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One kind sixth power mean of the three-term exponential sums

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Abstract: In this paper, we use the estimate for trigonometric sums and the properties of the congruence equations to study the computational problem of one kind sixth power mean of the three-term exponential sums. As a conclusion, we give an exact computational formula for it.

Keywords: The three-term exponential sums, Sixth power mean, Computational formula

MSC: 11L03, 11L07

1 Introduction

Let $q \geq 3$ be a positive integer, m and n are any integers. Then, the three-term exponential sum $C(m, n, k; q)$ can be defined as follows:

$$C(m, n, k; q) = \sum_{a=1}^{q-1} e\left(\frac{a^k + ma^2 + na}{q}\right),$$

where $k \geq 3$ is a given integer and $e(y) = e^{2\pi i y}$.

Many studies have been concerned with the properties of related exponential sums; we refer the readers to [1]–[10]. A series of interesting results have been obtained. For example, Xiancun Du and Di Han [5] proved the identity that for any integers $k \geq 3$, we have

$$\sum_{m=1}^p \sum_{n=1}^p \left| \sum_{a=1}^{p-1} e\left(\frac{ma^k + na^2 + a}{p}\right) \right|^4 = 2p^4 - 3p^3 - p^2 \cdot C(k, p), \quad (1)$$

where the constant $C(k, p)$ is defined as follows:

$$C(k, p) = \sum_{\substack{a=1 \\ a^k + b^k \equiv c^k + 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ a^2 + b^2 \equiv c^2 + 1 \pmod{p}}}^{p-1} \sum_{c=1}^{p-1} 1.$$

Especially, if $k = 6$, then one has the following identity

$$\sum_{m=1}^p \sum_{n=1}^p \left| \sum_{a=1}^{p-1} e\left(\frac{ma^6 + na^2 + a}{p}\right) \right|^4 = \begin{cases} 2p^4 - 11p^3 + 16p^2, & \text{if } p \equiv 3 \pmod{4}, \\ 2p^4 - 15p^3 + 36p^2, & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

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Wenpeng Zhang and Di Han [11] obtained the identity

$$\sum_{n=1}^{p-1} \left| \sum_{a=0}^{p-1} e \left(\frac{a^3 + na}{p} \right) \right|^6 = 5p^4 - 8p^3 - p^2,$$

if prime $p > 3$ satisfying $(3, p-1) = 1$.

Yahui Yu and Wenpeng Zhang [12] considered the sixth power mean of the three-term exponential sums

$$\sum_{m=1}^p \sum_{n=1}^p \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e \left(\frac{a^k + ma^2 + na}{p} \right) \right|^6,$$

and proved that for any integer $k \geq 3$, one has the identity

$$\sum_{m=1}^p \sum_{n=1}^p \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e \left(\frac{a^k + ma^2 + na}{p} \right) \right|^6 = p^2 \cdot (p-1)^2 \cdot (6p^2 - 21p + 19),$$

where χ be the Dirichlet character mod p .

However, regarding the sixth power mean of the three-term exponential sums:

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^k + ma^2 + na}{p} \right) \right|^6, \quad (2)$$

it seems that it has not been studied before or at least we have not found any relevant results. The problem is meaningful for it can reflect the upper bound approximation of $C(m, n, k; p)$. It is easy to find that the mean value (1) is the best possible. Thus we believe that (2) has similar asymptotic properties to (1). In fact, we can use the analytic method and the qualities of the congruence equation to propose an exact computational formula for (2) with a special integer $k = 3$. That is to say, we shall prove the following identity:

Theorem. *Let $p > 3$ be a prime with $(3, p-1) = 1$. Then we have*

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^3 + ma^2 + na}{p} \right) \right|^6 = \begin{cases} p^2(5p^3 - 9p^2 - 8p + 17), & \text{if } p = 12h + 5, \\ p^2(5p^3 - 9p^2 - 8p + 21), & \text{if } p = 12h + 11. \end{cases}$$

From this theorem we may immediately deduce the following estimate:

Corollary. *Let $p > 3$ be a prime with $(3, p-1) = 1$. Then for any integers m and n , we have the upper bound estimate*

$$\left| \sum_{a=1}^{p-1} e \left(\frac{a^3 + ma^2 + na}{p} \right) \right| < 5^{\frac{1}{6}} \cdot p^{\frac{5}{6}}.$$

Let $p > 3$ be an odd prime. For any fixed integers $h \geq 4$ and $k > 3$ with $(k, p-1) = 1$, whether there exist two computational formulae for the mean value

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^k + ma^2 + na}{p} \right) \right|^6 \quad \text{and} \quad \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^3 + ma^2 + na}{p} \right) \right|^{2h}$$

are two open problems, we will further study.

2 Some simple lemmas

In this section, we will put up several simple Lemmas which are necessary to prove the theorem. Hereunder, we will use many properties of trigonometric sums and congruence equation, which can be found in references [6] and [13], so we will not repeat them here. First we have the following:

Lemma 2.1. For any prime $p > 3$ with $(3, p-1) = 1$, we have the computational formula

$$\sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + a}{p} \right) \right|^6 = \begin{cases} p^4 + 2p^3 - 22p^2 - 20p, & \text{if } p = 12h + 5, \\ p^4 + 2p^3 - 14p^2 - 8p, & \text{if } p = 12h + 11. \end{cases}$$

Proof. For any integer n with $(n, p) = 1$, from [1] or [6] we know that

$$\sum_{a=0}^{p-1} e \left(\frac{na^2}{p} \right) = \left(\frac{n}{p} \right) \cdot G(p), \quad (3)$$

where is $\left(\frac{*}{p} \right)$ the Legendre symbol, $G(p) = \sum_{a=0}^{p-1} e \left(\frac{a^2}{p} \right) = \sum_{a=1}^{p-1} \left(\frac{a}{p} \right) e \left(\frac{a}{p} \right)$ and $G^2(p) = \left(\frac{-1}{p} \right) \cdot p$. So from (3) and the properties of the reduced residue system mod p we have the identity

$$\begin{aligned} & \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + a}{p} \right) \right|^6 = \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{\overline{m}a^2 + a}{p} \right) \right|^6 + 1 \\ &= \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{4ma^2 + 4ma}{p} \right) \right|^6 + 1 = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e \left(\frac{m(2a+1)^2 - m}{p} \right) - 1 \right|^6 + 1 \\ &= \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e \left(\frac{ma^2 - m}{p} \right) - 1 \right|^6 + 1 = \sum_{m=1}^{p-1} \left| \left(\frac{m}{p} \right) e \left(\frac{-m}{p} \right) G(p) - 1 \right|^6 + 1 \\ &= \sum_{m=1}^{p-1} \left(p + 1 - \left(\frac{m}{p} \right) e \left(\frac{-m}{p} \right) G(p) - \left(\frac{m}{p} \right) e \left(\frac{m}{p} \right) \overline{G(p)} \right)^3 + 1 \\ &= (p+1)^3(p-1) - 6(p+1)^2p - 6 \left(\frac{-1}{p} \right) (p+1)p + 6(p+1)p(p-1) - 2 \left(\frac{-3}{p} \right) p^2 - 6p^2 + 1 \\ &= \begin{cases} p^4 + 2p^3 - 22p^2 - 20p, & \text{if } p = 12h + 5, \\ p^4 + 2p^3 - 14p^2 - 8p, & \text{if } p = 12h + 11. \end{cases} \end{aligned}$$

This proves Lemma 2.1. □

Lemma 2.2. For any prime $p > 3$ with $(3, p-1) = 1$, we have the computational formula

$$\sum_{\substack{a=1 \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^2+c^2 \equiv d^2+e^2+1 \pmod{p}}}^{p-1} \sum_{\substack{c=1 \\ c^2 \equiv d^2+e^2+1 \pmod{p}}}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = \begin{cases} p^3 - 4p^2 + 13p - 17, & \text{if } p = 12h + 5, \\ p^3 - 4p^2 + 13p - 21, & \text{if } p = 12h + 11. \end{cases}$$

Proof. From the trigonometric identity

$$\sum_{a=0}^{p-1} e \left(\frac{na}{p} \right) = \begin{cases} p, & \text{if } (n, p) = p, \\ 0, & \text{if } (n, p) = 1 \end{cases} \quad (4)$$

we have

$$\begin{aligned} & \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + na}{p} \right) \right|^6 \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} \sum_{m=0}^{p-1} e \left(\frac{m(a^2 + b^2 + c^2 - d^2 - e^2 - f^2)}{p} \right) \\ & \quad \times \sum_{n=0}^{p-1} e \left(\frac{n(a + b + c - d - e - f)}{p} \right) \end{aligned}$$

$$\begin{aligned}
&= p^2 \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} 1 = p^2 \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} 1 \\
&\quad \begin{array}{l} a+b+c \equiv d+e+f \pmod{p} \\ a^2+b^2+c^2 \equiv d^2+e^2+f^2 \pmod{p} \end{array} \quad \begin{array}{l} a+b+c \equiv d+e+1 \pmod{p} \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod{p} \end{array} \\
&= p^2(p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1. \tag{5} \\
&\quad \begin{array}{l} a+b+c \equiv d+e+1 \pmod{p} \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod{p} \end{array}
\end{aligned}$$

On the other hand, from (3) and Lemma 2.1 we also have

$$\begin{aligned}
&\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + na}{p} \right) \right|^6 \\
&= (p-1)^6 + (p-1) + \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2}{p} \right) \right|^6 + \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + na}{p} \right) \right|^6 \\
&= (p-1)^6 + (p-1) + \sum_{m=1}^{p-1} \left| \left(\frac{m}{p} \right) G(p) - 1 \right|^6 \\
&\quad + \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{mn^2a^2 + a}{p} \right) \right|^6 \\
&= (p-1)^6 + (p-1) + \sum_{m=1}^{p-1} \left| \left(\frac{m}{p} \right) G(p) - 1 \right|^6 + (p-1) \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + a}{p} \right) \right|^6 \\
&= (p-1)^6 + (p-1) + (p+1)^3(p-1) + 6 \left(1 + \left(\frac{-1}{p} \right) \right) (p-1)p(p+1) \\
&\quad + (p-1) \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{ma^2 + a}{p} \right) \right|^6 \\
&= \begin{cases} (p-1)(p^5 - 4p^4 + 13p^3 - 21p^2), & \text{if } p = 12h + 11, \\ (p-1)(p^5 - 4p^4 + 13p^3 - 17p^2), & \text{if } p = 12h + 5. \end{cases} \tag{6}
\end{aligned}$$

From (5) and (6) we may immediately deduce

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = \begin{cases} p^3 - 4p^2 + 13p - 17, & \text{if } p = 12h + 5, \\ p^3 - 4p^2 + 13p - 21, & \text{if } p = 12h + 11. \end{cases}$$

$\begin{array}{l} a+b+c \equiv d+e+1 \pmod{p} \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod{p} \end{array}$

This proves Lemma 2.2. □

3 Proof of the theorem

In this section, we will give the proof of our theorem. Note that $(3, p-1) = 1$, from (4) and the properties of reduced residue system mod p we have

$$\begin{aligned}
&\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^3 + ma^2 + na}{p} \right) \right|^6 \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e \left(\frac{a^3 + b^3 + c^3 - d^3 - e^3 - f^3}{p} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} e \left(\frac{m(a^2 + b^2 + c^2 - d^2 - e^2 - f^2) + n(a + b + c - d - e - f)}{p} \right) \\
& = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e \left(\frac{f^3(a^3 + b^3 + c^3 - d^3 - e^3 - 1)}{p} \right) \\
& \quad \times \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} e \left(\frac{mf^2(a^2 + b^2 + c^2 - d^2 - e^2 - 1) + nf(a + b + c - d - e - 1)}{p} \right) \\
& = p^2 \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e \left(\frac{f(a^3 + b^3 + c^3 - d^3 - e^3 - 1)}{p} \right) \\
& \quad \substack{a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod p \\ a^3+b^3+c^3 \equiv d^3+e^3+1 \pmod p} \\
& = p^3 \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 \pmod p \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod p \\ a^3+b^3+c^3 \equiv d^3+e^3+1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 - p^2 \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 \pmod p \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 \\
& \equiv p^3 U - p^2 V.
\end{aligned} \tag{7}$$

Now we compute the values of U and V in (7) respectively. It is clear that from Lemma 2.2 we have

$$V = \begin{cases} p^3 - 4p^2 + 13p - 17, & \text{if } p = 12h + 5, \\ p^3 - 4p^2 + 13p - 21, & \text{if } p = 12h + 11. \end{cases} \tag{8}$$

To compute the value of U , we note that the identity

$$(a + b + c)^3 + 2(a^3 + b^3 + c^3) - 3(a + b + c)(a^2 + b^2 + c^2) = 6abc,$$

so the condition

$$\begin{cases} a + b + c \equiv d + e + 1 \pmod p, \\ a^2 + b^2 + c^2 \equiv d^2 + e^2 + 1 \pmod p, \\ a^3 + b^3 + c^3 \equiv d^3 + e^3 + 1 \pmod p \end{cases}$$

is equivalent to

$$\begin{cases} a + b + c \equiv d + e + 1 \pmod p, \\ a^2 + b^2 + c^2 \equiv d^2 + e^2 + 1 \pmod p, \\ abc \equiv de \pmod p \end{cases}$$

or

$$\begin{cases} abc + a + b + c - 1 \equiv ab + bc + ca \pmod p, \\ a + b + c \equiv d + e + 1 \pmod p, \\ a^2 + b^2 + c^2 \equiv d^2 + e^2 + 1 \pmod p \end{cases}$$

or

$$\begin{cases} (a - 1)(b - 1)(c - 1) \equiv 0 \pmod p, \\ a + b + c \equiv d + e + 1 \pmod p, \\ a^2 + b^2 + c^2 \equiv d^2 + e^2 + 1 \pmod p. \end{cases} \tag{9}$$

For all integers $1 \leq a, b, c, d, e \leq p - 1$, we compute the number of the solutions in (9). We separate the solutions in (9) into three cases:

(A): $a = 1, 2 \leq b, c \leq p - 1; b = 1, 2 \leq a, c \leq p - 1; c = 1, 2 \leq a, b \leq p - 1$.

(B): $a = b = 1, 2 \leq c \leq p - 1; a = c = 1, 2 \leq b \leq p - 1; b = c = 1, 2 \leq a \leq p - 1$.

(C): $a = b = c = 1$.

In the case (C), the equation (9) becomes $d + e \equiv 2 \pmod p$ and $d^2 + e^2 \equiv 2 \pmod p$, so $d = e = 1$. That is, in this case, the equation (9) has only one solution $(a, b, c, d, e) = (1, 1, 1, 1, 1)$.

In the case (B), if $a = b = 1$ and $2 \leq c \leq p-1$, then the equation (9) becomes $2 \leq c \leq p-1$, $d + e \equiv c + 1 \pmod{p}$, $d^2 + e^2 \equiv c^2 + 1 \pmod{p}$ and $de \equiv c \pmod{p}$, or $(d-1)(e-1) \equiv 0 \pmod{p}$ and $(d^2-1)(e^2-1) \equiv 0 \pmod{p}$, $2 \leq c \leq p-1$. In this case, the number of the solutions of the congruence equation is $2(p-2)$. So in the case (B), the number of all solutions of the congruence equation (9) is $3 \times 2(p-2) = 6(p-2)$.

It is clear that the number of the solutions of the congruence equation $d + e \equiv b + c \pmod{p}$, $d^2 + e^2 \equiv b^2 + c^2 \pmod{p}$, $1 \leq b, c, d, e \leq p-1$ is $(p-1)(2p-3)$. In fact, this congruence equation is equivalent to the congruence equation $d + e \equiv b + 1 \pmod{p}$, $d^2 + e^2 \equiv b^2 + 1 \pmod{p}$, $1 \leq b, c, d, e \leq p-1$. So from (B) we know that the number of the solutions is $(p-1)(2p-3)$. In the case (A), the congruence equation (9) becomes $2 \leq b, c \leq p-1$, $d + e \equiv b + c \pmod{p}$, $d^2 + e^2 \equiv b^2 + c^2 \pmod{p}$. So from (B) and (C) we know that the number of the solutions of the congruence equation (9) in case (A) is $3 \times [(p-1)(2p-3) - 4(p-2) - 1]$.

Combining three cases (A), (B) and (C), we deduce that the number of all solutions of the equation (9) is

$$3 \times [(p-1)(2p-3) - 4(p-2) - 1] + 6(p-2) + 1 = 6p^2 - 13p + 5.$$

That is, for any prime p with $(3, p-1) = 1$, the value of U is

$$U = \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 \pmod{p} \\ a^2+b^2+c^2 \equiv d^2+e^2+1 \pmod{p} \\ a^3+b^3+c^3 \equiv d^3+e^3+1 \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = 6p^2 - 13p + 5. \quad (10)$$

Combining (7), (8) and (10) we may immediately deduce the identity

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} e \left(\frac{a^3 + ma^2 + na}{p} \right) \right|^6 = \begin{cases} p^2(5p^3 - 9p^2 - 8p + 17), & \text{if } p = 12h + 5, \\ p^2(5p^3 - 9p^2 - 8p + 21), & \text{if } p = 12h + 11. \end{cases}$$

This completes the proof of our theorem.

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