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Research Article

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Supersolvable orders and inductively free arrangements

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Abstract: In this paper, we define the supersolvable order of hyperplanes in a supersolvable arrangement, and obtain a class of inductively free arrangements according to this order. Our main results improve the conclusion that every supersolvable arrangement is inductively free. In addition, we assert that the inductively free arrangement with the required induction table is supersolvable.

Keywords: Free arrangement, Inductively free arrangement, Supersolvable arrangement, Supersolvable order

MSC: 52C35, 32S22

1 Introduction and preliminaries

Let V be an ℓ-dimensional vector space on $\mathbb{K}$ with a coordinate system $\{x_1, \ldots, x_\ell\} \subset V^*$. A *central arrangement* (of hyperplanes) $\mathcal{A}$ is a finite set of linear hyperplanes in V. Let $S = S(V^*)$ be the symmetric algebra of V^* and $\mathrm{Der}_{\mathbb{K}}(S)$ be the module of derivations

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f),\ f, g \in S\}.$$

For a central arrangement $\mathcal{A}$, the *logarithmic derivation module* $D(\mathcal{A})$ is defined by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

$\mathcal{A}$ is called *free* if $D(\mathcal{A})$ is free. There are a lot of works on the freeness of central arrangements, especially on Coxeter arrangements and the cones over Catalan and Shi arrangements [1–6]. For proving the freeness of arrangements, Terao's Addition Theorem [7] provides a standard tool and this theorem leads to the notion of inductively freeness.

Definition 1.1 ([7]). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements, where $\mathcal{A}' = \mathcal{A}\backslash\{H_0\}$, $\mathcal{A}'' = \mathcal{A}^{H_0}$, $H_0 \in \mathcal{A}$. The class $\mathcal{IF}$ of inductively free arrangements is the smallest class of arrangements which satisfies*
(1) $\Phi_\ell \in \mathcal{IF}$ for $\ell \geq 0$,
(2) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp\mathcal{A}'' \subset \exp\mathcal{A}'$, then $\mathcal{A} \in \mathcal{IF}$.

The inductively free arrangements form a subclass of the free arrangements that usually come up in examples. In 1992, P. Orlik and H. Terao [7] conjectured that each reflection arrangement is inductively free. Barakat and Cuntz [8] completed this conjecture only by showing that every Coxeter arrangement is inductively free. Nevertheless, Orlik

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and Terao's conjecture is false in general. T. Hoge and G. Röhrle [9] gave two counterexamples, namely the reflection arrangements of G_{33} and G_{34} are not inductively free. And they also classified all inductively free reflection arrangements [10]. In [11], the authors extended the work [10] by determining all inductively free restrictions of reflection arrangements.

Next, we will introduce some basic definitions relative to supersolvable arrangements.

Let $L = L(\mathcal{A})$ be the set of nonempty intersections of hyperplanes in $\mathcal{A}$. Define $X \leq Y$ in L if $X \supseteq Y$. In other words, L is partially ordered by reverse inclusion. We call L the *intersection poset* of $\mathcal{A}$. A pair $(X, Y) \in L \times L$ is called a *modular pair* if $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$ for any $Z \leq Y$. An element $X \in L$ is called *modular element* if (X, Y) is a modular pair for all $Y \in L$.

Definition 1.2 ([12]). *Let $\mathcal{A}$ be a central arrangement, we call $\mathcal{A}$ supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements $V = X_0 < X_1 < \cdots < X_\ell = T$.*

Within the theory of hyperplane arrangements, supersolvability is a rather strong condition, as it implies essentially every desirable property. Firstly, the Poincaré polynomial of the lattice $L(\mathcal{A})$ of a supersolvable arrangement $\mathcal{A}$ factors into linear terms over $\mathbb{Z}[t]$ [7, 12]. Secondly, Jambu and Terao [13] showed that supersolvable arrangements form a proper subclass of the class of inductively free arrangements. T. Hoge and G. Röhrle [14] classified all supersolvable reflection arrangements. Thirdly, if $\mathcal{A}$ is supersolvable, then $\mathcal{A}$ is a $K(\pi, 1)$-arrangement.

For a given arrangement, if we want to show it is inductively free, we must start with some inductively free arrangement and add hyperplanes one at a time satisfying some suitable conditions. This process may be described in an induction table. But the most important and difficult thing is to determine in which order to add the hyperplanes. The aim of this paper is to introduce the supersolvable order of hyperplanes in a supersolvable arrangement, and we obtain a class of inductively free arrangements according to this order. Our main results generalize the conclusion that every supersolvable arrangement is inductively free. Moreover, we assert that the inductively free arrangement with the required induction table is supersolvable. Finally, for the Coxeter arrangements of type $A_{\ell-1}$ and B_ℓ, we analyze the supersolvable orders and construct the triangular bases for the logarithmic derivation modules of them.

2 Main results

Assume $\prec$ is an order on the hyperplanes of $\mathcal{A}$, that is, $H_i \prec H_j$ if and only if $1 \leq i < j \leq |\mathcal{A}|$.

Definition 2.1. *Let $\mathcal{A}$ be a supersolvable arrangement, we say the order $\prec$ is a supersolvable order on $\mathcal{A}$ (which will be denoted by $\prec_s$) if there exsits a maximal modular chain $V = X_0 < X_1 < \cdots < X_\ell = T$ in $L(\mathcal{A})$, such that for each $1 \leq i \leq \ell$, there exists $H(i) \in \mathcal{A}$, satisfying*

$$\mathcal{A}_{X_i} := \{H \in \mathcal{A} | X_i \subseteq H\} = \{H \in \mathcal{A} | H \preceq H(i)\}.$$

Let $\mathcal{A}$ be a supersolvable arrangement and $\prec_s$ be a supersolvable order on $\mathcal{A}$ respecting to the maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = T.$$

We define

$$b_i = |\mathcal{A}_{X_i} \backslash \mathcal{A}_{X_{i-1}}|,\ 1 \leq i \leq \ell,$$
$$p_j = |\mathcal{A}_{X_j}|,\ 1 \leq j \leq \ell.$$

For any integer $1 < k \leq |\mathcal{A}|$, there exists a unique $2 \leq i(k) \leq \ell$, such that $p_{i(k)-1} < k \leq p_{i(k)}$. Define

$$\mathcal{A}_k = \{H \cap H_k \mid H \in \mathcal{A}_{X_{i(k)-1}}\},$$
$$\mathcal{B}_k = \{H_j \cap H_k \mid 1 \leq j \leq k-1\},$$
$$\mathcal{C}_k = \{H_j \mid 1 \leq j \leq k\}.$$

For $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{C}_k$ we have the following results.

Lemma 2.2. *For any integer* $1 < k \leq |\mathcal{A}|$, $\mathcal{A}_k$ *and* $\mathcal{A}_{X_{i(k)-1}}$ *are lattice equivalent.*

Proof. For integer $1 < k \leq |\mathcal{A}|$, we define a map

$$\begin{aligned}\pi_{H_k} : L(\mathcal{A}_{X_{i(k)-1}}) &\longrightarrow L(\mathcal{A}_k)\\ Z &\longmapsto H_k \vee Z.\end{aligned}$$

It is clear that π_{H_k} is order preserving. Define another map

$$\begin{aligned}\widetilde{\pi}_{X_{i(k)-1}} : L(\mathcal{A}_k) &\longrightarrow L(\mathcal{A}_{X_{i(k)-1}})\\ Z &\longmapsto X_{i(k)-1} \wedge Z.\end{aligned}$$

As $\prec_s$ is a supersolvable order of hyperplanes, then

$$\mathcal{A}_{X_{i(k)-1}} = \{H \in \mathcal{A} | H \preceq_s H_{p_{i(k)-1}}\},$$

hence $X_{i(k)-1} \wedge H_k = V$. Since $X_{i(k)-1}$ is a modular element in $L(\mathcal{A})$, we have

$$\widetilde{\pi}_{X_{i(k)-1}} \circ \pi_{H_k}(Z) = \widetilde{\pi}_{X_{i(k)-1}}(H_k \vee Z) = X_{i(k)-1} \wedge (H_k \vee Z) = (X_{i(k)-1} \wedge H_k) \vee Z = Z$$

for any $Z \in L(\mathcal{A}_{X_{i(k)-1}})$, therefore,

$$\widetilde{\pi}_{X_{i(k)-1}} \circ \pi_{H_k} = id_{L(\mathcal{A}_{X_{i(k)-1}})}.$$

For any $Z \in L(\mathcal{A}_k)$,

$$\begin{aligned}\pi_{H_k} \circ \widetilde{\pi}_{X_{i(k)-1}}(Z) &= \pi_{H_k}(X_{i(k)-1} \wedge Z)\\ &= H_k \vee (X_{i(k)-1} \wedge Z)\\ &= (H_k \vee X_{i(k)-1}) \wedge Z\\ &= Z.\end{aligned}$$

Therefore,

$$\pi_{H_k} \circ \widetilde{\pi}_{X_{i(k)-1}} = id_{L(\mathcal{A}_k)}.$$

From the above, we obtain the result that $\mathcal{A}_k$ and $\mathcal{A}_{X_{i(k)-1}}$ are lattice equivalent. □

Lemma 2.3. *For any integer* $1 < k \leq |\mathcal{A}|$, $\mathcal{A}_k = \mathcal{B}_k$.

Proof. Since $p_{i(k)-1} \leq k-1$, it is clear that $\mathcal{A}_k \subseteq \mathcal{B}_k$. If $\mathcal{B}_k \backslash \mathcal{A}_k \neq \emptyset$, then there exists $p_{i(k)-1} < j_0 \leq k-1$, such that $Y = H_{j_0} \cap H_k \in \mathcal{B}_k \backslash \mathcal{A}_k$. Since $p_{i(k)-1} < j_0 \leq k-1$ and $p_{i(k)-1} < k \leq p_{i(k)}$, we have $Y \nleq X_{i(k)-1}$ and $Y \leq X_{i(k)}$. Therefore, $r(X_{i(k)-1} \vee Y) = i(k)$. As $X_{i(k)-1}$ is a modular element, then

$$r(X_{i(k)-1} \wedge Y) = r(X_{i(k)-1}) + r(Y) - r(X_{i(k)-1} \vee Y) = 1.$$

There exists a hyperplane $H_m \in \mathcal{A}(1 \leq m \leq p_{i(k)-1})$, such that $X_{i(k)-1} \wedge Y = H_m$. Therefore, $H_m < Y$, $H_m \cap H_k = H_{j_0} \cap H_k = Y \in \mathcal{A}_k$. This is a contradiction. Thus, $\mathcal{A}_k = \mathcal{B}_k$. □

Theorem 2.4. *For any integer* $1 < k \leq |\mathcal{A}|$, $\mathcal{C}_k$ *is inductively free with*

$$\exp\mathcal{C}_k = \{1, b_2, \ldots, b_{i(k)-1}, k - p_{i(k)-1}, 0^{\ell - i(k)}\}.$$

Proof. We add the hyperplanes in the induction table under the order $\prec_s$. We prove this theorem by induction on k.

For the case $k = 2$, we have $i(k) = 2$, $\mathcal{C}_2 = \{H_1, H_2\}$, it is obvious that $\mathcal{C}_2 \in \mathcal{IF}$ and $\exp\mathcal{C}_2 = \{1, 1, 0^{\ell-2}\}$. For any fixed k, $2 < k < |\mathcal{A}|$, we assume the conclusion holds for all $\mathcal{C}_n$, $n \leq k$.

Case 1: If $p_{i(k)-1} < k < p_{i(k)}$. By induction hypothesis, $\mathcal{C}_{p_{i(k)-1}} \in \mathcal{IF}$ and

$$\begin{aligned}\exp\mathcal{C}_{p_{i(k)-1}} &= \{1, b_2, \ldots, b_{i(k)-2}, p_{i(k)-1} - p_{i(k)-2}, 0^{\ell-(i(k)-1)}\} \\ &= \{1, b_2, \ldots, b_{i(k)-1}, 0^{\ell-(i(k)-1)}\}.\end{aligned}$$

As $p_{i(k)-1} < k < p_{i(k)}$, then $i(k+1) = i(k)$. Thus

$$\mathcal{A}_{X_{i(k+1)-1}} = \mathcal{A}_{X_{i(k)-1}} = \mathcal{C}_{p_{i(k)-1}},$$

the above result associated with Lemma 2.2 and Lemma 2.3 imply $\mathcal{B}_{k+1} \in \mathcal{IF}$ and

$$\exp\mathcal{B}_{k+1} = \exp\mathcal{A}_{k+1} = \{1, b_2, \ldots, b_{i(k)-1}, 0^{\ell-i(k)}\}.$$

Since $\mathcal{C}_{k+1} = \mathcal{C}_k \bigcup \{H_{k+1}\}$ and the restriction of $\mathcal{C}_{k+1}$ to H_{k+1} is $\mathcal{B}_{k+1}$, by addition theorem, we have

$$\begin{aligned}\exp\mathcal{C}_{k+1} &= \{1, b_2, \ldots, b_{i(k)-1}, k + 1 - p_{i(k)-1}, 0^{\ell-i(k)}\} \\ &= \{1, b_2, \ldots, b_{i(k+1)-1}, k + 1 - p_{i(k+1)-1}, 0^{\ell-i(k+1)}\}.\end{aligned}$$

Case 2: If $k = p_i$. By induction hypothesis, $\mathcal{C}_k = \mathcal{C}_{p_i}$ is inductively free with

$$\begin{aligned}\exp\mathcal{C}_k &= \exp\mathcal{C}_{p_i} \\ &= \{1, b_2, \ldots, b_{i-1}, p_i - p_{i-1}, 0^{\ell-i}\} \\ &= \{1, b_2, \ldots, b_i, 0^{\ell-i}\}.\end{aligned}$$

By Lemma 2.2, $\mathcal{A}_{p_i+1}$ and $\mathcal{A}_{X_i}$ are lattice equivalent. Since $\mathcal{A}_{X_i} = \mathcal{C}_{p_i}$, we have

$$\exp\mathcal{A}_{p_i+1} = \{1, b_2, \ldots, b_i, 0^{\ell-(i+1)}\}.$$

Since $\mathcal{C}_{p_i+1} = \mathcal{C}_{p_i} \bigcup \{H_{p_i+1}\}$ and the restriction of $\mathcal{C}_{p_i+1}$ to H_{p_i+1} is $\mathcal{A}_{p_i+1}$, by addition theorem, we have

$$\begin{aligned}\exp\mathcal{C}_{k+1} &= \exp\mathcal{C}_{p_i+1} \\ &= \{1, b_2, \ldots, b_i, 1, 0^{\ell-(i+1)}\} \\ &= \{1, b_2, \ldots, b_{i(k+1)-1}, k + 1 - p_{i(k+1)-1}, 0^{\ell-i(k+1)}\}.\end{aligned}$$

We complete the proof. □

Corollary 2.5. *If $\mathcal{A}$ is a supersolvable arrangement, then $\mathcal{A}$ is inductively free.*

Proof. Since $\mathcal{A} = \mathcal{C}_{p_\ell}$, the conclusion is clear by Theorem 2.4. □

Remark 2.6. *The above result agrees with Theorem 4.58 in [7].*

We know that supersolvable arrangements form a proper subclass of the class of inductively free arrangements. On the contrary, what kind of inductively free arrangements are supersolvable arrangements? In the following, we will give a characterization for them.

Lemma 2.7. *Assume $\mathcal{B}$ is an inductively free arrangement with*

$$\exp\mathcal{B} = \{d_1 = 1, \ldots, d_\ell\},$$

and there exists an induction table of $\mathcal{B}$ which is the same with Table 1. Let

$$q_0 = 0, q_i = \sum_{j=1}^{i} d_j, 1 \le i \le \ell,$$

and

$$\mathcal{D}_k = \{H_j \in \mathcal{B} \mid 1 \le j \le k\}, 1 \le k \le q_\ell.$$

For any $1 \le k \le q_\ell$, *there exists unique* $1 \le i \le \ell$, *such that* $q_{i-1} < k \le q_i$, *then under a suitable coordinate transformation, we have*

$$Q(\mathcal{D}_k) \in \mathbb{K}[x_1, \dots, x_i], \ H_{q_{i-1}+1} = \{x_i = 0\}, 1 \le i \le \ell.$$

Proof. From Table 1, we can see that $r(\mathcal{D}_k) = i$, then the conclusion is obvious. □

Table 1. Induction Table

$\exp\mathcal{B}'$	α_H	$\exp\mathcal{B}''$
0^ℓ	H_1	$0^{\ell-1}$
$1, 0^{\ell-1}$	H_2	$1, 0^{\ell-2}$
$1, 1, 0^{\ell-2}$	H_3	$1, 0^{\ell-2}$
$\vdots$	$\vdots$	$\vdots$
$1, d_2 - 1, 0^{\ell-2}$	H_{q_2}	$1, 0^{\ell-2}$
$\vdots$	$\vdots$	$\vdots$
$1, d_2, \dots, d_{i-1}, 0^{\ell-(i-1)}$	$H_{q_{i-1}+1}$	$1, d_2, \dots, d_{i-1}, 0^{\ell-i}$
$1, d_2, \dots, d_{i-1}, 1, 0^{\ell-i}$	$H_{q_{i-1}+2}$	$1, d_2, \dots, d_{i-1}, 0^{\ell-i}$
$\vdots$	$\vdots$	$\vdots$
$1, d_2, \dots, d_{i-1}, d_i - 1, 0^{\ell-i}$	H_{q_i}	$1, d_2, \dots, d_{i-1}, 0^{\ell-i}$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$1, d_2, \dots, d_{\ell-1}, d_\ell - 1$	H_{q_ℓ}	$1, d_2, \dots, d_{\ell-1}$
$1, d_2, \dots, d_{\ell-1}, d_\ell$		

Lemma 2.8. *Under the same assumption of the above lemma, the logarithmic derivation module* $D(\mathcal{D}_k)$ $(q_{i-1} < k \le q_i, 1 \le i \le \ell)$ *has a triangular basis* $\theta_1, \dots, \theta_\ell$ *as follows:*

$$\theta_m = \begin{cases} \sum_{j=1}^{i} x_j D_j & \text{if } m = 1, \\ Q(\mathcal{T}_m) D_m + \sum_{j=m+1}^{i} f_{m,j}^k D_j & \text{if } 2 \le m \le i-1, \\ Q_k \backslash Q_{q_{i-1}} D_i & \text{if } m = i, \\ D_m & \text{if } i+1 \le m \le \ell, \end{cases}$$

and

$$\exp\mathcal{D}_k = \{1, d_2, \dots, d_{i-1}, k - q_{i-1}, 0^{\ell-i}\},$$

where $D_j = \partial/\partial x_j$, $\mathcal{T}_m = \{H_j \mid q_{m-1} < j \le q_m\}$, $Q_k = Q(\mathcal{D}_k)$, $f_{m,j}^k \in \mathbb{K}[x_1, \dots, x_i]$ *is homogeneous and* $\deg f_{m,j}^k = \deg Q(\mathcal{T}_m)$, $1 \le k \le \ell$.

Proof. We prove this lemma by induction on k. For the case $k = 1$, the conclusion is clear. We assume $D(\mathcal{D}_k)$ has a triangular basis as above for $k > 1$.

Case 1: If $q_{i-1} < k < q_i$, $\mathcal{D}_k$ is free with

$$\exp\mathcal{D}_k = \{1, d_2, \dots, d_{i-1}, k - q_{i-1}, 0^{\ell-i}\}.$$

By Table 1, we have $(\mathcal{D}_k)^{H_{k+1}}$ is free with

$$\exp(\mathcal{D}_k)^{H_{k+1}} = \{1, d_2, \dots, d_{i-1}, 0^{\ell-i}\},$$

where $(\mathcal{D}_k)^{H_{k+1}}$ is the restricted arrangement of $\mathcal{D}_k$ by restricting on the hyperplane H_{k+1}. Since

$$\exp(\mathcal{D}_k)^{H_{k+1}} \subseteq \exp\mathcal{D}_k,$$

by addition theorem, there exists $c_m \in S\backslash\{0\}$, $2 \leq m \leq i-1$, such that $\eta_1, \ldots, \eta_\ell$ form a basis of $D(\mathcal{D}_{k+1})$ which are as follows.

(a) If $m = 1$,

$$\eta_m = \sum_{j=1}^{i} x_j D_j.$$

(b) If $2 \leq m \leq i-1$,

$$\begin{aligned} \eta_m &= \theta_m - c_m\theta_i \\ &= Q(\mathcal{T}_m)D_m + \sum_{j=m+1}^{i} f_{m,j}^k D_j - c_m Q_k\backslash Q_{q_{i-1}} D_i \\ &= Q(\mathcal{T}_m)D_m + \sum_{j=m+1}^{i} f_{m,j}^{k+1} D_j, \end{aligned}$$

where

$$f_{m,j}^{k+1} = \begin{cases} f_{m,j}^k & \text{If } m+1 \leq j \leq i-1, \\ f_{m,j}^k - c_m Q_k\backslash Q_{q_{i-1}} & \text{If } j = i, \end{cases}$$

and

$$\deg f_{m,j}^{k+1} = \deg Q(\mathcal{T}_m),\ \ f_{m,j}^{k+1} \in \mathbb{K}[x_1, \ldots, x_i].$$

(c) If $m = i$,

$$\eta_m = \alpha_{k+1} Q_k\backslash Q_{q_{i-1}} D_i = Q_{k+1}\backslash Q_{q_{i-1}} D_i,\ \ker(\alpha_{k+1}) = H_{k+1}.$$

(d) If $i+1 \leq m \leq \ell$,

$$\eta_m = D_m.$$

It is obvious that $\eta_1, \ldots, \eta_\ell$ form a triangular basis of $D(\mathcal{D}_{k+1})$ with

$$\exp\mathcal{D}_{k+1} = \{1, d_2, \ldots, d_{i-1}, k+1-q_{i-1}, 0^{\ell-i}\}.$$

Case 2: $k = q_i$. By Lemma 2.7, under a suitable coordinate transformation, $H_{q_i+1} = \{x_{i+1} = 0\}$. Since $Q_{q_i} \in \mathbb{K}[x_1, \ldots, x_i]$, we can say, $D(\mathcal{D}_{q_i+1})$ has a triangular basis as follows,

$$\theta_m = \begin{cases} \sum_{j=1}^{i} x_j D_j & \text{if } m = 1, \\ Q(\mathcal{T}_m)D_m + \sum_{j=m+1}^{i} f_{m,j}^{q_i} D_j & \text{if } 2 \leq m \leq i-1, \\ x_{i+1} Q_{q_i}\backslash Q_{q_{i-1}} D_i & \text{if } m = i, \\ D_m & \text{if } i+1 \leq m \leq \ell. \end{cases}$$

From the above, we complete the induction. □

Lemma 2.9 ([15], Theorem 6.6)**.** *Let $\mathcal{A}$ be a free arrangement, $D(\mathcal{A})$ has a triangular basis for some choice of coordinates if and only if $\mathcal{A}$ is supersolvable.*

Theorem 2.10. *If $\mathcal{A}$ is an inductively free arrangement with* $\exp\mathcal{A} = \{d_1 = 1, \ldots, d_\ell\}$*, and its induction table is the same as Table 1, then $\mathcal{A}$ is supersolvable.*

Proof. The conclusion is clear according to Lemma 2.8 and Lemma 2.9. □

In the following, we will focus on the Coxeter arrangements of type $A_{\ell-1}$ and B_ℓ, analyze the supersolvable orders and construct the triangular bases for the logarithmic derivation modules.

Example 2.11. *Consider the Coxeter arrangement of type $A_{\ell-1}$:*

$$Q(\mathcal{A}_{\ell-1}) = \prod_{1\le i<j\le \ell} (x_i - x_j).$$

$\mathcal{A}_{\ell-1}$ is supersolvable with a maximal modular chain in $L(\mathcal{A}_{\ell-1})$:

$$\mathbb{R}^\ell < \{x_1 = x_2\} < \{x_1 = x_2 = x_3\} < \cdots < \{x_1 = x_2 = \cdots = x_\ell\}.$$

The order of the following hyperplanes is a supersolvable order on $\mathcal{A}_{\ell-1}$:

$$\begin{aligned}
&x_1 - x_2 = 0,\\
&x_1 - x_3 = 0,\ x_2 - x_3 = 0,\\
&\qquad\vdots\\
&x_1 - x_m = 0,\ x_2 - x_m = 0,\ \cdots,\ x_{m-1} - x_m = 0,\\
&\qquad\vdots\\
&x_1 - x_\ell = 0,\ x_2 - x_\ell = 0,\ \cdots,\ x_{\ell-1} - x_\ell = 0.
\end{aligned}$$

According to Theorem 2.4, $\mathcal{A}_{\ell-1}$ is inductively free with $\exp \mathcal{A}_{\ell-1} = \{0, 1, 2, \ldots, \ell-1\}$, *and the subarrangements of $\mathcal{A}_{\ell-1}$ are also inductively free according to the above supersolvable order. By Lemma 2.8, there exists a triangular basis for $D(\mathcal{A}_{\ell-1})$ as follows:*

$$(\theta_1, \theta_2, \ldots, \theta_\ell) = (D_1, D_2, \ldots, D_\ell)\begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & 0\\
1 & x_1 - x_2 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots\\
1 & x_1 - x_m & \cdots & \prod_{1\le i<m}(x_i - x_m) & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots\\
1 & x_1 - x_{\ell-1} & \cdots & \prod_{1\le i<m}(x_i - x_{\ell-1}) & \cdots & 0\\
1 & x_1 - x_\ell & \cdots & \prod_{1\le i<m}(x_i - x_\ell) & \cdots & \prod_{1\le i<\ell}(x_i - x_\ell)
\end{pmatrix}.$$

Example 2.12. *Consider the Coxeter arrangement of type B_ℓ:*

$$Q(\mathcal{B}_\ell) = \prod_{1\le k\le \ell} x_k \prod_{1\le i<j\le \ell} (x_i + x_j)(x_i - x_j).$$

$\mathcal{B}_\ell$ is supersolvable with a maximal modular chain in $L(\mathcal{B}_\ell)$:

$$\mathbb{R}^\ell < \{x_1 = 0\} < \{x_1 = x_2 = 0\} < \cdots < \{x_1 = x_2 = \cdots = x_\ell = 0\}.$$

The order of the following hyperplanes is a supersolvable order on $\mathcal{B}_\ell$:

$$\begin{aligned}
&x_1 = 0,\\
&x_2 = 0,\ x_1 + x_2 = 0,\ x_1 - x_2 = 0,\\
&\quad\vdots\\
&x_m = 0,\ x_1 + x_m = 0,\ x_1 - x_m = 0,\ \cdots,\ x_{m-1} + x_m = 0,\ x_{m-1} - x_m = 0,\\
&\quad\vdots
\end{aligned}$$

$$x_\ell = 0,\ x_1 + x_\ell = 0,\ x_1 - x_\ell = 0,\ \cdots,\ x_{\ell-1} + x_\ell = 0,\ x_{\ell-1} - x_\ell = 0.$$

We observe that the orders are still the supersolvable orders even if we adjust the orders of hyperplanes in the same rows above. According to Theorem 2.4 and Lemma 2.8, $\mathcal{B}_\ell$ is inductively free with $\exp \mathcal{B}_\ell = \{1, 3, \ldots, 2\ell - 1\}$, *and there exists a triangular basis for $D(\mathcal{B}_\ell)$ as follows:*

$$\theta_m = \begin{cases} \sum_{j=1}^{\ell} x_j D_j & \text{if } m = 1, \\ \sum_{j=m}^{\ell} \Big[\prod_{1 \le i < m} (x_i + x_j)(x_i - x_j) \Big] x_j D_j & \text{if } 2 \le m \le \ell. \end{cases}$$

That is

$$(\theta_1, \theta_2, \ldots, \theta_\ell) = (D_1, D_2, \ldots, D_\ell)T,$$

where the lower triangular matrix T is as follows:

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 & \cdots & 0 \\ x_2 & x_2(x_1^2 - x_2^2) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_m & x_m(x_1^2 - x_m^2) & \cdots & \prod_{1 \le i < m} x_m(x_i^2 - x_m^2) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{\ell-1} & x_{\ell-1}(x_1^2 - x_{\ell-1}^2) & \cdots & \prod_{1 \le i < m} x_{\ell-1}(x_i^2 - x_{\ell-1}^2) & \cdots & 0 \\ x_\ell & x_\ell(x_1^2 - x_\ell^2) & \cdots & \prod_{1 \le i < m} x_\ell(x_i^2 - x_\ell^2) & \cdots & \prod_{1 \le i < \ell} x_\ell(x_i^2 - x_\ell^2) \end{pmatrix}.$$

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