DE GRUYTER OPEN Open Math. 2017; 15: 611–615

Open Mathematics **Open Access**

Research Article

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Finite groups whose all second maximal subgroups are cyclic

DOI 10.1515/math-2017-0054
Received September 20, 2016; accepted March 21, 2017.

Abstract: In this paper, we give a complete classification of the finite groups G whose second maximal subgroups are cyclic.

Keywords: Cyclic subgroups, 2-maximal subgroups, Solvable groups

MSC: 20D10, 20D20

1 Introduction

All groups considered in this paper are finite. Throughout the following, G always denotes a finite group. The symbol $\pi(G)$ denotes the set of the prime divisors of $|G|$. In 1903, Miller and Moreno [1] gave a complete classification of finite groups in which all maximal subgroups are abelian. In 1924, Shmidt [2] described finite groups whose maximal subgroups are all nilpotent. Suzuki [3] and Janko [4] have described finite unsolvable groups whose 2-maximal subgroups are nilpotent. There are only two such groups: A_5 and the special linear group $SL(2,5)$. In 1968, V. A. Belonogov [5] described finite solvable groups whose 2-maximal subgroups are all nilpotent. In 1979, De Vivo [6] investigated finite groups whose 2-maximal subgroups are all Sylow tower groups. In 1988, S.R. Li [7] investigated finite unsolvable groups whose all 2-maximal $3d$-subgroups are super solvable.

The aim of this paper is to describe finite groups whose second maximal subgroups are all cyclic. For convenience, we introduce the definition as follows:

Definition 1.1. *A group G is called an SMC-group if every second maximal subgroup of G is cyclic.*

All unexplained notations and terminologies are standard and can be found in [8–10].

2 Main results

For the proof of the Main Theorem, we need some known results. Below we give the result of Janko, Miller and Moreno.

Lemma 2.1 ([4]). *Let G be an unsolvable group. If every second maximal subgroup of G is nilpotent, then G is isomorphic to A_5 or $SL(2,5)$.*

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Lemma 2.2 ([1]). *Let G be a non-cyclic group all of whose proper subgroups are cyclic. Then one of the following holds:*
(1) $G \cong Z_p \times Z_p$, *p is a prime.*
(2) $G \cong Q_8$.
(3) $G \cong \langle a, b : a^p = b^{q^m} = 1, b^{-1}ab = a^s \rangle$, *where* $s \not\equiv 1 (\mathrm{mod}\, p)$, $s^q \equiv 1\ (\mathrm{mod}\, p)$, *$p$ and q are distinct primes.*

The following Theorem shows that SMC-groups are solvable.

Theorem 2.3. *Let G be a non-cyclic SMC-group. Then G is solvable and $|\pi(G)| \leq 3$.*

Proof. Suppose G is an unsolvable SMC-group, then all second maximal subgroups of G are cyclic and hence they are nilpotent. By Lemma 2.1, we know $G \cong A_5$ or $SL(2,5)$. Since each of the groups A_5 and $SL(2,5)$ possesses one non-cyclic second maximal subgroup, we have that G is solvable.

Let G be a solvable non-cyclic group having cyclic 2-maximal subgroups. Then every proper subgroup of G is a cyclic group or a minimal non-cyclic group, and each minimal non-cyclic group is maximal in G. Because minimal non-cyclic groups satisfy the thesis of Lemma 2.2, we can assume that one maximal subgroup M of G is a minimal non-cyclic group, and since the index of evert maximal subgroup in a solvable group is a power of a prime, we have that $|\pi(G)| \leq 3$. □

Corollary 2.4. *Let G be an SMC-group. If $|\pi(G)| \geq 4$, then G is cyclic.*

By Corollary 2.4, we only determine the structure of SMC-groups G with $|\pi(G)| \leq 3$. Firstly, we show the structure of SMC-groups with $|\pi(G)| = 3$.

Lemma 2.5. *Let G be a non-cyclic SMC-group with $|\pi(G)| = 3$. Then all Sylow subgroups of G are cyclic.*

Proof. Let G be a non-cyclic SMC-group and $\pi(G) = \{p_1, p_2, p_3\}$, where p, q and r are distinct primes. By Theorem 2.3, we know that G is solvable and hence G possesses a Sylow system $\{P_1, P_2, P_3\}$, where $P_i \in Syl_{p_i}(G)$. Thus, $P_i < P_iP_j < G$ for all $i \neq j$. Since every 2-maximal subgroup of G is cyclic, we get each P_i is cyclic for $i = 1, 2, 3$. □

A famous result of Burnside, Hölder and Zassenhaus is recalled below.

Lemma 2.6. *For an odd $m \geq 1$ and an arbitrary $n \geq 1$ such that $r^n \equiv 1 \bmod m$, $1 \leq r < m$ and $gcd(n(r-1), m) = 1$, the group*

$$M(m \cdot n) = \langle a, b | a^m = b^n = 1, b^{-1}ab = a^r \rangle$$

is meta-cyclic and all its Sylow subgroups are cyclic. Conversely, each group with such a property has a presentation of the form of $M(m \cdot n)$.

Suppose that G is a non-cyclic SMC-group with $|\pi(G)| = 3$, then G is a meta-cyclic group by Lemma 2.6. Furthermore, the following results hold.

Theorem 2.7. *Let G be a non-cyclic group with $|\pi(G)| = 3$. If G is an SMC-group, then one of the following statements holds.*
(1) $G = \langle a, b, c \rangle$, *where* $a^{p^m} = b^q = c^r = [b, c] = 1$, $a^b = a^s, a^c = a^t$, $s \not\equiv 1 (\mathrm{mod}\, q)$, $s^p \equiv 1\ (\mathrm{mod}\, q)$, $t \not\equiv 1 (\mathrm{mod}\, r)$, $t^p \equiv 1\ (\mathrm{mod}\, r)$, *$p$, q and r are distinct primes.*
(2) $G = H \times Z_r$, *where* $H \cong \langle a, b : a^{p^m} = b^q = 1, a^{-1}ba = b^s, s \not\equiv 1 (\mathrm{mod}\, q), s^p \equiv 1\ (\mathrm{mod}\, q) \rangle$, *$p$, q and r are distinct primes.*
(3) $G = \langle a, b, c \rangle$, *where* $a^p = b^{q^m} = c^r = [a, b] = [a, c] = 1, a^c = a^s$, $s \not\equiv 1 (\mathrm{mod}\, r)$, $s^q \equiv 1\ (\mathrm{mod}\, r)$, *$p$, q and r are distinct primes.*

(4) $G = \langle a, b, c\rangle$*, where* $a^{p^m} = b^q = c^r = [a,b] = [b,c] = 1$*,* $a^c = a^s$ *,* $s \not\equiv 1 (\bmod q)$*,* $s^p \equiv 1 \ (\bmod q)$*,* p*,* q *and* r *are distinct primes.*
(5) $G = \langle a, b, c\rangle$*, where* $a^{p^m} = b^q = c^r = [a,b] = 1$*,* $a^c = a^s, b^c = b^t$ *,* $s \not\equiv 1 (\bmod q)$*,* $s^p \equiv 1 \ (\bmod q)$*,* $t \not\equiv 1 (\bmod r)$*,* $t^q \equiv 1 \ (\bmod r)$ p*,* q *and* r *are distinct primes.*

Proof. Suppose that G is a non-cyclic SMC-group and $\pi(G) = \{p,q,r\}(p < q < r)$. As G is solvable, we know that G possesses a Sylow system $\{P, Q, R\}$. Assume that $P = \langle a\rangle$, $Q = \langle b\rangle$ and $R = \langle c\rangle$.

Firstly, suppose $[P, Q] \neq 1$, then PQ is non-cyclic and hence PQ is a maximal subgroup of G and $\delta(PQ) = 1$. By Lemma 2.2, we get that $PQ = \langle a, b : a^{p^m} = b^q = 1, b^a = b^s\rangle$. Suppose that $[P, R] \neq 1$. As in the above argument, then $PR = \langle a, c : a^{p^m} = c^r = 1, c^a = c^t\rangle$. By Lemma 2.6, we get $[Q, R] = 1$ and hence the conclusion (1) holds.

Assume $[P, R] = 1$. Then PR is a cyclic group. We claim that $[Q, R] = 1$. Suppose that $[Q, R] \neq 1$, we get one of Q and PR is normal in G by Lemma 2.6. If PR is normal in G, then P is normal in G, which is contrary to $[P, Q] \neq 1$. Thus, Q is normal in G. In another, R is normal in G. Hence we have $QR = Q \times R$, that is, $[Q, R] = 1$. The conclusion (2) holds.

Secondly, suppose $[P, Q] = 1$. If $[P, R] = 1$, then $[Q, R] \neq 1$ and QR is a minimal non-cyclic group. Thus, we get the conclusion (3). In the following, suppose $[P, R] \neq 1$. Similarly to the above argument, we get the conclusion (4) and (5). □

The following Theorem shows the structure of SMC-groups G with $|\pi(G)| = 2$.

Theorem 2.8. *Let* G *be a non-cyclic group with* $|\pi(G)| = 2$*. If* G *is an* SMC*-group, then one of the following statements hold.*
(1) G *is a minimal non-cyclic group.*
(2) $G = (Z_p \times Z_p)Z_q$*, where* $Z_p \times Z_p \trianglelefteq G$*,* p *and* q *are distinct primes.*
(3) $G = Q_8 \times Z_p$*,* p *is an odd prime.*
(4) $G = Q_8 \rtimes Z_3$*.*
(5) $G = \langle a, b\rangle$*, where* $a^p = b^{q^m} = 1$*,* $b^{-1}ab = a^s$*,* $s^q \not\equiv 1 (\bmod p)$*,* $s^{q^2} \equiv 1 \ (\bmod p)$*,* $m \geq 2$*,* p *and* q *are distinct primes.*
(6) $G = \langle a, b, c\rangle$*, where* $a^p = b^2 = [a,b] = 1, b^2 = c^2, b^{-1}cb = c^{-1}, c^{-1}ac = ca^t, t \not\equiv 1 (\bmod p)$*,* $t^2 \equiv 1 \ (\bmod p)$*.*
(7) $G \cong \langle a, b\rangle$*, where* $a^{p^2} = b^{q^m} = 1, , b^{-1}ab = a^t, t^q \not\equiv 1 (\bmod p)$*,* $t^{q^2} \equiv 1 \ (\bmod p)$*.*

Proof. Let G be an SMC-group with $\pi(G) = \{p, q\}$. Since G is solvable, there exists a maximal subgroup M of G such that M is normal in G. Hence $|G : M|$ is a prime. Suppose that $|G : M| = q$, then there exists a q-element c such that $G = M\langle c\rangle$ with $c^q \in M$. Suppose G is not a minimal non-cyclic group, then every maximal subgroup of G is either a cyclic group or a minimal non-cyclic group. Thus, we need to treat the following two cases for M.

Case I: M is a minimal non-cyclic group.

By Lemma 2.2, we need to treat the following three cases for M.

(1) $M \cong Z_p \times Z_p$.

Since G is not a p-group, we have $G = MZ_q$ for some prime $q(\neq p)$. Thus, G proves to be a group of type (2).

(2) $M \cong Q_8$.

In this case, $G = Q_8Z_q$, where $Q_8 \trianglelefteq G$ and q is an odd prime. If $G = Q_8 \times Z_q$, then we get conclusion (3). Suppose Z_q is not normal in G, then Z_q induces an automorphism of Q_8 of order q. We know that $Aut(Q_8) \cong S_4$. Hence $q = 3$, which gives conclusion (4): $G = Q_8 \rtimes Z_3$.

(3) $M = \langle a, b\rangle, a^p = b^{q^m} = 1, b^{-1}ab = a^s, s \not\equiv 1 (\bmod p), s^q \equiv 1 \ (\bmod p)$

In this case, let $H = \langle a\rangle$ be normal of order p in G and $a \notin Z(G)$. Thus $C_G(H) < G$. Moreover, $G/C_G(H)$ is cyclic of order dividing $p - 1$. Hence $G/C_G(H)$ is a cyclic q-group. Since $C_G(H) \neq M$, we see that $C_G(H)$ is cyclic. Also, as $C_G(H) \trianglelefteq G$, we can assume that $C_G(H)$ is not maximal in G. Consequently $|G/C_G(H)| = q^t$, $t \geq 2$. Moreover, by the definition of M, we have $b^q \in C_G(H)$ but $b \notin C_G(H)$. This shows that $\langle b \rangle C_G(H)$ is non-abelian of index q^{t-1}. As $M = \langle a, b \rangle \subseteq \langle b \rangle C_G(H)$, we get that $\langle b \rangle C_G(H) = M$ and hence $t = 2$. Now, $G/C_G(H)$ is cyclic of order q^2 and the Sylow q-subgroup $Q = \langle x \rangle$ of G is cyclic of order q^{m+1}, $x^{q^2} \in C_G(H)$ but $x^q \notin C_G(H)$. The above argument implies conclusion(5).

$$G = \langle a, x : a^p = x^{q^m} = 1, x^{-1}ax = a^s, s^q \not\equiv 1(\text{mod } p), s^{q^2} \equiv 1 \ (\text{mod } p) \rangle, m \geq 2.$$

Case II: M is a cyclic group.

(1) $\pi(M) = \pi(G)$.

Write $|M| = p^{m_1} q^{m_2}$. We consider the Sylow decomposition of M as

$$M = \langle a \rangle \times \langle b \rangle, \text{ where } a^{p^{m_1}} = b^{q^{m_2}} = 1.$$

Firstly, we suppose that $[a, c] = [b, c] = 1$. Then G is abelian. Hence $H = \langle b, c \rangle$ is a non-cyclic proper subgroup of order a power of q. So H is a maximal subgroup of G and is a minimal non-cyclic subgroup. By Lemma 2.2, $H \cong Z_q \times Z_q$ and hence $G \cong Z_q \times Z_q \times Z_p$. Thus, G proves to be a group of type (2).

Secondly, suppose that G is non-abelian.

Assume that $[b, c] \neq 1$. Consider the non-abelian subgroup $H = \langle b, c \rangle$, then H is a maximal subgroup of G of order a power of q. By Lemma 2.2, we see that $H \cong Q_8$ and $q = 2$. If $[a, c] = 1$, then $G \cong Q_8 \times Z_p$, which yields conclusion (3).

Assume that $[a, c] \neq 1$. Set $K = \langle a, c \rangle$, then K is a non-cyclic subgroup of G and hence H is a minimal non-cyclic group. By Lemma 2.2, we get $K = \langle a, c : c^4 = a^p = 1, c^{-1}ac = a^t, t \not\equiv 1(\text{mod } p), s^2 \equiv 1 \ (\text{mod } q) \ \rangle$.

$$G = \langle a, b, c : a^p = b^2 = [a, b] = 1, b^2 = c^2, b^{-1}cb = c^{-1}, c^{-1}ac = ca^t, t \not\equiv 1(\text{mod } p), t^2 \equiv 1 \ (\text{mod } p) \ \rangle.$$

Therefore G is of type (6).

(2) $|M| = p^n$.

Let $M = \langle a \rangle$, where $o(a) = p^n$. Then the $G = \langle a, b \rangle$ is a non-abelian group, where $\langle a \rangle$ is normal in G, $\langle b \rangle$ is non-normal with order q^m, $n \geq 1$, $m \geq 1$. By a simple theorem, we know that $q < p$ and so p is an odd prime. Thus b as an automorphism of $\langle a \rangle$ is fixed point free. If $n \geq 2$, then the subgroup B generalized by $a^{p^{n-1}}$ and b is non-abelian of order pq^m. Hence B is a minimal non-cyclic group. By Lemma 2.2, we know that $n = 2$. and we obtain

$$G \cong \langle a, b : a^{p^2} = b^{q^m} = 1, , b^{-1}ab = a^t, t^q \not\equiv 1(\text{mod } p), t^{q^2} \equiv 1 \ (\text{mod } p) \rangle.$$

Therefore G is of type (7).

The proof is now complete. □

To determine the structure of SMC-p-groups, we need the following known results.

Lemma 2.9 ([11]). *Let G be a group of order 2^4 and $M \cong Q_8$ be a maximal subgroup of G. Then one of the following statements is true:*
(1) $G \cong Q_{16}$, a generalized quaternion 2-group of order 2^4.
(2) $G \cong Q_8 \times Z_2$.
(3) $G \cong Q_8 * Z_4$, where $*$ denotes a central product.

Theorem 2.10. *Let G be a p-group. Then G is a non-cyclic SMC-group if and only if either $|G| \leq p^3$ or $G \cong Q_{16}$.*

Proof. If all maximal subgroups of G are cyclic, then G is a minimal non-cyclic p-group. By Lemma 2.2, we know that G is isomorphic to Q_8 or $Z_p \times Z_p$, hence $|G| \leq p^3$. Let M be a non-cyclic maximal subgroup of G, then M is a minimal non-cyclic group and hence $|M| \leq p^3$. So we have $|G| \leq p^4$. If $|G| = p^4$, then $|M| = p^3$. By Lemma 2.2, we know $M \cong Q_8$. Hence G is a group of order 2^4 and $M \cong Q_8$ is a maximal subgroup of G. By Lemma

2.9, we know that G is one of the groups Q_{16}, $Q_8 \times Z_2$ or $Q_8 * Z_4$. It can be easily shown that both $Q_8 \times Z_2$ and $Q_8 * Z_4$ have a non-cyclic 2-maximal subgroups. Thus, we have $G \cong Q_{16}$. The proof is now complete. □

Theorem. *Let G be a non-cyclic SMC-group. Then G is solvable and $|\pi(G)| \leq 3$. Furthermore, one of the following statements is true:*
(1) G is a minimal non-cyclic group.
(2) G is a p-group of order p^3.
(3) G is a generalized quaternion 2-group of order 2^4.
(4) $G = (Z_p \times Z_p)Z_q$, where $Z_p \times Z_p \trianglelefteq G$, p and q are distinct primes.
(5) $G = Q_8 \times Z_p$, p is an odd prime.
(6) $G = \langle a, b\rangle$, where $a^{p^2} = b^{q^m} = 1$, $b^{-1}ab = a^s$, $s \not\equiv 1(\mathrm{mod}\, p^2)$, $s^q \equiv 1 \ (\mathrm{mod}\, p^2)$, p and q are distinct primes.
(7) $G = Q_8 \rtimes Z_3$.
(8) $G = \langle a, b\rangle$, where $a^p = b^{q^m} = 1$, $b^{-1}ab = a^s$, $s^q \not\equiv 1(\mathrm{mod}\, p)$, $s^{q^2} \equiv 1 \ (\mathrm{mod}\, p)$, $m \geq 2$, p and q are distinct primes.
(9) $G = \langle a, b, c\rangle$, where $a^p = b^2 = [a, b] = 1, b^2 = c^2, b^{-1}cb = c^{-1}, c^{-1}ac = ca^t, t \not\equiv 1(\mathrm{mod}\, p)$, $t^2 \equiv 1 \ (\mathrm{mod}\, p)$.
(10) $G = \langle a, b, c\rangle$, where $a^{p^m} = b^q = c^r = [b, c] = 1$, $a^b = a^s, a^c = a^t$, $s \not\equiv 1(\mathrm{mod}\, q)$, $s^p \equiv 1 \ (\mathrm{mod}\, q)$, $t \not\equiv 1(\mathrm{mod}\, r)$, $t^p \equiv 1 \ (\mathrm{mod}\, r)$, p, q and r are distinct primes.
(11) $G = H \times Z_r$, where $H \cong \langle a,\ b : a^{p^m} = b^q = 1,\ a^{-1}ba = b^s,\ s \not\equiv 1(\mathrm{mod}\, q),\ s^p \equiv 1 \ (\mathrm{mod}\, q)\ \rangle$, p, q and r are distinct primes.
(12) $G = \langle a, b, c\rangle$, where $a^p = b^{q^m} = c^r = [a, b] = [a, c] = 1, a^c = a^s$, $s \not\equiv 1(\mathrm{mod}\, r)$, $s^q \equiv 1 \ (\mathrm{mod}\, r)$, p, q and r are distinct primes.
(13) $G = \langle a, b, c\rangle$, where $a^{p^m} = b^q = c^r = [a, b] = [b, c] = 1$, $a^c = a^s$, $s \not\equiv 1(\mathrm{mod}\, q)$, $s^p \equiv 1 \ (\mathrm{mod}\, q)$, p, q and r are distinct primes.
(14) $G = \langle a, b, c\rangle$, where $a^{p^m} = b^q = c^r = [a, b] = 1$, $a^c = a^s, b^c = b^t$, $s \not\equiv 1(\mathrm{mod}\, q)$, $s^p \equiv 1 \ (\mathrm{mod}\, q)$, $t \not\equiv 1(\mathrm{mod}\, r)$, $t^q \equiv 1 \ (\mathrm{mod}\, r)$ p, q and r are distinct primes.

Proof. The proof of Main Theorem comes from the Theorem 2.7, 2.8 and 2.10. □

Acknowledgement: W. Meng is supported by National Natural Science Foundation of China (11361075)and the Project of Guangxi Colleges and Universities Key Laboratory of Mathematical and Statistical Model(2016GXKLMS002). L. Ma is supported by National Natural Science Foundation of China (11601263).

References

[1] Miller G.A. and Moreno H.C., Non-abelian groups in which every subgroup is abelian. Trans. Amer. Math. Soc. 1903,4, 398-404.
[2] Schmidt O. Yu., Groups whose subgroups are all special, Matem. sb. 1924, 31, 366-372.
[3] Suzuki M., The nonexistence of a certain type of simple groups of odd order, Proc. Am. Math. Soc. 1957,8, 686-695.
[4] Janko Z., Endliche Gruppen mit lauter nilpotenten zweitmaximalen Untergruppen, Math. Z. 1962,79,422-424.
[5] Belonogov V. A., Finite solvable groups with nilpotent 2-maximal subgroups, Math. Z. 1968, 3, 21-32.
[6] Devivo C., A minimality problem concerning Sylow towers in finite groups, Rend. Accad. Sci. Fis. Mat. Napoli, 1979,46, 43-61.
[7] Li S.R., Finite unsolvable groups whose every 2-maximal $3d$-subgroup is super solvable, Chinese Annals of Mathematics(In Chinese), 1988, 9A,32-37.
[8] Gorenstein D., Finite groups, Harper & Row Pubishers, New York, 1980.
[9] Huppert B., Endliche Gruppen I, Springer-Verlag, Berlin, Heidelberg, New York, 1967.
[10] D. J. S. Robinson, A Course in the Theory of Groups. 2nd ed. New York: Springer-Verlag, 1996.
[11] M. Y. Xu and H. P. Qu, Finite p-groups, Peking University Press(in Chinese), 2010.