbb{N}$, such that*

$$\alpha := \sup_k \alpha_k < \beta := \liminf_k \beta_k \tag{30}$$

then the solution semigroup generated by A is Devaney chaotic.

In a similar way, the general "birth-and-death" process can be described as:

$$\begin{aligned} (f_1)_t &= \alpha_n f_1 + \beta_n f_2, \\ (f_n)_t &= \gamma_n f_{n+1} + \alpha_n f_n + \beta_n f_{n+1}, \quad n \geq 2. \end{aligned} \tag{31}$$

In this case, the Abstract Cauchy Problem is also stated as (28) but with

$$B = \begin{pmatrix} \alpha & \beta & & & \\ \gamma & \alpha & \beta & & \\ & \gamma & \alpha & \beta & \\ & & \gamma & \alpha & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{32}$$

The case in which we have constant coefficients, $\alpha_n := \alpha_0, \beta_n := \beta_0$ and $\gamma_n := \gamma_0$, $n \in \mathbb{N}_0$, was considered in [129] and by Banasiak and Moszynski [133] in the more general form.

Theorem 4.2. *If $0 < |\gamma| < |\beta|$ and $|\alpha| < |\beta + \gamma|$, then the semigroup generated by B is Devaney chaotic.*

The general case of the "birth-and-death" with variable coefficients was studied in [134], obtaining conditions for sub-chaos. The generation of these semigroups was obtained in [135]. More information on kinetic models can be found in [136].

5 Dynamics of semigroups on Herzog spaces

In [137] Herzog introduced the following spaces when studying the universality of the solutions of the classical Fourier heat equation:

$$u_t(x,t) = u_{x,x}(x,t). \tag{33}$$

In order to have well-defined orbits, he considered spaces of analytic functions regulated by a parameter, or a tuner, that allows to control their growth at infinity. The dynamics of the heat semigroup was also considered in L^p-symmetric spaces of non-compact type [138] and on Damek-Ricci spaces [139].

Searching for universal elements for the solution family of operators

$$(T_t\varphi)(x)\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-s)^2}{4t}\right)\varphi(s)ds \quad t>0, \tag{34}$$

Herzog defined the following spaces of analytic functions. For every $\rho > 0$ he considered

$$X_\rho = \left\{ f:\mathbb{R}\to\mathbb{C};\, f(x)=\sum_{n=0}^{\infty}\frac{a_n\rho^n}{n!}x^n, (a_n)_{n\geq 0}\in c_0(\mathbb{N}_0)\right\} \tag{35}$$

endowed with the norm $||f|| = \sup_{n\geq 0}|a_n|$. This space is isometrically isomorphic to $c_0(\mathbb{N}_0) := \{a_n : \mathbb{N}_0 \to \mathbb{C} : \lim_{n\to\infty}|a_n| = 0\}$.

In this way the operators $T_t : X_\rho \to C(\mathbb{R})$, where $C(\mathbb{R})$ is the space of continuous functions endowed with the compact open topolgy, are well defined. Herzog achieved to show that there is a huge amount of universal functions for $\{T_t\}_{t\geq 0}$.

Theorem 5.1 ([137, Th. 1.1])**.** *The set of universal functions of X_ρ that are universal for $\{T_t\}_{t\geq 0}$ is residual.*

As a generalization, deLaubenfels, Emamirad and Grosse-Erdmann studied the dynamics of semigroups of C-regularized semigroups of unbounded operators [140].

5.1 Solution semigroups associated to second order PDEs

The classical Fourier heat equation is not the right governing equation to model heat transfer processes in which extremely short periods of time or extreme temperature gradients are involved. In these situations, in order to obtain the adequate temperature profiles, we need to use the hyperbolic heat transfer equation (HHTE).

$$\begin{aligned} \tau u_{tt}(x,t) + u_t(x,t) &= \alpha u_{xx}(x,t) \\ u(x,0) &= \varphi_1(x) \\ u_t(x,0) &= \varphi_2(x). \end{aligned} \tag{36}$$

This last equation predicts a finite speed of heat conduction and assumes a wavy character of the heat transfer, contrary to the FHTE.

This equation can be reduced to a first order differential equation as follows,

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & -\frac{1}{\tau}I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x\in\mathbb{R}. \end{cases} \tag{37}$$

which permits to express its solution in terms of semigroups.

Theorem 5.2 ([8, 141]). *If $\rho > 0$ and $\alpha\tau\rho^2 > 2$, then the solution semigroup $\{e^{tA}\}_{t\geq 0}$ is Devaney chaotic on $X_\rho \oplus X_\rho$.*

In the case of the wave equation,

$$u_{tt}(x,t) = \alpha u_{xx}(x,t) \text{ with } \alpha > 0, \tag{38}$$

the result is even more compelling. The solution semigroup $\{e^{tA}\}_{t\geq 0}$ is Devaney chaotic on $X_\rho \oplus X_\rho$ for every $\alpha, \rho > 0$ [141, Th. 2.3].

For the general case of second order partial differential equations of the form

$$u_{tt}(x,t) + \gamma u_t(x,t) + \theta u(x,t) = \alpha u_{xx}(x,t) \quad \text{with } t \geq 0, x \in \mathbb{R} \tag{39}$$

we have recenlty obtained the following results in [142]:

Theorem 5.3. *Let $\gamma, \theta, \alpha > 0$. Suppose that $\gamma^2 = 4\theta$. Then the solution semigroup of the abstract Cauchy problem given by* (39) *is Devaney chaotic.*

The results can be also extended for the non-homogenous version of equation (33), that is also known as the bioheat equation, which corresponds to the HHTE but with an additional source term [143].

5.2 Semigroups associated to PDEs of order greater than or equal to 2

Following a similar approach as described in the previous section, one can analyze the chaotic behavior of different linear partial differential equations that describe physical phenomena that include an underlying propagation process. We gather here some of these examples.

5.2.1 Moore-Gibson-Thompson equation

The Kuznetsov's equation had been considered by many authors as the "classical" acoustics equation. This equation for the velocity potential ψ is:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}(\psi_t)^2 + |\nabla\psi|^2\right)_t, \tag{40}$$

where c is the sound speed, δ is the diffusivity of the sound and B/A is the parameter of nonlinearity.

If the heat flux is described by the classical Fourier transfer heat equation, the energy propagation has infinite speed. To avoid this paradox, other equations were considered to model the heat transfer in order to obtain a nonlinear acoustics wave equation. The Maxwell-Cattaneo equation combined with fluid physics equations leads to a third order in time partial differential equation model, known as the Jordan-Moore-Gibson-Thompson equation:

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}(\psi_t)^2 + |\nabla\psi|^2\right)_t, \tag{41}$$

where $b = \delta + \tau c^2$. We consider the one-dimensional version of (41), which is usually referred to as the Moore-Gibson-Thompson equation:

$$\tau u_{ttt}(x,t) + \alpha u_{tt}(x,t) - c^2 u_{xx}(x,t) - b u_{xxt}(x,t) = 0, \quad \text{for every } t \geq 0, x \in \mathbb{R}. \tag{42}$$

with the initial conditions given by $u(0,x) = \varphi_1(x), u_t(0,x) = \varphi_2(x), u_{tt}(0,x) = \varphi_3(x),\ x \in \mathbb{R}$ and where τ, α, c^2 and b are positive constants. Its solutions exhibit very different qualitative behavior from the familiar second order complete equation ($\tau = 0, \alpha > 0$). For third order in time equations, the critical parameter

$$\gamma \equiv \alpha - \frac{\tau c^2}{b}, \tag{43}$$

plays a fundamental role in the asymptotic behavior, energy estimates and regularity of solutions [144]. Indeed, all studies require the positivity assumption $\gamma > 0$. This is the common case considered in nonlinear acoustics, where $b\gamma$ is equal to the Lighthill's diffusivity of sound, which is always positive [145, 146].

However, and excepting few results on the subject, the analysis of the behavior of (42) in case $\gamma \leq 0$ remains largely open. In [147] we have considered the dynamics in this case. Indeed, we prove the remarkable fact that for $\gamma < 0$ the corresponding initial value problem (42) exhibits chaotic behaviour (Theorem 5.4). Our arguments are analytical rather than numerical and give new insights about the dynamical behaviour in more general situations.

Theorem 5.4. *Let $\tau, b > 0$ and $\alpha \geq 0$ be given. Assume $\gamma := \alpha - \frac{\tau c^2}{b} < 0$. Then the solution semigroup to* (42) *is uniformly continuous and Devaney chaotic on $X_\rho \oplus X_\rho$ for each $\rho^2 > \frac{2c^4\tau\alpha}{b^2(\tau c^2 - b\alpha)}$.*

5.2.2 Lighthill-Whitham-Richards equation

Chaos is present in mathematical nonlinear models that describe traffic flows, see [148, 149]. Apart from the presence at microscopic level in [120, 121], we have also shown the existence of chaos for the macroscopic model in [122], given by the Lighthill-Whitmam-Richards equation [150, 151]. It is described by the following continuity equation:

$$F_t(x,t) + q_x(x,t) = 0, \tag{44}$$

where F is the flow rate of traffic and q is the traffic density, that considers that the number of cars is preserved along the track between any pair of points and times, which permits to model shocking waves [152]. Flow and density are related by the velocity as follows

$$F(x,t) = v(x,t)q(x,t), \quad \text{for every } t \geq 0, x \in \mathbb{R}. \tag{45}$$

However, this model does not consider inertial effects, and speeds of vehicles are adapted instantaneously. The addition of a diffusive term would model how drivers look ahead to adjust their speed. Therefore, Lighthill and Whitham proposed this second order linear model:

$$u_t(x,t) + Cu_x + Tu_{tt} - Du_{xx} = 0 \tag{46}$$

where, T is the inertial time constant for speed variation, C is the wave speed, and D denotes the diffusion coefficient that shows how drivers respond to changes far away from their position. In this situation, there exists $\rho_0 > 0$ such that the solution semigroup of (46) exhibits Devaney chaos on $X_\rho \oplus X_\rho$ for all $\rho \geq \rho_0$, as was shown in [122].

5.2.3 van Wijgaarden-Eringen

The linearized version of viscous van Wijngaarden–Eringen equation models the acoustic planar propagation in bubbly liquids.

$$u_{tt}(x,t) - u_{xx}(x,t) = (\mathrm{Re}_d)^{-1}u_{xxt}(x,t) + a_0^2 u_{xxtt}(x,t) \text{ for all } x \in \mathbb{R}, t \geq 0 \tag{47}$$

being $\mathrm{Re}_d = c_e L/\delta$ a *Reynolds number*, where $c_e(>0)$ is the adiabatic sound speed, δ is the diffusivity of sound [153], and L is a characteristic (macroscopic length). The constant a_0 is a *Knudsen number* that corresponds to the dimensionless bubble radius. The interest in studying the propagation of pressure waves of small amplitude in bubbly liquids appeared in order to try to take advantage of these acoustical properties to control the sound produced by propellers, both of surface ships and submerged ship. This linear version is known as the viscous (or dissipative) van Wijngaarden–Eringen equation, see [154, 155]. More details on the formulation of this equation can be found in [156] and [157].

Theorem 5.5. *Suppose that $a_0 < 1$ and*

$$0.3726 \approx \frac{\sqrt{5}}{6} < a_0 \,\mathrm{Re}_b < \frac{1}{2}. \tag{48}$$

Then for each ρ satisfying

$$\rho > \frac{r_0}{(\frac{1}{2a_0^2 \operatorname{Re} b} - 3r_0)a_0}, \tag{49}$$

where $r_0 := \frac{1}{2}\frac{\sqrt{1-4a_0^2 \operatorname{Re}^2 b}}{2a_0^2 \operatorname{Re} b}$ the solution semigroup to (47) *is a uniformly continuous Devaney chaotic on $X_\rho \oplus X_\rho$.*

Remark 5.6. *For the sake of completeness, we recall that the semigroups generated are distributionally chaotic and topologically mixing, too. Besides, they admit a strongly mixing measure with full support on $X_\rho \oplus X_\rho$ [158].*

6 Some open problems

In this section we pose some open problems that can be considered in the study of the dynamics of semigroups.

It is well-known that if x is a hypercyclic vector for T, then its orbit $\{x, Tx, T^2x, \dots\}$ is a linearly independent set. However, we do not know if this property also holds for the hypercyclic vectors of a semigroup.

Problem 6.1 ([91])**.** *Is the orbit of a hypercyclic vector for a C_0-semigroup $\{T_t\}_{t\geq 0}$ always a linearly independent set?*

In contrast to the case of hypercyclicity, Bayart and Bermúdez show the existence of Devaney chaotic semigroups that do not contain any single Devaney chaotic operator. Based on a result of the authors [159], Muñoz-Fernández et al. proved the following result [160, 161].

Theorem 6.2. *Let $\mathcal{P}$ be the set of periods of a given semigroup $\{T_t\}_{t\geq 0}$.*

1. *If $i(0, 2\pi c] \subseteq \sigma_p(A)$ for some $c > 0$, then $[1/c, \infty) \subseteq \mathcal{P}$.*
2. *If $i(0, 2\pi c] = \sigma_p(A)$ for some $c > 0$, then $\mathcal{P} = [1/c, \infty)$.*
3. *If $i[2\pi c, \infty] \subseteq \sigma_p(A)$ for some $c > 0$, then $\mathcal{P} = (0, \infty)$.*

Additionally, based on a result of Kalisch [162] generalized by Bayart et al. in [163, Th. 2.11] and [164, Lem. 2.5], these authors obtained the next result:

Theorem 6.3. *Let $c > 0$. Then there exists a uniformly continuous, Devaney chaotic semigroup $\{T_t\}_{t\geq 0}$ such that the set of periods of the semigroup is $\{0\} \cup [c, \infty)$.*

This result leads us to recall the following question of [160].

Problem 6.4. *For which subsets $P \subseteq [0; \infty)$ does there exist a Devaney chaotic semigroup whose set of periods is P?*

In Section 5 we have seen many examples of partial differential equations related to acoustics and fluid dynamics, that describe an underlying propagation phenomena. We wonder what other examples can be found in this line:

Problem 6.5. *Is it possible to find more examples of partial differential equations of order higher than two that present a chaotic behaviour on Herzog spaces?*

There are several problems whose associated semigroups are positive, and whose only solutions that make sense are positive too. Therefore, it is natural to ask (Banasiak, Desch and Rudnicki in personal communication) for positive solutions whose orbits are dense in the positive cone in these cases. Sufficient conditions that partially solve this problem were provided in [165]. They are expressed in terms of the operators of the semigroup, and it would be desirable to obtain them in terms of the generator.

Problem 6.6. *Given a semigroup $\{T_t\}_{t\geq 0}$ of positive operators on a Banach lattice, is it possible to find conditions expressed in terms of the infinitesimal generator which ensure the existence of positive vectors whose orbit is dense in the positive cone?*

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