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Research Article

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The uniqueness of meromorphic functions in k-punctured complex plane

DOI 10.1515/math-2017-0063
Received November 3, 2016; accepted February 28, 2016.

Abstract: The main purpose of this paper is to investigate the uniqueness of meromorphic functions that share two finite sets in the k-punctured complex plane. It is proved that there exist two sets S_1, S_2 with $\sharp S_1 = 2$ and $\sharp S_2 = 5$, such that any two admissible meromorphic functions f and g in Ω must be identical if $E_\Omega(S_j, f) = E_\Omega(S_j, g)(j = 1, 2)$.

Keywords: Shared-set, k-puncture, Admissible meromorphic function

MSC: 30D30, 30D35

1 Introduction

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, the first and second main theorem, lemma on the logarithmic derivatives etc. of Nevalinna theory, (see Hayman [9], Yang [18] and Yi and Yang [19]).

In 1926, R.Nevanlinna [15] proved the following well-known theorems.

Theorem 1.1 (see [15]). *If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $X = \mathbb{C}$, then $f(z) \equiv g(z)$.*

Due to this theorem, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many researchers (see [19]). In 1999, Fang [5] investigated the uniqueness of admissible functions in the unit disc that shared some finite sets. In [20, 21], Zheng studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$.

In fact, the whole complex plane, unit disc and angular domain can be regarded as simply connected regions. Thus, it is very interesting to consider the uniqueness of meromorphic functions on doubly and multiply connected regions. For the double connected region, Khrystiyanyn and Kondratyuk [10, 11] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [12]) in 2005. In 2010, Fernández [6] further investigated the value distribution of meromorphic functions on annuli. In 2009 and 2011, Cao [2, 3] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-value theorem. In 2012, Cao and Deng [1], Xu and Xuan [16] studied the uniqueness of meromorphic functions sharing some finite sets and four values on the annulus, respectively.

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However, there is no paper on uniqueness of meromorphic functions in a multiply connected region. The main purpose of this article is to investigate the uniqueness of meromorphic functions in a special multiply connected region—m-punctured complex plane.

The structure of this paper is as follows. In Section 2, we introduce the basic notations and fundamental theorems of meromorphic functions m-punctured complex plane. Section 3 is devoted to study the uniqueness of meromorphic functions that share some finite sets in m-punctured complex planes.

2 Nevanlinna theory in m-punctured complex planes

Given a set of distinct points $c_j \in \mathbb{C}$, $j \in \{1, 2, \dots, k\}, k \in \mathbb{N}_+$, we call that $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^{k} \{c_j\}$ is a k-punctured complex plane. The annulus is regarded as a special k-punctured plane if $k = 1$ see [10, 11]. The main purpose of this article is to study meromorphic functions of those k-punctured planes for which $k \geq 2$.

Denote $d = \frac{1}{2} \min\{|c_s - c_j| : j \neq s\}$ and $r_0 = \frac{1}{d} + \max\{|c_j| : j \in \{1, 2, \dots, k\}\}$. Then $\frac{1}{r_0} < d$, $\overline{D}_{1/r_0}(c_j) \bigcap \overline{D}_{1/r_0}(c_s) = \emptyset$ for $j \neq s$ and $\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0)$ for $j \in \{1, 2, \dots, k\}$, where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. For an arbitrary $r \geq r_0$, we define

$$\Omega_r = D_r(0) \setminus \bigcup_{j=1}^{m} \overline{D}_{1/r}(c_j).$$

Thus, it follows that $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$. It is easy to see that Ω_r is $k + 1$ connected region.

In 2007, Hanyak and Kondratyuk [8] gave some extension of the Nevanlinna value distribution theory for meromorphic functions in k-punctured complex planes and proved a series of theorems which is an analog of the result on the whole plane $\mathbb{C}$.

Let f be a meromorphic function in a k-punctured plane Ω, we use $n_0(r, f)$ to denote the counting function of its poles in $\overline{\Omega}_r$, $r_0 \leq r < +\infty$ and

$$N_0(r, f) = \int_{r_0}^{r} \frac{n_0(t, f)}{t} dt,$$

and we also define

$$\begin{aligned} m_0(r, f) = & \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta + \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \log^+ \left| f(c_j + \frac{1}{r} e^{i\theta}) \right| d\theta - \\ & - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(r_0 e^{i\theta}) \right| d\theta - \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \log^+ \left| f(c_j + \frac{1}{r_0} e^{i\theta}) \right| d\theta, \end{aligned}$$

where $\log^+ x = \max\{\log x, 0\}$ and $r_0 \leq r < +\infty$, then

$$T_0(r, f) = m_0(r, f) + N_0(r, f)$$

is called as the Nevanlinna characteristic of f.

Theorem 2.1 (see [8, Theorem 3])**.** *Let f, f_1, f_2 be meromorphic functions in a k-punctured plane Ω. Then*

(i) the function $T_0(r, f)$ is non-negative, continuous, non-decreasing and convex with respect to $\log r$ on $[r_0, +\infty)$, $T_0(r_0, f) = 0$;

(ii) if f identically equals a constant, then $T_0(r, f)$ vanishes identically;

(iii) if f is not identically equal to zero, then $T_0(r, f) = T_0(r, 1/f), r_0 \leq r < +\infty$;

(iv) $T_0(r, f_1 f_2) \leq T_0(r, f_1) + T_0(r, f_2) + O(1)$ and $T_0(r, f_1 + f_2) \leq T_0(r, f_1) + T_0(r, f_2) + O(1)$, for $r_0 \leq r < +\infty$.

Theorem 2.2 (see [8, Theorem 4]). *Let f be a non-constant meromorphic function in a k-punctured plane Ω. Then*

$$T_0(r, \frac{1}{f-a}) = T_0(r, f) + O(1),$$

for any fixed $a \in \mathbb{C}$ and all r, $r_0 \leq r < +\infty$.

Theorem 2.3 (see [8, Theorem 6], The second fundamental theorem in k-punctured planes). *Let f be a non-constant meromorphic function in a k-punctured plane Ω, and let $a_1, a_2, \ldots, a_q$ be distinct complex numbers. Then*

$$m_0(r, f) + \sum_{\nu=1}^{q} m_0(r, \frac{1}{f-a_\nu}) \leq 2T_0(r, f) - \widehat{N}_0(r, f) + S(r, f), \quad r_0 \leq r < +\infty,$$

where $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, \frac{1}{f'})$ and

$$S(r, f) = O(\log T_0(r, f)) + O(\log^+ r), \quad r \to +\infty,$$

outside a set of finite measure.

Remark 2.4. *For non-constant meromorphic function f in a k-punctured plane Ω, and any $a \in \mathbb{C}$, we use $\widetilde{n}_0(r, \frac{1}{f-a})$ to denote the counting function of zeros of $f - a$ with the multiplicities reduced by 1, then $n_0(r, \frac{1}{f'}) = \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, \frac{1}{f-a})$ for $r_0 \leq r < +\infty$, and*

$$\widehat{n}_0(r, f) := \widetilde{n}_0(r, f) + \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, \frac{1}{f-a}) = n_0(r, \frac{1}{f'}) + 2n_0(r, f) - n_0(r, \frac{1}{f'}),$$

and $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, \frac{1}{f'})$, where $\widehat{N}_0(r, f) = \int_1^r \frac{\widehat{n}_0(t,f)}{t} dt$, $r \geq 1$, holds for $r_0 \leq r < +\infty$.

The following theorem is the other interesting form of the second fundamental theorem in k-punctured planes, and plays an important role in this paper.

Theorem 2.5 ([17, Theorem 2.5]). *Let f be a non-constant meromorphic function in an m-punctured plane Ω, and let $a_1, a_2, \ldots, a_q (q \geq 3)$ be distinct complex numbers in the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then for $r_0 \leq r < +\infty$,*

$$(i)\ \ (q-2)T_0(r, f) \leq \sum_{\nu=1}^{q} N_0\left(r, \frac{1}{f-a_\nu}\right) - N_0(r, \frac{1}{f'}) + S(r, f),$$

$$(ii)\ \ (q-2)T_0(r, f) \leq \sum_{\nu=1}^{q} \widetilde{N}_0\left(r, \frac{1}{f-a_\nu}\right) + S(r, f),$$

where $\widetilde{N}_0(r, \frac{1}{f-a_\nu}) = \int_1^r \frac{\widetilde{n}_0(t, \frac{1}{f-a_\nu})}{t} dt$, $r \geq 1$ and $S(r, f)$ is stated as in Theorem 2.3.

Proof. To facilitate the reading and save the readers' time, we show the proof of this theorem as follows. If z_0 is a pole of f in k-punctured plane Ω_r with multiply s, then $\widetilde{n}_0(r, f)$ counts $s - 1$ times at z_0, and if z_0 is a zero of $f - a$ in Ω_r with multiply s, then $\widetilde{n}_0(r, f)$ also counts $s - 1$ times at z_0. Then we have

$$\sum_{\nu=1}^{q} N_0(r, \frac{1}{f-a_\nu}) - \widehat{N}_0(r, f) \leq \sum_{\nu=1}^{q} \widetilde{N}_0(r, \frac{1}{f-a_\nu}), \quad r_0 \leq r < +\infty. \tag{1}$$

By Theorem 2.2, for any $a \in \widehat{\mathbb{C}}$ and $r_0 \leq r < +\infty$, we have

$$m_0(r, \frac{1}{f-a}) = T_0(r, f) - N_0(r, \frac{1}{f-a}) + O(1), \tag{2}$$

where $m_0(r, \frac{1}{f-a}) = m_0(r, f)$ and $N_0(r, \frac{1}{f-a}) = N_0(r, f)$ as $a = \infty$. From (1),(2) and Theorem 2.3, we can get Theorem 2.5 (ii). Noting that $2N_0(r, f) - N_0(r, \frac{1}{f'}) \geq 0$, from (2) and Theorem 2.3, we can easily get Theorem (i).

Thus, this completes the proof of Theorem 2.5. □

3 The uniqueness for meromorphic functions in k-punctured planes

In this section, the uniqueness of meromorphic functions in k punctured planes that shared some values and sets will be investigated. So, we firstly introduced some basic notations of uniqueness of meromorphic functions as follows.

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega \subseteq \mathbb{C}$. Define

$$E_\Omega(S, f) = \bigcup_{a \in S} \{z \in \Omega | f_a(z) = 0, \quad counting \quad multiplicities\},$$

$$\overline{E}_\Omega(S, f) = \bigcup_{a \in S} \{z \in \Omega | f_a(z) = 0, \quad ignoring \quad multiplicities\},$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = 1/f(z)$.

For two non-constant meromorphic functions f and g in $\mathbb{C}$, we say that f and g share the set S CM(counting the multiplicities) in Ω if $E_\Omega(S, f) = E_\Omega(S, g)$; we say that f and g share the set S IM(ignoring the multiplicities) in Ω if $\overline{E}_\Omega(S, f) = \overline{E}_\Omega(S, g)$. In particular, when $S = \{a\}$, where $a \in \widehat{\mathbb{C}}$, we say that f and g share the value a CM in Ω if $E_\Omega(S, f) = E_\Omega(S, g)$, and we say that f and g share the value a IM in Ω if $\overline{E}_\Omega(S, f) = \overline{E}_\Omega(S, g)$.

Definition 3.1. *Let f be a nonconstant meromorphic function in k-punctured plane Ω. The function f is called admissible in k-punctured plane Ω provided that*

$$\limsup_{r \to +\infty} \frac{T_0(r, f)}{\log r} = +\infty, \quad r_0 \le r < +\infty.$$

Similar to the proof of Five-Values theorems [15, 19] of Nevanlinna theory, we can easily get the following theorem by Theorem 2.5.

Theorem 3.2. *Let f and g be two admissible meromorphic functions in Ω, if f, g share five distinct values a_1, a_2, a_3, a_4, a_5 IM in Ω, then $f(z) \equiv g(z)$.*

Remark 3.3. *A question is: does the conclusion of Theorem 2.5 still hold if a_j $(j = 1, \dots, 5)$ are replaced by small functions $a_j(z)(j = 1, \dots, 5)$, where $a(z)$ is called a small function of f if $T_0(r, a(z)) = o(T_0(r, f))$ as $r \to +\infty$.*

Now, we will show the main theorem of this article as follows.

Theorem 3.4. *Let f and g be two admissible meromorphic functions in Ω, and let $S_1 = \{0, 1\}$, $S_2 = \{w : \frac{(n-1)(n-2)}{2} w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2} w^{n-2} + 1 = 0\}$. If $E_\Omega(S_i, f) = E_\Omega(S_i, g)$ and $n \ge 5$, then $f(z) \equiv g(z)$.*

Corollary 3.5. *There exist two sets S_1, S_2 with $\sharp S_1 = 2$ and $\sharp S_2 = 5$, such that any two admissible meromorphic functions f and g must be identical if $E_\Omega(S_j, f) = E_\Omega(S_j, g)(j = 1, 2)$, where $\sharp S$ is to denote the cardinality of a set S.*

To prove this theorem, we require some lemmas as follows.

Lemma 3.6 ([17, Lemma 3.1]). *Let f, g be two non-constant meromorphic functions in m-punctured plane Ω, and let z_0 be a common pole of f, g in Ω with multiply 1, then z_0 is a zero of $\frac{f''}{f'} - \frac{g''}{g'}$ in Ω with multiply $k \ge 1$.*

Lemma 3.7 (see [7, Page 192]). *Let*

$$Q(w) = (n-1)^2(w^n - 1)(w^{n-2} - 1) - n(n-2)(w^{n-1} - 1)^2,$$

then

$$Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6}),$$

where $\beta_j \in \mathbb{C}\backslash\{0,1\}(j=1,2,\ldots,2n-6)$*, which are distinct respectively.*

By a similar discussion as in [14], we can obtain a stand and Valiron-Mohonzko type theorem in Ω as follows.

Lemma 3.8. *Let* f *be a nonconstant meromorphic function in* m*-punctured plane* Ω*, and let*

$$R(f)=\sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f *with coefficients* $\{a_k\}$ *and* $\{b_j\}$*, where* $a_n \neq 0$ *and* $b_m \neq 0$*. Then*

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=max\{n,m\}$*.*

Proof of Theorem 3.4. Here, we only give the proof of Theorem 3.2 as $n=5$, because the case $n\geq 6$ is similar to the case $n=5$.

Set $F=6f^5-15f^4+10f^3+1$ and $G=6g^5-15g^4+10g^3+1$. Since $E_\Omega(S_j,f)=E_\Omega(S_j,g)$, then we have F,G to share 0, 1 CM in Ω and $F'=30f^2(f-1)2f', G'=30g^2(g-1)^2g'$. From Lemma 3.3, we have $T_0(r,F)=5T_0(r,f)+S(r,f), T_0(r,G)=5T_0(r,g)+S(r,g)$ and $S(r,F)=S(r,f), S(r,G)=S(r,g)$.

We consider the following two cases.

Case 1: Suppose that there exists a constant $\lambda(>\frac{1}{2})$ and a set $I\subset[r_0,+\infty)(mesI=+\infty)$ such that

$$N_0(r,\frac{1}{f})+N_0(r,\frac{1}{f-1})\geq\lambda(T_0(r,f)+T_0(r,g))+S(r,f)+S(r,g),\quad(r\to+\infty,r\in I). \tag{3}$$

Setting $U=\frac{F'}{F}-\frac{G'}{G}$, from [8, Lemma 6] we have $m_0(r,U)=S(r,F)+S(r,G)=S(r,f)+S(r,g)$. It is easy to see that the pole of U may occur at the poles of F,G or the zeros of F,G. However, if z_0 is a common zero of F,G, by simple calculating we get that U is analytic at z_0. Since F,G share 0 CM in Ω, then it follows that $N_0(r,U)\leq\widetilde{N}_0(r,f)+\widetilde{N}_0(r,g)$. Hence, $T_0(r,U)\leq\widetilde{N}_0(r,f)+\widetilde{N}_0(r,g)+S(r,f)+S(r,g)$. On the other hand, if $U\not\equiv 0$, the zeros of U may occur at the zeros of F',G', and since $E_\Omega(S_1,f)=E_\Omega(S_1,g)$, we have

$$2N_0(r,\frac{1}{f})+2N_0(r,\frac{1}{f-1})\leq N_0(r,\frac{1}{U}). \tag{4}$$

From (3) and (4), it follows that

$$\begin{aligned}2\lambda(T_0(r,f)+T_0(r,g))+S(r,f)+S(r,g)&\leq N_0(r,\frac{1}{U})\leq T_0(r,U)+O(1)\\&\leq T_0(r,f)+T_0(r,g)+S(r,f)+S(r,g),\ r\to+\infty,r\in I.\end{aligned} \tag{5}$$

Since $\mu>\frac{1}{2}$ and f,g are admissible functions in Ω, we can get a contradiction. Thus, it follows that $U\equiv 0$, and by integration we have

$$G=KF, \tag{6}$$

where K is a non-zero constant. From Lemma 3.3, we have

$$T_0(r,f)=T_0(r,g)+S(r,g). \tag{7}$$

The following four subcases will be considered.

Subcase 1.1. Suppose that there exists $z_0\in\Omega$ such that $f(z_0)=0$ and $g(z_0)=0$. From (6), we have $K=1$, that is,

$$6f^5-15f^4+10f^3+1=6g^5-15g^4+10g^3+1. \tag{8}$$

Let α_1,α_2 be two distinct roots of equation $w^2-\frac{5}{2}w+\frac{5}{3}=0$, obviously, $\alpha_1,\alpha_2\neq 0,1$. Then, it follows from (7) that

$$f^3(f-\alpha_1)(f-\alpha_2)=g^3(g-\alpha_1)(g-\alpha_2).$$

From the above equation and $E_\Omega(S_1, f) = E_\Omega(S_1, g)$, we have f, g to share 0,1,∞ CM in Ω. Thus, let $h = \frac{f}{g}$, then h is analytic in Ω. From (8), we have

$$6(h^5 - 1)g^2 - 15(h^4 - 1)g^3 + 10(h^3 - 1)g^3 \equiv 0, \tag{9}$$

it follows that

$$[4(h^5 - 1)g - 5(h^4 - 1)]^2 = -\frac{5}{3}Q(h), \tag{10}$$

where $Q(h)$ is stated as in Lemma 3.2 and

$$Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_4),$$

where $\beta_j \in \mathbb{C}\backslash\{0,1\}(j = 1,2,3,4)$, which are distinct respectively. From (10) we know that every zero of $h - \beta_j(j = 1,2,3,4)$ is of order at least 2. By Theorem 2.5 we have $n \leq 4$, which is a contradiction. Hence h is a constant. Then from (9) we can get that $h = 1$ i.e., $f \equiv g$.

Subcase 1.2. Suppose that there exists $z_0 \in \Omega$ such that $f(z_0) = 0$ and $g(z_0) = 1$. Then from (6) we have $K = 2$, that is,

$$2(6f^5 - 15f^4 + 10f^3 + 1) = 6g^5 - 15g^4 + 10g^3 + 1. \tag{11}$$

It follows that 1 is a Picard exceptional value of f and 0 is a Picard exceptional value of g. Since $E_\Omega(S_1, f) = E_\Omega(S_1, g)$, it follows that 0,1 are all Picard exceptional values of f, g, which contradicts with (1).

Subcase 1.3. Suppose that there exists $z_0 \in \Omega$ such that $f(z_0) = 1$ and $g(z_0) = 0$. From (6), we have $K = \frac{1}{2}$. Similarly to the argument in Subcase 1.2, we can get a contradiction.

Subcase 1.4. Suppose that there exists $z_0 \in \Omega$ such that $f(z_0) = 1$ and $g(z_0) = 1$. From (6), we have $K = 1$. Similarly to the argument in Subcase 1.1, we can get that $f \equiv g$.

Case 2. Suppose that there exist a constant $\kappa(\frac{1}{2} \leq \kappa < \frac{3}{4})$ and a set $I \subset [r_0, +\infty)$ $(mesI = +\infty)$ such that

$$N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{f-1}) \leq \kappa(T_0(r, f) + T_0(r, g)) + S(r, f) + S(r, g), \tag{12}$$

as $r \to +\infty, r \in I$. Set

$$H = \frac{(\frac{1}{F})''}{(\frac{1}{F})'} - \frac{(\frac{1}{G})''}{(\frac{1}{G})'} = (\frac{F''}{F'} - \frac{2F'}{F}) - (\frac{G''}{G'} - \frac{2G'}{G}). \tag{13}$$

From [8, Lemma 6] we have $m_0(r, H) = S(r, F) + S(r, G) = S(r, f) + S(r, g)$.

Suppose that $H \not\equiv 0$, since F, G share 0 CM in Ω, we know that the pole of H may occur at the simple zeros of F', G' which are not the zeros of F, G in Ω, and the poles of F, G. Since the simple zeros of F' are only the simple zeros of f' and the simple zeros of G' are only the simple zeros of g', then we have

$$N_0(r, H) \leq \widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) + \widetilde{N}_0(r, f) + \widetilde{N}(r, g), \tag{14}$$

where $\widetilde{N}_0^*(r, \frac{1}{f'})$ is the reduced counting function of those zeros of f' in Ω which are not the zeros of $f(f-1)$ and $\widetilde{N}_0^*(r, \frac{1}{g'})$ is similarly defined. From Lemma 3.1, we have $N_0^{1)}(r, \frac{1}{F}) \leq N_0(r, \frac{1}{H})$ where $N_0^{1)}(r, \frac{1}{F})$ is the counting function of those zeros of F with multiply 1. Then for $r_0 \leq r < +\infty$, we have

$$\begin{aligned} N_0(r, \frac{1}{F}) &= N_0^{1)}(r, \frac{1}{F}) + N_0^{[2}(r, \frac{1}{F}) \leq N_0(r, \frac{1}{H}) + N_0^{[2}(r, \frac{1}{F}) \\ &\leq T_0(r, H) + N_0^{[2}(r, \frac{1}{F}) + O(1) \leq N_0(r, H) + N_0^{[2}(r, \frac{1}{F}) + S(r, f) \\ &\leq N_0^0(r, \frac{1}{f'}) + N_0^0(r, \frac{1}{g'}) + \widetilde{N}_0(r, f) + \widetilde{N}(r, g) + S(r, f) + S(r, g), \end{aligned} \tag{15}$$

where $N_0^{[2}(r, \frac{1}{F})$ is the counting function of those zeros of F with multiply ≥ 2, $N_0^0(r, \frac{1}{f'})$ is the counting function of those zeros of f' in Ω which are not the zeros of $f(f-1)$ and $N_0^0(r, \frac{1}{g'})$ is similarly defined.

Similarly, we have

$$N_0(r, \frac{1}{G}) \leq N_0^0(r, \frac{1}{f'}) + N_0^0(r, \frac{1}{g'}) + \widetilde{N}_0(r, f) + \widetilde{N}(r, g) + S(r, f) + S(r, g), \tag{16}$$

as $r_0 \leq r < +\infty$. Let $V = \frac{f'}{f} + \frac{f'}{f-1}$, from [8, Lemma 6] we have $m_0(r, V) = S(r, f)$ as $r \to +\infty$ and $N_0(r, V) \leq \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, f)$. Noting the zeros of V, we have $N_0(r, \frac{1}{f-\frac{1}{2}}) + N_0^0(r, \frac{1}{f'}) \leq N_0(r, \frac{1}{V})$. Thus, it follows that

$$\begin{aligned} N_0(r, \frac{1}{f-\frac{1}{2}}) + N_0^0(r, \frac{1}{f'}) &\leq N_0(r, \frac{1}{V}) \leq T_0(r, \frac{1}{V}) \leq T_0(r, V) + O(1) \\ &\leq N_0(r, V) + S(r, f) \\ &\leq \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, f) + S(r, f). \end{aligned} \tag{17}$$

Similarly, we have

$$N_0(r, \frac{1}{g-\frac{1}{2}}) + N_0^0(r, \frac{1}{g'}) \leq \widetilde{N}_0(r, \frac{1}{g}) + \widetilde{N}_0(r, \frac{1}{g-1}) + \widetilde{N}_0(r, g) + S(r, g). \tag{18}$$

Noting that $P(w) = 6w^5 - 15w^4 + 10w^3 + 1 = 0$ have five roots, then by using Theorem 2.5 and from (15) and (17), we have

$$\begin{aligned} 6T_0(r, f) \leq{}& N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{f-1}) + N_0(r, \frac{1}{f-\frac{1}{2}}) - N_0(r, \frac{1}{f'}) + S(r, f) \\ \leq{}& 2\widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + 2(N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{f-1})) + N_0^0(r, \frac{1}{g'}) - N_0(r, \frac{1}{f'}) + S(r, f) + S(r, g). \end{aligned}$$

Similarly, we have

$$6T_0(r, g) \leq 2\widetilde{N}_0(r, g) + \widetilde{N}_0(r, f) + 2(N_0(r, \frac{1}{g}) + N_0(r, \frac{1}{g-1})) + N_0^0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r, f) + S(r, g).$$

Noting that $N_0^0(r, \frac{1}{f'}) - N_0(r, \frac{1}{f'}) \leq 0$, $N_0^0(r, \frac{1}{g'}) - N_0(r, \frac{1}{g'}) \leq 0$, and f, g share 0,1 CM in Ω, then it follows from (12) that

$$\begin{aligned} 6[T_0(r, f) + T_0(r, g)] \leq{}& 3(\widetilde{N}_0(r, f) + \widetilde{N}_0(r, g)) + 4(N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{f-1})) + S(r, f) + S(r, g) \\ \leq{}& (3 + 4\kappa)[T_0(r, f) + T_0(r, g)] + S(r, f) + S(r, g), \quad r \in I, \quad r \to +\infty, \end{aligned}$$

which is a contradiction with $\kappa < \frac{3}{4}$ and f, g are admissible functions in Ω. Thus, $H \equiv 0$, i.e.,

$$\frac{F''}{F'} - \frac{2F'}{F} \equiv \frac{G''}{G'} - \frac{2G'}{G}. \tag{19}$$

By integration, we have from (19) that $\frac{1}{F} = \frac{A}{G} + B$ where A, B are constants which are not equal to zero at the same time. Thus, it follows that

$$F - 1 = \frac{(1-B)(G-1) + 1 - A - B}{B(G-1) + A + B}. \tag{20}$$

and $T_0(r, f) + S(r, f) = T_0(r, g) + S(r, g)$ by Lemma 3.3.

We consider two subcases as follows.

Subcase 2.1. Suppose that $B = 0$. Thus $G = AF$ and $A \neq 0$.

If $A = 1$, that is, $F \equiv G$. Similarly to Subcase 1.1, we get $f \equiv g$.

If $A = \frac{1}{2}$, that is, $\frac{1}{2}F = G$. Similarly to Subcase 1.3, we get a contradiction.

If $A \neq 1$ and $A \neq \frac{1}{2}$. Since $AF = G$, we have

$$A(6f^5 - 15f^4 + 10f^3) + A - 1 = 6g^3(g - \alpha_1)(g - \alpha_2). \tag{21}$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_5$ be five distinct roots of equation $Aw^5 - 15Aw^4 + 10Aw^3 + A - 1 = 0$, then from (21) and Theorem 2.5, we have

$$5T_0(r, f) \leq \sum_{v=1}^{5} \widetilde{N}_0(r, \frac{1}{f-\gamma_v}) + N_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + S(r, f)$$

$$\begin{aligned}&\le \widetilde{N}_0(r,\frac{1}{g})+\widetilde{N}_0(r,\frac{1}{g-\alpha_1})+\widetilde{N}_0(r,\frac{1}{g-\alpha_2})+\\&+N_0(r,\frac{1}{f})+\widetilde{N}_0(r,\frac{1}{f-1})+S(r,f)\\&\le (3+2\kappa)T_0(r,f)+S(r,f),\ \ r\in I,\ \ r\to+\infty,\end{aligned}$$

which is a contradiction with $\kappa<\frac{3}{4}$ and f is an admissible function in Ω.

Subcase 2.2. Suppose that $B\neq 0$.

If $B=1$, then $\frac{1}{F}\equiv\frac{A}{G}+1$, that is, $(F-1)(G+A)\equiv -A$. Thus, it follows that

$$6f^3(f-\alpha_1)(f-\alpha_2)(6g^5-15g^4+10g^3+A+1)\equiv -A. \tag{22}$$

Note that the zeros of $f-\alpha_1$ or $f-\alpha_2$ in Ω must be the poles of g in Ω with multiply ≥ 5, then by Theorem 2.2, Theorem 2.5 and (12) we have

$$\begin{aligned}2T_0(r,f)&\le \widetilde{N}_0(r,\frac{1}{f})+\widetilde{N}_0(r,\frac{1}{f-1})+\widetilde{N}_0(r,\frac{1}{f-\alpha_1})+\widetilde{N}_0(r,\frac{1}{f-\alpha_2})+S(r,f)\\&\le(2\kappa+\frac{2}{5})T_0(r,f)+S(r,f),\quad r\in I,\ \ r\to+\infty,\end{aligned}$$

which implies a contradiction with $\kappa<\frac{3}{4}$ and f is an admissible function in Ω.

If $B=\frac{1}{2}$, then

$$F-1=\frac{G-2A}{G+2A}=\frac{(G-1)+1-2A}{(G-1)+1+2A}. \tag{23}$$

Note that 1 is a multiply zero of F' in Ω, and 1 is also a zero of $F-2$ in Ω, then 1 is a zero of $F-2$ with multiply ≥ 3, that is, $F-2=6(f-1)^3(f-\alpha_1')(f-\alpha_2')$. From (23), we have

$$\begin{aligned}\widetilde{N}_0(r,g)&=\widetilde{N}_0(r,G-1)=\widetilde{N}_0(r,\frac{1}{F-2})\\&=\widetilde{N}_0(r,\frac{1}{f-1})+\widetilde{N}_0(r,\frac{1}{f-\alpha_1'})+\widetilde{N}_0(r,\frac{1}{f-\alpha_2'}),\ \ r_0\le r<+\infty.\end{aligned} \tag{24}$$

By Theorem 2.5 we have

$$\begin{aligned}2T_0(r,f)&\le \widetilde{N}_0(r,\frac{1}{f})+\widetilde{N}_0(r,\frac{1}{f-1})+\widetilde{N}_0(r,\frac{1}{f-\alpha_1'})+\widetilde{N}_0(r,\frac{1}{f-\alpha_2'})+S(r,f)\\&\le \widetilde{N}_0(r,\frac{1}{f})+\widetilde{N}_0(r,g)+S(r,f),\ \ r_0\le r<+\infty.\end{aligned}$$

Since $T_0(r,f)=T_0(r,g)+S(r,g)$ and f,g are admissible functions in Ω, it follows that 0 is not a Picard exceptional value of f in Ω. Thus, there exists $z_0\in\Omega$ such that $f(z_0)=0$. Since $E_\Omega(S_1,f)=E_\Omega(S_1,g)$, we have $g(z_0)=0$, it follows that $A=\frac{1}{2}$, that is, $F-1=\frac{G-1}{G}$. Thus

$$(6f^5-15f^4+10f^3)(6g^5-15g^4+10g^3+1)=6g^3(g-\alpha_1)(g-\alpha_2). \tag{25}$$

If $\gamma_1,\gamma_2,\dots,\gamma_5$ are five distinct roots of equation $6w^5-15w^4+10w^3+1=0$, from (25), we can see that the zeros of $g-\gamma_j\,(j=1,\dots 5)$ in Ω must be the poles of f in Ω. Thus by applying Theorem 2.5 for g, we have

$$3T_0(r,g)\le\sum_{j=1}^{5}\widetilde{N}_0(r,\frac{1}{g-\gamma_j})+S(r,g)\le\widetilde{N}_0(r,f)+S(r,g),\quad r_0\le r<+\infty.$$

Since $T_0(r,f)=T_0(r,g)+S(r,g)$ and f,g are admissible functions in Ω, it is easy to get a contradiction from the above inequality.

If $B\neq\frac{1}{2}$ and $B\neq 1$, then $\frac{1-B}{B}\neq 0,1,\infty$. Thus, for $r_0\le r<+\infty$, we have

$$\widetilde{N}_0(r,\frac{1}{F-1})=\widetilde{N}_0(r,\frac{1}{f})+\widetilde{N}_0(r,\frac{1}{f-\alpha_1})+\widetilde{N}_0(r,\frac{1}{f-\alpha_2}),$$

$$\widetilde{N}_0(r, \frac{1}{F-1-1}) = \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, \frac{1}{f-\alpha_1'}) + \widetilde{N}_0(r, \frac{1}{f-\alpha_2'}),$$
$$\widetilde{N}_0(r, \frac{1}{F-1-\frac{1-B}{B}}) = \widetilde{N}_0(r, G-1) = \widetilde{N}_0(r, g), \quad \widetilde{N}_0(r, F-1) = \widetilde{N}_0(r, f).$$

Then by applying Theorem 2.5 and Lemma 3.3 for $F-1$, we have

$$\begin{aligned} 10T_0(r, f) + S(r, f) = T_0(r, F-1) &\leq \widetilde{N}_0(r, \frac{1}{F-1}) + \widetilde{N}_0(r, \frac{1}{F-1-1}) + \\ &+ \widetilde{N}_0(r, \frac{1}{F-1-\frac{1-B}{B}}) + \widetilde{N}_0(r, F-1) + S(r, F) \\ &\leq (6+2\kappa)T_0(r, f) + S(r, f), \quad r \to +\infty, \ r \in I, \end{aligned}$$

which is a contradiction with $\kappa < \frac{3}{4}$ and f is admissible in Ω.

Therefore, from Case 1 and Case 2, we complete the proof of Theorem 3.2. □

Acknowledgement: The authors were supported by the National Natural Science Foundation of China (61373174,11561033,61662037), the Natural Science Foundation of Jiangxi Province in China (20151BAB201008), and the Foundation of Education Department of Jiangxi (GJJ150902,GJJ150918, GJJ160914) of China.

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