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Research Article

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Existence of solutions for delay evolution equations with nonlocal conditions

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Abstract: In this paper, we are devoted to study the existence of mild solutions for delay evolution equations with nonlocal conditions. By using tools involving the Kuratowski measure of noncompactness and fixed point theory, we establish some existence results of mild solutions without the assumption of compactness on the associated semigroup. Our results improve and generalize some related conclusions on this issue. Moreover, we present an example to illustrate the application of the main results.

Keywords: Evolution equation, Nonlocal conditions, Delay, Mild solution, Kuratowski measure of noncompactness, Equicontinuous semigroup

MSC: 34G20, 34K30, 35D35

1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$. We denote by $C([a,b],X)$ the Banach space of all the continuous functions from $[a,b]$ to X endowed with the sup-norm $\|u\|_{[a,b]} = \sup_{t\in[a,b]}\|u(t)\|$, where a, $b \in \mathbb{R}$ are two constants with $a < b$.

The aim of this paper is to study the existence of mild solutions for the following delay evolution equations with nonlocal conditions

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u_t), & t \in [0,a], \\ u_0 = \phi + g(u) \in C([-q,0], X), \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ is the infinitesimal generator of an equicontinuous semigroup $T(t)$ $(t \geq 0)$ in X, $a, q > 0$ are two constants, $\phi \in C([-q,0],X)$, $f : [0,a] \times X \times C([-q,0],X) \to X$ and $g : C([0,a],X) \to C([-q,0],X)$ are appropriate given functions.

For any $u \in C([-q,a],X)$ and $t \in [0,a]$, we denote by u_t the element of $C([-q,0],X)$ defined by

$$u_t(\theta) = u(t+\theta), \quad \theta \in [-q,0].$$

Here u_t represents the history of the state from time $t-q$ up to the present time t.

The existence problem on the compact interval $[0,a]$, in the very simplest case when $q = 0$, i.e., when the delay is absent, was studied by Liu and Yuan [1]. In this case $C([-q,0],X)$ is identified with X, $f(t,u,u_0)$ is identified with a function f from $[0,a] \times X$ to X, and so in the paper [1] the following problem was studied:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t \in [0,a], \\ u(0) = \phi + g(u) \in X. \end{cases} \tag{2}$$

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On the basis of Schauder's fixed point theorem and approximation theory, the authors established some existence results of mild solutions for the problem (2) in [1]. As we can easily see, the general problem (1) contains as particular case (2). The nonlocal problem was motivated by physical problems. Indeed, it is demonstrated that the nonlocal condition can be more useful than the classical initial condition to describe some physical phenomena. There are many papers concerning the nonlocal problems, see [2–11]. A very important specific case of (2) concerns ω-periodic problems, which corresponds to the choice of g as $g(u) = u(\omega)$ and $\phi \equiv 0$.

On the other hand, the study of the existence of solutions for the differential equations with delay was initiated by Travis and Webb [12] and Webb [13]. Since the theory of differential equations with delay has extensive physical background and realistic mathematical model, it has been considerably developed and numerous properties of solutions have been studied, see [14–16] and references therein. Therefore, the nonlocal Cauchy problems for evolution equations with delay are an important area of investigation and have been studied extensively in recent years. However, as far as we know, the assumptions that the associated semigroup is compact, the perturbation function f is compact, or f satisfies Lipschitz-type conditions play key roles in many of known consequences.

Inspired by the above-mentioned works, in this paper, we apply the Kuratowski measure of noncompactness and fixed point theory to obtain the existence of mild solutions for evolution equation with nonlocal conditions and delay (1) without the assumption of compactness on the associated semigroup. Our results improve and extend some related results in this direction. And in the technical framework developed in this paper, we can study several cases simultaneously, such as the semigroup $T(t)(t \geq 0)$ is compact, the perturbation function f is compact, or f satisfies Lipschitz-type conditions.

The paper is divided into 5 sections. Section 2 contains some notations, definitions and preliminary facts which will be used throughout this paper. In Section 3, we consider the existence of mild solutions for the problem (1.1) under the situation that g satisfies Lipschitz-type condition. By utilizing fixed point theorem and Kuratowski measure of noncompactness, we establish some existence results of mild solutions for the problem (1). In Section 4, we discuss the existence of mild solutions for the problem (1) under the case that g is completely continuous. Results in the section are based on fixed point theorem. Finally, an example is presented in the last section to illustrate the applications of the obtained results.

2 Preliminaries

Let $-A : D(A) \subset X \to X$ be the infinitesimal generator of an equicontinuous semigroup $T(t)$ $(t \geq 0)$ in Banach space X. This fact means that $T(t)$ is continuous by operator norm for every $t > 0$. Let $\mathcal{L}(X)$ be the Banach space of all linear and bounded operators in X endowed with the topology defined by the operator norm. We denote by

$$M := \sup_{t\in[0,a]} \|T(t)\|_{\mathcal{L}(X)},$$

then one can easily see that $M \geq 1$ is a constant. It is well known that the compact or analytic semigroup is equicontinuous. The following lemma is obvious.

Lemma 2.1. *If the semigroup $T(t)$ $(t \geq 0)$ is equicontinuous and $m \in L([0, a], \mathbb{R}^+)$, then the set*

$$\Big\{ \int_0^t T(t-s)u(s)ds,\ \|u(s)\| \leq m(s) \text{ for a.e. } s \in [0,a] \Big\}$$

is equicontinuous for $t \in [0, a]$.

For each $u \in C([-q, a], X)$, the restriction of u on $[0, a]$ denoted by $u|_{[0,a]}$ is an element of $C([0, a], X)$. For simplicity, we also write $g(u|_{[0,a]})$ as $g(u)$. For any subset D of the space $C([-q, a], X)$, denote $D|_{[0,a]} = \{u|_{[0,a]} : u \in D\}$. Similarly, we also write $g(D|_{[0,a]})$ as $g(D)$.

Next, we recall some properties about the Kuratowski measure of noncompactness, which will be used in the sequel. In this article, we denote by $\alpha(\cdot)$ and $\alpha_{[a,b]}(\cdot)$ the Kuratowski measure of noncompactness on the bounded subsets of X and $C([a,b],X)$, respectively. For any $t \in [a,b]$ and $D \subset C([a,b],X)$, set $D(t) = \{u(t) \mid u \in D\} \subset X$. If $D \subset C([a,b],X)$ is bounded, then $D(t)$ is bounded in X and $\alpha(D(t)) \leq \alpha_{[a,b]}(D)$. For the details of the definition and properties for the Kuratowski measure of noncompactness, we refer to the monographs [17] and [18].

To discuss the existence of mild solutions for the problem (1) we also need the following lemmas.

Lemma 2.2 ([17])**.** *Let X be a Banach space, and let $D \subset C([a,b],X)$ be a bounded and equicontinuous set. Then $\alpha(D(t))$ is continuous on $[a,b]$, and*

$$\alpha_{[a,b]}(D) = \max_{t\in[a,b]} \alpha(D(t)).$$

Lemma 2.3 ([19])**.** *Let X be a Banach space, and let $D = \{u_n\} \subset C([a,b],X)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[a,b]$, and*

$$\alpha\Big(\Big\{\int_a^b u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2\int_a^b \alpha(D(t))dt.$$

Lemma 2.4 ([4, 20])**.** *Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable subset $D^* \subset D$, such that $\alpha(D) \leq 2\alpha(D^*)$.*

Lemma 2.5 ([21])**.** *Let A with norm $\|\cdot\|_A$ and C with norm $\|\cdot\|_C$ be bounded sets in Banach space X. If there is surjective map $Q : C \to A$ such that for any $c, d \in C$ one has $\|Q(c) - Q(d)\|_A \leq \|c - d\|_C$, then $\alpha(A) \leq \alpha(C)$.*

In order to introduce the concept of mild solution for problem (1), by comparison with the linear initial value problem

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in [0,a], \\ u(0) = \overline{u}, \end{cases} \tag{3}$$

whose properties are well known [22], we associate (1) to the integral equation

$$u(t) = T(t)(\phi(0) + g(u)(0)) + \int_0^t T(t-s)f(s,u(s),u_s)ds, \quad t \in [0,a]. \tag{4}$$

Definition 2.6. *A continuous function $u : [-q,a] \to X$ is called a mild solution of the problem* (1) *if $u_0 = \phi + g(u)$ and the integral equation* (4) *is satisfied.*

It is supposed that for each positive number R:

$$D_R(C([-q,a],X)) = \{u \in C([-q,a],X)) : \|u\|_{[-q,a]} \leq R\},$$

then D_R is clearly a bounded closed convex set in $C([-q,a],X))$. Obviously, if $u \in D_R$, then $u|_{[0,a]}$ is an element of $D_R(C([0,a],X))$.

3 Existence results under the situation that g is Lipschitz continuous

In this section, we discuss the existence of mild solutions for the problem (1) under the situation that the nonlocal function g is Lipschitz continuous. For this purpose, we suppose that the nonlinear term f and nonlocal term g satisfy the following conditions:

(P_1) The function $f : [0,a] \times X \times C([-q,0],X) \to X$ is continuous and there exist positive constants C_1, C_2 and C_0 such that

$$\|f(t,x,y)\| \leq C_1\|x\| + C_2\|y\|_{[-q,0]} + C_0, \quad t \in [0,a],\ x \in X,\ y \in C([-q,0],X).$$

(P_2) There exist positive constants L_1 and L_2 such that for every $t \in [0, a]$ and bounded subset $D \subset C([-q, a], X)$, we have

$$\alpha(f(t, D(t), D_t)) \le L_1\alpha(D(t)) + L_2\alpha_{[-q,0]}(D_t),$$

where $D(t) = \{x(t) \mid x \in D\} \subset X$ and $D_t = \{x_t \mid x \in D\} \subset C([-q, 0], X)$.

(P_3) There exists a constant $0 < L < \frac{1}{M}$ such that

$$\|g(x) - g(y)\|_{[-q,0]} \le L\|x - y\|_{[0,a]}, \quad x, y \in C([0, a], X).$$

Now, we are prepared to state and prove our main results of this section.

Theorem 3.1. *Assume that the conditions (P_1), (P_2) and (P_3) are satisfied. Then for every $\phi \in C([-q, 0], X)$, problem* (1) *has at least one mild solution provided that*

$$M(L + 4a(L_1 + L_2)) < 1 \tag{5}$$

and

$$M(a(C_1 + C_2) + L) < 1. \tag{6}$$

Proof. We consider the operator $Q : C([-q, a], X) \to C([-q, a], X)$ defined by

$$(Qu)(t) = (Q_1u)(t) + (Q_2u)(t), \quad t \in [-q, a],$$

where

$$(Q_1u)(t) = \begin{cases} \phi(t) + g(u)(t), & t \in [-q, 0], \\ T(t)(\phi(0) + g(u)(0)), & t \in [0, a], \end{cases}$$

$$(Q_2u)(t) = \begin{cases} 0, & t \in [-q, 0], \\ \int_0^t T(t - s)f(s, u(s), u_s)ds, & t \in [0, a]. \end{cases}$$

With the help of Definition 2.6, we know that the mild solution of problem (1) is equivalent to the fixed point of operator Q. Next, we shall show that Q has at least one fixed point by using the famous Sadovskii's fixed point theorem, which can be found in [23]. To do this, we first prove that there exists a positive constant R_0 big enough such that $Q(D_{R_0}(C([-q, a], X))) \subset D_{R_0}(C([-q, a], X))$. For every $u \in C([-q, a], X)$, by condition (P_1), we have

$$\|f(t, u(t), u_t)\| \le C_1\|u(t)\| + C_2\|u_t\|_{[-q,0]} + C_0, \quad t \in [0, a]. \tag{7}$$

Choose a positive constant

$$R_0 \ge \frac{M(\|\phi\|_{[-q,0]} + \|g(0)\|_{[-q,0]} + aC_0)}{1 - M(a(C_1 + C_2) + L)}. \tag{8}$$

For every $u \in D_{R_0}(C([-q, a], X))$, then $\|u_t\|_{[-q,0]} \le \|u\|_{[-q,a]} \le R_0$ for all $t \in [0, a]$. By the condition (P_3) we get that for $t \in [-q, 0]$,

$$\begin{aligned} \|(Qu)(t)\| &\le \|\phi\|_{[-q,0]} + \|g(u)\|_{[-q,0]}\| \le \phi\|_{[-q,0]} + LR_0 + \|g(0)\|_{[-q,0]} \\ &:= K_1. \end{aligned}$$

Hence,

$$\|Qu\|_{[-q,0]} \le K_1.$$

On the other hand, by the condition (P_3) and (7) we know that for every $t \in [0, a]$,

$$\begin{aligned} \|(Qu)(t)\| &\le M(\|\phi(t)\| + \|g(u)(t)\|) + M\int_0^t \|f(s, u(s), u_s)\|ds \\ &\le M(\|\phi\|_{[-q,0]} + \|g(u)\|_{[-q,0]}]) \end{aligned}$$

$$\begin{aligned}
&+M\int_0^t (C_1\|u(s)\| + C_2\|u_s\|_{[-q,0]} + C_0)ds \\
\le\ & M(\|\phi\|_{[-q,0]} + L\|u\|_{[0,a]} + \|g(0)\|_{[-q,0]}) \\
&+M\int_0^a ((C_1 + C_2)\|u\|_{[-q,a]} + C_0)ds \\
\le\ & M(\|\phi\|_{[-q,0]} + \|g(0)\|_{[-q,0]} + aC_0) + M(a(C_1 + C_2) + L)R_0 \\
:=\ & K_2,
\end{aligned}$$

which means that

$$\|Qu\|_{[0,a]} \le K_2.$$

Notice that $M \ge 1$ yields that $K_1 \le K_2$. Therefore, by (6) and (8) we obtain

$$\|Qu\|_{[-q,a]} \le R_0.$$

Hence, we have proved that

$$Q(D_{R_0}(C([-q,a],X))) \subset D_{R_0}(C([-q,a],X)).$$

Next, we will prove that Q is continuous on $D_{R_0}(C([-q,a],X))$. Let $u_n \subset D_{R_0}(C([-q,a],X))$ with $u_n \to u$ $(n \to \infty)$ in $D_{R_0}(C([-q,a],X))$. Then $u_{nt} \to u_t$ as $n \to \infty$ for all $t \in [0,a]$, where $u_{nt} = (u_n)_t$. Applying the condition (P_3) we get that for each $t \in [-q,0]$,

$$\begin{aligned}
\|(Qu_n)(t) - (Qu)(t)\| &= \|(gu_n)(t) - (gu)(t)\| \\
&\le \|g(u_n) - g(u)\|_{[-q,0]} \\
&\le L\|u_n - u\|_{[0,a]} \\
&\le L\|u_n - u\|_{[-q,a]} \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

By the facts $u_{nt} \to u_t$ when $n \to \infty$ for every $t \in [0,a]$ and the function f is continuous, we have

$$f(t,u_n(t),u_{nt}) \to f(t,u(t),u_t) \quad \text{as} \quad n \to \infty \quad \text{for} \quad \forall\ t \in [0,a].$$

Combined this fact with Lebesgue dominated convergence theorem, we know that

$$\begin{aligned}
\|(Qu_n)(t) - (Qu)(t)\| &\le ML\|u_n - u\|_{[-q,a]} + M\int_0^t \|f(s,u_n(s),u_{ns}) - f(s,u(s),u_s)\|ds \\
&\to 0 \quad \text{as} \quad n \to \infty, \quad \forall\ t \in [0,a].
\end{aligned}$$

Hence

$$\|Qu_n - Qu\|_{[-q,a]} \to 0 \quad \text{as} \quad n \to \infty,$$

which means that the operator Q is continuous on $D_{R_0}(C([-q,a],X))$.

In what follows, we will prove that Q is a condensing operator. For this purpose, we firstly prove that Q_1 is Lipschitz continuous on $D_{R_0}(C([-q,a],X))$. Taking u and v in $D_{R_0}(C([-q,a],X))$. Then $u|_{[0,a]}$ and $v|_{[0,a]}$ in $D_{R_0}(C([0,a],X))$, by the hypothesis (P_3) we get that

$$\begin{aligned}
\|(Q_1u)(t) - (Q_1v)(t)\| &= \|(gu)(t) - (gv)(t)\| \\
&\le \|g(u) - g(v)\|_{[-q,0]} \\
&\le L\|u - v\|_{[0,a]} \\
&\le L\|u - v\|_{[-q,a]} \quad \text{for} \quad t \in [-q,0]
\end{aligned}$$

and

$$\begin{aligned}\|(Q_1u)(t)-(Q_1v)(t)\| &= M\|(gu)(0)-(gv)(0)\|\\ &\le M\|g(u)-g(v)\|_{[-q,0]}\\ &\le ML\|u-v\|_{[0,a]}\\ &\le ML\|u-v\|_{[-q,a]} \quad \text{for} \quad t\in[0,a].\end{aligned}$$

From the above discussion we get that

$$\|Q_1u-Q_1v\|_{[-q,a]}\le ML\|u-v\|_{[-q,a]}$$

for all $u,v\in D_{R_0}(C([-q,a],X))$, which means that $Q_1 : D_{R_0}(C([-q,a],X))\to D_{R_0}(C([-q,a],X))$ is Lipschitz continuous with Lipschitz constant ML. Therefore, combining this fact with Lemma 2.5 we know that for any bounded subset $D\subset D_{R_0}(C([-q,a],X))$,

$$\alpha_{[-q,a]}(Q_1(D))\le ML\alpha_{[-q,a]}(D). \tag{9}$$

On the other hand, by the fact that the semigroup $T(t)$ $(t\ge 0)$ generated by $-A$ is equicontinuous and Lemma 2.1, it is easily to see that $\{Q_2u : u\in D_{R_0}(C([-q,a],X))\}$ is a family of equicontinuous functions. Then for every bounded subset $D\subset D_{R_0}(C([-q,a],X))$, we know that $Q_2(D)$ is bounded and equicontinuous. It follows from Lemma 2.4 that there exists a countable set $D^*=\{u_n\}\subset D$ such that

$$\alpha_{[-q,a]}(Q_2(D))\le 2\alpha_{[-q,a]}(Q_2(D^*)). \tag{10}$$

For each $t\in[0,a]$, we denote by

$$D^*(t)=\{u(t)\mid u\in D^*\},$$

$$D^*_t=\{u_t\mid u\in D^*\}\subset D_{R_0}(C([-q,0],X)).$$

Then, for every $u,v\in D^*$ we have $u_t, v_t\in D^*_t$, and

$$\|u_t-v_t\|_{[-q,0]}=\sup_{\theta\in[-q,0]}\|u(t+\theta)-v(t+\theta)\|\le\|u-v\|_{[-q,a]}. \tag{11}$$

By (11) and Lemma 2.5 we know that

$$\alpha_{[-q,0]}(D^*_t)\le\alpha_{[-q,a]}(D^*). \tag{12}$$

For every $t\in[-q,0]$, $(Q_2D^*)(t)=0$, and therefore $\alpha((Q_2D^*)(t))=0$. For every $t\in[0,a]$, taking the assumption (P_2), Lemma 2.3 and (12) into account, we get that

$$\begin{aligned}\alpha((Q_2D^*)(t)) &= \alpha\Big(\Big\{\int_0^t T(t-s)f(s,u_n(s),u_{ns})ds : n\in\mathbb{N}\Big\}\Big)\\ &\le 2M\int_0^t \alpha(\{f(s,u_n(s),u_{ns}) : n\in\mathbb{N}\})ds\\ &\le 2M\int_0^t (L_1\alpha(D^*(s))+L_2\alpha_{[-q,0]}(D^*_s))ds\\ &\le 2aM(L_1+L_2)\alpha_{[-q,a]}(D^*)\\ &\le 2aM(L_1+L_2)\alpha_{[-q,a]}(D).\end{aligned}$$

Therefore, for every $t\in[-q,a]$, we get that

$$\alpha((Q_2D^*)(t))\le 2aM(L_1+L_2)\alpha_{[-q,a]}(D). \tag{13}$$

Since $Q_2(D^*)$ is equicontinuous, by Lemma 2.1, we know that

$$\alpha_{[-q,a]}(Q_2(D^*)) = \max_{t\in[-q,a]} \alpha(Q_2(D^*)(t)).$$

Hence by (10) and (13), we have

$$\alpha_{[-q,a]}(Q_2(D)) \le 4aM(L_1 + L_2)\alpha_{[-q,a]}(D). \tag{14}$$

For every subset $D \subset D_{R_0}(C([-q,a],X))$, by (5), (9) and (14), we have

$$\begin{aligned}\alpha_{[-q,a]}(Q(D)) &\le \alpha_{[-q,a]}(Q_1(D)) + \alpha_{[-q,a]}(Q_2(D))\\ &\le (ML + 4aM(L_1 + L_2))\alpha_{[-q,a]}(D)\\ &< \alpha_{[-q,a]}(D).\end{aligned}$$

Therefore, $Q : D_{R_0}(C([-q,a],X)) \to D_{R_0}(C([-q,a],X))$ is a condensing operator. By Sadovskii's fixed point theorem, we know that the operator Q has at least one fixed point u in $D_{R_0}(C([-q,a],X))$, which means that the problem (1) has at least one mild solution $u \in C([-q,a],X)$. This completes the proof of Theorem 3.1. □

If the nonlinear term f and the nonlocal term g satisfy the following growth conditions:

(P_4)The function $f : [0,a] \times X \times C([-q,0],X) \to X$ is continuous and there exist positive constants $\overline{C}_1, \overline{C}_2, \overline{C}_0$ and $\gamma \in [0,1)$ such that

$$\|f(t,x,y)\| \le \overline{C}_1\|x\|^\gamma + \overline{C}_2\|y\|^\gamma_{[-q,0]} + \overline{C}_0,\ \ t \in [0,a],\ x \in X,\ y \in C([-q,0],X);$$

(P_5)There exist positive constants $\overline{N}_1, \overline{N}_0$ and $\mu \in [0,1)$ such that

$$\|g(x)\|_{[-q,0]} \le \overline{N}_1\|x\|^\mu_{[0,a]} + \overline{N}_0, \quad x \in C([0,a],X);$$

we have the following existence result:

Theorem 3.2. *Assume that the conditions (P_2), (P_3), (P_4) and (P_5) are satisfied. Then for every $\phi \in C([-q,0],X)$, nonlocal problem* (1) *has at least one mild solution provided that*

$$M(L + 4a(L_1 + L_2)) < 1.$$

Proof. Let Q be the operator defined in the proof of Theorem 3.1. By conditions (P_3) and (P_4), one can use the same argument as in the proof of Theorem 3.1 to deduce that Q is continuous on $C([-q,a],X)$ and the mild solution of problem (1) is equivalent to the fixed point of the operator Q.

Next, we will demonstrate that Q maps bounded sets of $C([-q,a],X)$ into bounded sets. For any $R > 0$ and $u \in D_R(C([-q,a],X))$, by the conditions (P_4) and (P_5), we know that there exist constants $\overline{C}_3$ and $\overline{N}_2$ such that

$$\|f(t,u(t),u_t)\| \le (\overline{C}_1 + \overline{C}_2)R^\gamma + \overline{C}_0 := \overline{C}_3 \tag{15}$$

for every $t \in [0,a]$, and

$$\|g(u)\|_{[-q,0]} \le \overline{N}_1R^\mu + \overline{N}_0 := \overline{N}_2. \tag{16}$$

Taking (15) and (16) into account, we obtain that

$$\|(Qu)(t)\| \le \|\phi\|_{[-q,0]} + \overline{N}_2 \quad \text{for} \quad t \in [-q,0]$$

and

$$\begin{aligned}\|(Qu)(t)\| &\le M(\|\phi\|_{[-q,0]} + \overline{N}_2) + M\int_0^t \|f(s,u(s),u_s)\|ds\\ &\le M(\|\phi\|_{[-q,0]} + \overline{N}_2) + aM\overline{C}_3\end{aligned}$$

$$= M(\|\phi\|_{[-q,0]} + \overline{N}_2 + a\overline{C}_3) \quad \text{for} \quad t \in [0, a].$$

Hence,

$$\|Qu\|_{[-q,a]} \le M(\|\phi\|_{[-q,0]} + \overline{N}_2 + a\overline{C}_3).$$

It means that $Q(D_R(C([-q, a], X)))$ is a bounded set in $C([-q, a], X)$.

Using similar argument as getting the proof of Theorem 3.1 deduce that $Q : C([-q, a], X) \to C([-q, a], X)$ is a condensing operator.

Next, we show that the set $S = \{u \in C([-q, a], X) \mid u = \lambda Qu,\ 0 < \lambda < 1\}$ is bounded. Let $u \in S$. Then $u = \lambda Qu$ for every $0 < \lambda < 1$. Applying the conditions (P_4) and (P_5) we get

$$\|u(t)\| = \lambda\|(Qu)(t)\| < \|(Qu)(t)\| \le \|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0$$

for $t \in [-q, 0]$, and

$$\begin{aligned}
\|u(t)\| &= \lambda\|(Qu)(t)\| < \|(Qu)(t)\| \\
&\le M(\|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0) + M\int_0^t \|f(s, u(s), u_s)\|ds \\
&\le M(\|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0) + aM((\overline{C}_1 + \overline{C}_2)\|u\|^{\gamma}_{[-q,a]} + \overline{C}_0) \\
&= M(\|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0 + a((\overline{C}_1 + \overline{C}_2)\|u\|^{\gamma}_{[-q,a]} + \overline{C}_0))
\end{aligned}$$

for $t \in [0, a]$. Hence,

$$\|u\|_{[-q,a]} \le M(\|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0 + a((\overline{C}_1 + \overline{C}_2)\|u\|^{\gamma}_{[-q,a]} + \overline{C}_0)). \tag{17}$$

By (17) and the fact that $\gamma,\ \mu \in [0, 1)$, we can prove that S is a bounded set. If this is not true, then for any $u \in S$ we have that $\|u\|_{[-q,a]} \to +\infty$. Dividing both sides of (17) by $\|u\|_{[-q,a]}$ and taking the limits as $\|u\|_{[-q,a]} \to +\infty$, we get

$$\begin{aligned}
1 &< \lim_{\|u\|_{[-q,a]}\to+\infty} \frac{M(\|\phi\|_{[-q,0]} + \overline{N}_1\|u\|^{\mu} + \overline{N}_0 + a((\overline{C}_1 + \overline{C}_2)\|u\|^{\gamma}_{[-q,a]} + \overline{C}_0))}{\|u\|_{[-q,a]}} \\
&= \lim_{\|u\|_{[-q,a]}\to+\infty} \left(\frac{aM(\overline{C}_1 + \overline{C}_2)}{\|u\|^{1-\gamma}_{[-q,a]}} + \frac{M\overline{N}_1}{\|u\|^{1-\mu}_{[-q,a]}}\right) \\
&= 0.
\end{aligned}$$

This is a contradiction. It means that the set S is bounded. By condensing mapping fixed point theorem of topological degree, which can be found in [24], there is a fixed point u of Q on $C([-q, a], X)$, which is just the mild solution of problem (1). The proof is completed. □

4 Existence results under the situation that g is compact

In the section, we consider the existence of mild solutions for the problem (1) under the situation that the nonlocal function g is compact. To do this, we suppose that the nonlinear term f and nonlocal function g satisfy the following conditions:

(P_6)The function $f : [0, a] \times X \times C([-q, 0], X) \to X$ is continuous and there exist an integrable function $p : [0, a] \to [0, +\infty)$ and a continuous nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\|f(t, x, y)\| \le p(t)\big(\varphi(\|x\|) + \varphi(\|y\|_{[-q,0]})\big),\ t \in [0, a],\ x \in X,\ y \in C([-q, 0], X).$$

(P_7)The nonlocal function $g : C([0, a], X) \to C([-q, 0], X)$ is continuous and compact, and there exists a positive constant N such that

$$\|g(x)\|_{[-q,0]} \le N, \quad x \in C([0, a], X).$$

Now we are in the position to state our main conclusion of this section.

Theorem 4.1. *Assume that the conditions* (P_2), (P_6) *and* (P_7) *are satisfied. Then for every* $\phi \in C([-q,0],X)$, *nonlocal problem* (1) *has at least one mild solution provided that*

$$4aM(L_1+L_2)<1 \tag{18}$$

and

$$\int_0^t p(s)ds < \frac{1}{2M}\int_c^\infty \frac{1}{\varphi(\tau)}d\tau, \tag{19}$$

where $c=\|\phi\|_{[-q,0]}+N$.

Proof. Consider the map $Q: C([-q,a],X)\to C([-q,a],X)$ defined in the proof of Theorem 3.1. It is easily seen that the fixed point of Q is equivalent to the mild solution of the problem (1). Similar to the proof of Theorem 3.1, we can show that Q is continuous by usual technique.

Firstly, we see that if $u\in C([-q,a],X)$ with $\|u\|_{[-q,a]}\le k$ for some positive constant k, then $\|u_t\|_{[-q,0]}\le k$ for all $t\in[0,a]$. Therefore, by the conditions (P_6) and (P_7) we know that

$$\|(Qu)(t)\|\le\|\phi\|_{[-q,0]}+\|g(u)\|_{[0,a]}\le\|\phi\|_{[-q,0]}+N \quad \text{for} \quad t\in[-q,0] \tag{20}$$

and

$$\begin{aligned}\|(Qu)(t)\| &\le M\|(\phi(0)+g(u)(0))\|+M\int_0^t\|f(s,u(s),u_s)\|ds\\ &\le M(\|\phi\|_{[-q,0]}+\|g(u)\|_{[0,a]})+M\int_0^t p(s)(\varphi(\|u(s)\|)+\varphi(\|u_s\|_{[-q,0]}))ds\\ &\le M(\phi\|_{[-q,0]}+N)+M\int_0^t 2p(s)\varphi(k)ds\\ &\le M\Big(\phi\|_{[-q,0]}+N+2\varphi(k)\int_0^t p(s)ds\Big) \quad \text{for} \quad t\in[0,a].\end{aligned}$$

This together with (20) imply that Q maps bounded subsets of $C([-q,a],X)$ into bounded subsets.

By lemma 2.1 and the compactness of g involving Ascoli-Arzela's theorem, it is easily seen that the operator Q maps bounded subsets of $C([-q,a],X)$ into equicontinuous sets.

Next, we shall demonstrate that Q ia a condensing operator. Let D be a bounded subset of $C([-q,a],X)$. From the proof of Theorem 3.1, we know that

$$\alpha_{[-q,a]}(Q_2(D))\le 4aM(L_1+L_2)\alpha_{[-q,a]}(D). \tag{21}$$

On the other hand, the compactness of g implies that

$$\alpha_{[-q,a]}(Q_1(D))=0. \tag{22}$$

Therefore, for every subset $D\subset D_R(C([-q,a],X))$, due to (18), (21) and (22), we get that

$$\begin{aligned}\alpha_{[-q,a]}(Q(D)) &\le \alpha_{[-q,a]}(Q_1(D))+\alpha_{[-q,a]}(Q_2(D))\\ &\le 4aM(L_1+L_2)\alpha_{[-q,a]}(D)\\ &<\alpha_{[-q,a]}(D),\end{aligned}$$

which means that $Q: C([-q,a],X)\to C([-q,a],X)$ is a condensing operator.

Now, we show that the set

$$S = \{u \in C([-q,a],X) \mid u = \lambda Qu,\ 0 < \lambda < 1\}$$

is bounded. Let $u \in S$. Then $u = \lambda Qu$ for every $0 < \lambda < 1$. Thus

$$u(t) = \lambda T(t)(\phi(0) + g(u)(0)) + \lambda \int_0^t T(t-s) f(s,u(s),u_s)ds$$

for $t \in [0,a]$. It follows from the conditions (P_6) and (P_7) that

$$\|u(t)\| < M(\phi\|_{[-q,0]} + N) + M \int_0^t p(s)(\varphi(\|u(s)\|) + \varphi(\|u_s\|_{[-q,0]}))ds \tag{23}$$

for $t \in [0,a]$. We consider the function ρ defined by

$$\rho(t) = \sup\{\|u(s)\| : -q \le s \le t\}, \quad t \in [0,a].$$

Then $\|u(t)\| \le \rho(t)$ and $\|u_t\|_{[-q,0]} \le \rho(t)$ for $t \in [0,a]$. By the previous inequality (23), we have

$$\rho(t) \le M(\phi\|_{[-q,0]} + N) + 2M \int_0^t p(s)\varphi(\rho(s))ds. \tag{24}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c := v(0) = M(\phi\|_{[-q,0]} + N)$$

and

$$v'(t) = 2Mp(t)\varphi(\rho(t)).$$

Using the nondecreasing character of φ and (24) we get that

$$v'(t) \le 2Mp(t)\varphi(v(t)). \tag{25}$$

By (19) and (25), we have

$$\int_c^{v(t)} \frac{1}{\varphi(\tau)} d\tau \le 2M \int_0^t p(s)ds < \int_c^{\infty} \frac{1}{\varphi(\tau)} d\tau.$$

It follows that there exists a constant L^* such that $v(t) \le L^*$, and therefore $\rho(t) \le v(t) \le L^*$ for $t \in [0,a]$. Since for every $t \in [0,a]$, $\|u_t\|_{[-q,0]} \le \rho(t)$, we have

$$\|u\|_{[-q,a]} = \sup\{u(t) : -q \le t \le a\} \le L^*,$$

where L^* depends only on a and the functions p and φ. Noticing that $\rho(t) \le \|\phi\|_{[-q,0]} + N$ for $t \in [-q,0]$. Therefore, the set S is bounded. By condensing mapping fixed point theorem of topological degree, there exists a fixed point u of Q on $C([-q,a],X)$, which is just the mild solution of problem (1). The proof is completed. □

5 Example

In this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to a specific problem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$ for $n \in \mathbb{N}$. We consider the following nonlocal problem for the semilinear parabolic equation with delay

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = f(x,t,u(x,t),u_t(x)), & (x,t) \in \Omega \times [0,a], \\ u|_{\partial\Omega} = 0, \\ u(x,t) = \int_q^a K(\theta)u(t+\theta)(x)d\theta, & (x,t) \in \Omega \times [-q,0], \end{cases} \tag{26}$$

where Δ is a Laplace operator, $g : \Omega \times [0,a] \times \mathbb{R} \times C([-q,0], L^2(\Omega)) \to \mathbb{R}$ is continuous, q, $a > 0$ are two real numbers with $q < a$, and $K(\cdot) \in L^1([q,a], \mathbb{R}^+)$ with $\int_q^a K(\theta)d\theta := \tau \in (0,1)$.

In order to write the nonlocal problem for the semilinear parabolic equation with delay (26) in the form of the problem (1), let $X = L^2(\Omega)$ with the norm $\|\cdot\|_2$. Then X is reflexive Banach space. Define an operator A in reflexive Banach space X by

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad Au = -\Delta u.$$

From [22], we know that A is a sectorial operator and $-A$ is the infinitesimal generator of compact and analytic semigroup $T(t)$ $(t \geq 0)$ on X, and $T(t)$ $(t \geq 0)$ is contractive. Hence $\|T(t)\|_2 \leq M := 1$ for every $t \geq 0$. Let

$$u(t) = u(\cdot,t), \quad f(t,u(t),u_t) = f(\cdot,t,u(\cdot,t),u_t(\cdot)),$$

$$\phi \equiv 0, \quad g(u) = \int_q^a K(\theta)u(t+\theta)(\cdot)d\theta.$$

Then the nonlocal problem for the semilinear parabolic equation with delay (26) can be rewritten into the abstract form of problem (1).

Theorem 5.1. *Assume that the following condition holds:*

(P_9) *There exist positive constants a_1, a_2 and a_0 with $a(a_1+a_2)+\tau < 1$ such that*

$$|f(x,t,\eta,\zeta)| \leq a_1|\eta| + a_2\|\zeta\|_{[-q,0]} + a_0, \; x \in \Omega, \; t \in [0,a], \; \eta \in \mathbb{R}, \; \zeta \in C([-q,0], L^2(\Omega)).$$

Then the nonlocal problem for the semilinear parabolic equation with delay (26) *has at least one mild solution.*

Proof. By the condition (P_9) and the fact that $T(t)$ $(t \geq 0)$ is a compact semigroup one can easily verify that the conditions (P_1) and (P_2) are satisfied with $C_1 = a_1$, $C_2 = a_2$, $C_0 = a_0$ and $L_1 = L_2 = 0$. Furthermore, from the definition of the nonlocal function g and the fact that $\tau := \int_q^a K(\theta)d\theta < 1$ we know that the condition (P_3) is satisfied with $L = \tau$. At last, combined $a(a_1+a_2)+\tau < 1$ with the fact that $L_1 = L_2 = 0$ and $M = 1$ one can easily get that (5) and (6) are satisfied. Therefore, our conclusion follows from Theorem 3.1. This completes the proof of Theorem 5.1. □

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