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Asymptotically almost automorphic solutions of differential equations with piecewise constant argument

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Abstract: This paper is concerned with the existence and uniqueness of asymptotically almost automorphic solutions to differential equations with piecewise constant argument. To study that, we first introduce several notions about asymptotically almost automorphic type functions and obtain some properties of such functions. Then, on the basis of a systematic study on the associated difference system, the existence and uniqueness theorem is established. Compared with some earlier results, we do not assume directly that the Green's function is a *Bi*-almost automorphic type function.

Keywords: Asymptotically almost automorphic, Almost automorphic, Differential equations with piecewise constant argument, DEPCA

MSC: 34C27

1 Introduction

In this paper, we aim to study the existence and uniqueness of asymptotically almost automorphic solution for the following differential equation with piecewise constant argument (DEPCA):

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t, y(t), y([t])), \quad t \in \mathbb{R}, \quad (1)$$

where $y(t)$ is a p -dimensional complex vector (p is a fixed positive integer), $A(t)$ and $B(t)$ are $p \times p$ complex matrices, and the coefficients satisfy some conditions recalled in the sequel.

As noted in some earlier works (cf. [2, 3]), differential equations with piecewise constant argument (DEPCA) are of considerable importance in applications to some biomedical dynamics, physical phenomena, discretization problems, etc., and there is a large literature on qualitative properties of solutions to DEPCA, like uniqueness, boundedness, periodicity, almost periodicity, pseudo almost periodicity, stability, etc. However, it seems that there are only few results concerning almost automorphic type solutions for DEPCA (cf. [2–5, 7, 11]).

Recall that since Bochner [1] introduced the concept of almost automorphy, the automorphic functions have been applied to many areas including ordinary as well as partial differential equations, abstract differential equations, functional-differential equations, integral equations, dynamical systems, etc. We refer the reader to the monographs of N'Guéréata [8, 9] for the basic theory of almost automorphic functions and their applications.

Stimulated by a recent work of Chávez, Castillo, and Pinto [3], in this paper we investigate the existence and uniqueness of asymptotically almost automorphic solution to equation (1). In order to establish our main results,

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we recall some notions about asymptotically almost automorphic type functions. Also, we make a systematic study on the properties of the introduced asymptotically almost automorphic type functions and the associated difference system for equation (1). Compared with some earlier results (e.g., [2, 3]), we do not assume directly that the Green's function is a *Bi*-almost automorphic type function. In fact, under some assumptions, we prove that the Green's function is *Bi*-asymptotically almost automorphic (see Lemma 3.4).

Before closing this section, we recall what is understood by a solution of DEPCA (1).

Definition 1.1 ([2, 3]). *A function $y : \mathbb{R} \rightarrow \mathbb{C}^p$ is called a solution of DEPCA (1) if the following assertions are satisfied:*

- (a) *y is continuous on $\mathbb{R}$;*
- (b) *y is differentiable on $\mathbb{R} \setminus \mathbb{Z}$ and for every $n \in \mathbb{Z}$, $y'_+(n)$ and $y'_-(n)$ exist;*
- (c) *there hold $y'_+(n) = A(n)y(n) + B(n)y(n) + f(t, y(n), y(n))$ for every $n \in \mathbb{Z}$ and*

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t, y(t), y([t])), \quad t \in (n, n+1), \quad n \in \mathbb{Z}.$$

2 Almost automorphic type functions

In this paper, we denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{R}$ the set of all real numbers, by $\mathbb{C}$ the set of all complex numbers, by $\mathbb{C}^p$ the set of all p -dimensional complex vectors, by $M_{p \times p}(\mathbb{C})$ the set of all $p \times p$ complex matrices, and by $BC(\mathbb{Y}, \mathbb{X})$ the Banach space of all bounded and continuous functions from $\mathbb{Y}$ to $\mathbb{X}$ with norm $\|f\| = \sup_{t \in \mathbb{Y}} \|f(t)\|$ for every two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$. Moreover, for convenience, for every $c = \{c_i\}_{i=1}^p \in \mathbb{C}^p$ and $C = \{c_{ij}\}_{i,j=1}^p \in M_{p \times p}(\mathbb{C})$, we denote their norms by the followings:

$$|c| = \max_{1 \leq i \leq p} |c_i|, \quad |C| = \max_{1 \leq i, j \leq p} |c_{ij}|.$$

Next, let us recall some basic definitions and results about almost automorphic functions and asymptotically almost automorphic functions. For more details, we refer the reader to [8, 9].

Definition 2.1. *A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if given any sequence $\{\bar{s}_n\} \subset \mathbb{R}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that*

$$\tilde{f}(t) := \lim_{n \rightarrow \infty} f(t + s_n)$$

is well defined for every $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} \tilde{f}(t - s_n) = f(t)$$

for every $t \in \mathbb{R}$. We denote the set of all such functions by $AA(\mathbb{R}, \mathbb{X})$.

Remark 2.2. *The following is a typical example of almost automorphic function:*

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}, \quad t \in \mathbb{R}.$$

We refer the reader to the monographs of N'Guérékata [8, 9] for the basic theory of almost automorphic functions and their applications.

Definition 2.3. *A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be asymptotically almost automorphic iff it admits a decomposition*

$$f = g + h,$$

where $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in C_0(\mathbb{R}, \mathbb{X})$. Here,

$$C_0(\mathbb{R}, \mathbb{X}) := \{h : \mathbb{R} \rightarrow \mathbb{X} : h \text{ is continuous on } \mathbb{R} \text{ and } \lim_{t \rightarrow \infty} h(t) = 0\}.$$

We denote the set of all such functions by $AAA(\mathbb{R}, \mathbb{X})$.

Lemma 2.4. *The following assertions hold:*

- (i) $\{g(t) : t \in \mathbb{R}\} \subset \{\overline{f(t)} : t \in \mathbb{R}\}$, where $f = g + h$, $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in C_0(\mathbb{R}, \mathbb{X})$.
- (ii) *The decomposition of every asymptotically almost automorphic function is unique.*
- (iii) $AA(\mathbb{R}, \mathbb{X})$ and $AAA(\mathbb{R}, \mathbb{X})$ are both Banach spaces under the supremum norm $\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|$.

Definition 2.5. A function $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called *almost automorphic* if given any compact set $K \subset \mathbb{Y}$ and any sequence $\{\bar{s}_n\} \subset \mathbb{R}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that

$$\widetilde{f}(t, x) := \lim_{n \rightarrow \infty} f(t + s_n, x)$$

is well defined for every $t \in \mathbb{R}$ and $x \in K$, and

$$\lim_{n \rightarrow \infty} \widetilde{f}(t - s_n, x) = f(t, x)$$

for every $t \in \mathbb{R}$ and $x \in K$. We denote the set of all such functions by $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Definition 2.6. A function $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be *asymptotically almost automorphic* iff it admits a decomposition

$$f = g + h,$$

where $g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $h \in C_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$. Here, $C_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ denote the set of all continuous functions from $\mathbb{R} \times \mathbb{Y}$ to $\mathbb{X}$ satisfying $\lim_{t \rightarrow \infty} h(t, y) = 0$ uniformly for y in any compact subset of $\mathbb{Y}$. We denote the set of all such functions by $AAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Next, let us recall an interesting notion of $\mathbb{Z}$ -almost automorphic functions, which is introduced recently by Chávez, Castillo, and Pinto [2, 3].

Let $BPC(\mathbb{R}, \mathbb{X})$ be the space of all bounded functions $f : \mathbb{R} \rightarrow \mathbb{X}$ satisfying that f is continuous in $\mathbb{R} \setminus \mathbb{Z}$ with finite lateral limits in $\mathbb{Z}$.

Definition 2.7. A function $f \in BPC(\mathbb{R}, \mathbb{X})$ is said to be *$\mathbb{Z}$ -almost automorphic* if given any sequence $\{\bar{s}_n\} \subset \mathbb{Z}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that

$$\widetilde{f}(t) := \lim_{n \rightarrow \infty} f(t + s_n)$$

is well defined for every $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} \widetilde{f}(t - s_n) = f(t)$$

for every $t \in \mathbb{R}$. We denote the set of all such functions by $ZAA(\mathbb{R}, \mathbb{X})$.

Example 2.8. *It is not difficult to verify that*

$$f(t) = \sin \frac{1}{2 + \cos[t] + \cos(\sqrt{2}[t])}, \quad t \in \mathbb{R}$$

is $\mathbb{Z}$ -almost automorphic.

Stimulated by the notion of $\mathbb{Z}$ -almost automorphic functions, in the following, we introduce the notion of $\mathbb{Z}$ -asymptotically almost automorphic functions and study some basic properties of such functions.

Definition 2.9. A function $f \in BPC(\mathbb{R}, \mathbb{X})$ is said to be *$\mathbb{Z}$ -asymptotically almost automorphic* iff it admits a decomposition

$$f = g + h,$$

where $g \in ZAA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$. Here,

$$\mathbb{Z}C_0(\mathbb{R}, \mathbb{X}) = \{h \in BPC(\mathbb{R}, \mathbb{X}) : \lim_{t \rightarrow \infty} h(t) = 0\}.$$

We denote the set of all such functions by $\mathbb{Z}AAA(\mathbb{R}, \mathbb{X})$.

Lemma 2.10. *The following assertions hold:*

- (i) $\{g(t) : t \in \mathbb{R}\} \subset \{f(t) : t \in \mathbb{R}\}$, where $f = g + h$, $g \in \mathbb{Z}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$.
- (ii) *The decomposition of every $\mathbb{Z}$ -asymptotically almost automorphic function is unique.*
- (iii) *Let $f = g + h$, $g \in \mathbb{Z}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$. If f is uniformly continuous on $\mathbb{R}$, then g is also uniformly continuous on $\mathbb{R}$.*
- (iv) $f \in AAA(\mathbb{R}, \mathbb{X})$ implies $f([\cdot]) \in ZAAA(\mathbb{R}, \mathbb{X})$.
- (v) *Let f be uniformly continuous on $\mathbb{R}$ and $f \in ZAAA(\mathbb{R}, \mathbb{X})$. Then $f \in AAA(\mathbb{R}, \mathbb{X})$.*

Proof. The assertion (i) follows from the pointwise limits in the definition of $\mathbb{Z}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$. Then, (i) yields (ii) since 0 has a unique decomposition.

By the definition of $\mathbb{Z}AA(\mathbb{R}, \mathbb{X})$, there exists a sequence $\{s_n\} \subset \mathbb{Z}$ with $\lim_{n \rightarrow \infty} s_n = \infty$, such that

$$\lim_{n \rightarrow \infty} g(t + s_n) = \tilde{g}(t), \quad \lim_{n \rightarrow \infty} \tilde{g}(t - s_n) = g(t), \quad t \in \mathbb{R}.$$

For every $t_1, t_2 \in \mathbb{R}$, there holds

$$\begin{aligned} |\tilde{g}(t_1) - \tilde{g}(t_2)| &\leq |\tilde{g}(t_1) - g(t_1 + s_n)| + |g(t_1 + s_n) - g(t_2 + s_n)| + |g(t_2 + s_n) - \tilde{g}(t_2)| \\ &\leq |\tilde{g}(t_1) - g(t_1 + s_n)| + |f(t_1 + s_n) - f(t_2 + s_n)| + |g(t_2 + s_n) - \tilde{g}(t_2)| \\ &\quad + |h(t_1 + s_n)| + |h(t_2 + s_n)|. \end{aligned}$$

Combining the above observations and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$, in view of the fact that f is uniformly continuous on $\mathbb{R}$, we conclude that $\tilde{g}$ is uniformly continuous on $\mathbb{R}$, which yields that g is also uniformly continuous on $\mathbb{R}$. This completes the proof of (iii).

The assertion (iv) follows from the fact that $f([t + k]) = f([t] + k)$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Now, let's come to the proof of (v). Let $f = g + h$, $g \in \mathbb{Z}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{X})$. Then, by (iii), g and h are both uniformly continuous on $\mathbb{R}$. It is easy to see that $h \in C_0(\mathbb{R}, \mathbb{X})$. By [2, Lemma 2.8], $g \in AA(\mathbb{R}, \mathbb{X})$. Thus, $f \in AAA(\mathbb{R}, \mathbb{X})$. $\square$

Throughout the rest of this paper, we denote $\mathfrak{Lip}(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$ be the set of all functions $f : \mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p \rightarrow \mathbb{C}^p$ satisfying the following property: there exists a constant $L_f > 0$ such that

$$|f(t, x, y) - f(t, z, w)| \leq L_f(|x - z| + |y - w|), \quad t \in \mathbb{R}, (x, y), (z, w) \in \mathbb{C}^p \times \mathbb{C}^p.$$

Lemma 2.11. *Let $f \in AAA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p) \cap \mathfrak{Lip}(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$ and $\psi \in AAA(\mathbb{R}, \mathbb{C}^p)$. Then, $f(\cdot, \psi(\cdot), \psi([\cdot])) \in ZAAA(\mathbb{R}, \mathbb{C}^p)$.*

Proof. Let

$$f = f_1 + f_2 \quad \psi = \psi_1 + \psi_2,$$

where $f_1 \in AA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$, $\psi_1 \in AA(\mathbb{R}, \mathbb{C}^p)$ and $f_2 \in C_0(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$, $\psi_2 \in C_0(\mathbb{R}, \mathbb{C}^p)$. Then, we have

$$\begin{aligned} f(t, \psi(t), \psi([t])) &= f_1(t, \psi_1(t), \psi_1([t])) + f(t, \psi(t), \psi([t])) - f_1(t, \psi_1(t), \psi_1([t])) \\ &= f_1(t, \psi_1(t), \psi_1([t])) + f_1(t, \psi(t), \psi([t])) - f_1(t, \psi_1(t), \psi_1([t])) + f_2(t, \psi(t), \psi([t])). \end{aligned}$$

By a similar proof to that of (iii) in Lemma 2.10, one can show that $f_1 \in \mathfrak{Lip}(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$ with Lipschitz constant L_f . Combining this with [2, Lemma 2.7], we get $f_1(\cdot, \psi_1(\cdot), \psi_1([\cdot])) \in \mathbb{Z}AA(\mathbb{R}, \mathbb{C}^p)$. In addition, it follows from

$$|f_1(t, \psi(t), \psi([t])) - f_1(t, \psi_1(t), \psi_1([t]))| \leq L_f(|\psi_2(t)| + |\psi_2([t])|), \quad t \in \mathbb{R},$$

that $t \rightarrow f_1(t, \psi(t), \psi([t])) - f_1(t, \psi_1(t), \psi_1([t]))$ belongs to $\mathbb{Z}C_0(\mathbb{R}, \mathbb{C}^p)$. Moreover, it is easy to see that $f_2(\cdot, \psi(\cdot), \psi([\cdot])) \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{C}^p)$. This completes the proof. $\square$

Next, we recall some notions about discrete almost automorphic type functions (cf. [2, 3]).

Definition 2.12. A function $f: \mathbb{Z} \rightarrow \mathbb{X}$ is said to be discrete almost automorphic if given any sequence $\{\bar{s}_n\} \subset \mathbb{Z}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that

$$\tilde{f}(k) := \lim_{n \rightarrow \infty} f(k + s_n)$$

is well defined for every $k \in \mathbb{Z}$, and

$$\lim_{n \rightarrow \infty} \tilde{f}(k - s_n) = f(k)$$

for every $k \in \mathbb{Z}$. We denote the set of all such functions by $AA(\mathbb{Z}, \mathbb{X})$.

Definition 2.13. A function $G: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{X}$ is said to be discrete Bi-almost automorphic if given any sequence $\{\bar{s}_n\} \subset \mathbb{Z}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that

$$\tilde{G}(k, m) := \lim_{n \rightarrow \infty} G(k + s_n, m + s_n)$$

is well defined for every $k, m \in \mathbb{Z}$, and

$$\lim_{n \rightarrow \infty} \tilde{G}(k - s_n, m - s_n) = G(k, m)$$

for every $k, m \in \mathbb{Z}$. We denote the set of all such functions by $BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$.

Definition 2.14. A function $f: \mathbb{Z} \rightarrow \mathbb{X}$ is said to be discrete asymptotically almost automorphic iff it admits a decomposition

$$f = g + h,$$

where $g \in AA(\mathbb{Z}, \mathbb{X})$ and $h \in C_0(\mathbb{Z}, \mathbb{X}) := \{h: \mathbb{Z} \rightarrow \mathbb{X} : \lim_{k \rightarrow \infty} h(k) = 0\}$. Denote the set of all such functions by $AAA(\mathbb{Z}, \mathbb{X})$.

In order to establish our main results, we introduce the following notion of Bi-asymptotically almost automorphic functions.

Definition 2.15. A function $G: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{X}$ is said to be discrete Bi-asymptotically almost automorphic iff it admits a decomposition

$$G = H + I,$$

where $H \in BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$ and $I \in BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. Here,

$$BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X}) = \{I: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{X} : \lim_{n \rightarrow \infty} I(n + k, n + m) = 0 \text{ for every } k, m \in \mathbb{Z}\}.$$

We denote the set of all such functions by $BAAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$.

Remark 2.16. Note that $I \in BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$ does not imply that

$$\lim_{k, m \rightarrow \infty} I(k, m) = 0.$$

In fact, letting $I(k, m) = (k - m) \sin \frac{\pi}{k + m}$, for every $k, m \in \mathbb{Z}$, we have

$$I(n + k, n + m) = (k - m) \sin \frac{\pi}{k + m + 2n} \rightarrow 0, \quad n \rightarrow \infty.$$

However, $I(2m, m) = m \sin \frac{\pi}{3m} \rightarrow \frac{\pi}{3}$ as $m \rightarrow \infty$.

Lemma 2.17. The following assertions hold:

- (i) Let $G \in BAAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. Then, for every $k, m \in \mathbb{Z}$, $G(k + \cdot, m + \cdot)$ is bounded on $\mathbb{Z}$.
- (ii) Let X be a Banach algebra and $G_1, G_2 \in BAAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. Then, $G_1 \cdot G_2 \in BAAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$.
- (iii) Let $G = H + I$, where $H \in BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$ and $I \in BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. Then, for every $(k, m) \in \mathbb{Z} \times \mathbb{Z}$, there holds

$$\{H(k + n, m + n) : n \in \mathbb{Z}\} \subset \overline{\{G(k + n, m + n) : n \in \mathbb{Z}\}}.$$

Proof. Fix $k, m \in \mathbb{Z}$. Let $G = H + I$, where $H \in BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$ and $I \in BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. We claim that $H(k + \cdot, m + \cdot)$ is bounded on $\mathbb{Z}$. In fact, if this is not true, there exists a sequence $\{s_n\} \subset \mathbb{Z}$ such that

$$\lim_{n \rightarrow \infty} H(k + s_n, m + s_n) = \infty.$$

This contradicts with $H \in BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$. Moreover, it follows from the definition of $BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$ that $I(k + \cdot, m + \cdot)$ is bounded on $\mathbb{Z}$. Thus, $G(k + \cdot, m + \cdot)$ is bounded on $\mathbb{Z}$.

The proof of (ii) follows from (i) and the definition of $BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$.

It remains to show (iii). Fix $(k, m) \in \mathbb{Z} \times \mathbb{Z}$. By the definition of $BAA(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$, there exists $\{s_l\} \subset \mathbb{Z}$ with $\lim_{l \rightarrow \infty} s_l = \infty$ and a function $\tilde{H}$ such that

$$\lim_{l \rightarrow \infty} H(k + n + s_l, m + n + s_l) = \tilde{H}(k + n, m + n), \quad n \in \mathbb{Z}, \quad (2)$$

and

$$\lim_{l \rightarrow \infty} \tilde{H}(k + n - s_l, m + n - s_l) = H(k + n, m + n), \quad n \in \mathbb{Z}. \quad (3)$$

It is easy to see from (3) that

$$\{H(k + n, m + n) : n \in \mathbb{Z}\} \subset \overline{\{\tilde{H}(k + n, m + n) : n \in \mathbb{Z}\}}.$$

On the other hand, we have

$$\begin{aligned} & \|\tilde{H}(k + n, m + n) - G(k + n + s_l, m + n + s_l)\| \\ & \leq \|\tilde{H}(k + n, m + n) - H(k + n + s_l, m + n + s_l)\| + \|I(k + n + s_l, m + n + s_l)\|. \end{aligned}$$

Combining this with (2) and $I \in BC_0(\mathbb{Z} \times \mathbb{Z}, \mathbb{X})$, we get

$$\lim_{l \rightarrow \infty} G(k + n + s_l, m + n + s_l) = \tilde{H}(k + n, m + n), \quad n \in \mathbb{Z},$$

which means that

$$\{\tilde{H}(k + n, m + n) : n \in \mathbb{Z}\} \subset \overline{\{G(k + n, m + n) : n \in \mathbb{Z}\}}.$$

Thus, $\{H(k + n, m + n) : n \in \mathbb{Z}\} \subset \overline{\{G(k + n, m + n) : n \in \mathbb{Z}\}}$. $\square$

Before closing this section, we recall and introduce another two notions (the first one has been mentioned in [3]).

Definition 2.18. A function $G : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{X}$ is said to be *Bi-almost automorphic* if for every $\{\bar{s}_n\} \subset \mathbb{R}$, there exists a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ such that

$$\tilde{G}(s, t) := \lim_{n \rightarrow \infty} G(s + s_n, t + s_n)$$

is well defined for every $(s, t) \in \mathbb{R}^2$, and

$$\lim_{n \rightarrow \infty} \tilde{G}(s - s_n, t - s_n) = G(s, t)$$

for every $(s, t) \in \mathbb{R}^2$. We denote the set of all such functions by $BAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Definition 2.19. A function $G : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{X}$ is said to be *Bi-asymptotically almost automorphic* iff it admits a decomposition

$$G = H + I,$$

where $H \in BAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ and $I \in BC_0(\mathbb{R} \times \mathbb{R}, \mathbb{X})$. Here,

$$BC_0(\mathbb{R} \times \mathbb{R}, \mathbb{X}) = \{I : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{X} : \lim_{r \rightarrow \infty} I(s + r, t + r) = 0, (s, t) \in \mathbb{R}^2\}.$$

We denote the set of all such functions by $BAAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

3 Difference equations

To study equation (1), let us first consider the following linear nonhomogeneous DEPCA:

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t). \quad (4)$$

Let y be an arbitrary solution of (4) on $\mathbb{R}$. Then, by the variation of constants formula, there holds,

$$y(t) = \left[\Phi(t, n) + \int_n^t \Phi(t, u)B(u)du \right] y(n) + \int_n^t \Phi(t, u)f(u)du, \quad t \in [n, n+1), \quad n \in \mathbb{Z}, \quad (5)$$

where $\Phi(t, s) := \Phi(t)\Phi^{-1}(s)$ and $\Phi(t)$ is a fundamental matrix solution of the system

$$x'(t) = A(t)x(t).$$

Since y is continuous on $\mathbb{R}$, taking $t \rightarrow (n+1)^-$ in the equation (5), we obtain the difference system

$$y(n+1) = C(n)y(n) + h(n), \quad n \in \mathbb{Z}, \quad (6)$$

where $h(n) = \int_n^{n+1} \Phi(n+1, u)f(u)du$, and

$$C(n) = \Phi(n+1, n) + \int_n^{n+1} \Phi(n+1, u)B(u)du.$$

From the above observations, naturally, we need to consider the difference system (6). Firstly, we investigate if $C(n)$ and $h(n)$, $n \in \mathbb{Z}$, are discrete asymptotically almost automorphic when the coefficients $A(t)$, $B(t)$ and $f(t)$, $t \in \mathbb{R}$, are asymptotically almost automorphic.

Lemma 3.1. *Let $A = A_1 + A_2 \in AAA(\mathbb{R}, M_{p \times p}(\mathbb{C}))$, $B \in AAA(\mathbb{R}, M_{p \times p}(\mathbb{C}))$, and $f \in ZAAA(\mathbb{R}, \mathbb{C}^p)$, where $A_1 \in AA(\mathbb{R}, M_{p \times p}(\mathbb{C}))$ and $A_2 \in C_0(\mathbb{R}, M_{p \times p}(\mathbb{C}))$. Also, let Φ and Φ_1 be the fundamental matrix solutions of systems $x'(t) = A(t)x(t)$ and $x'(t) = A_1(t)x(t)$, respectively. Moreover, let $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$, $\Phi_1(t, s) = \Phi_1(t)\Phi_1^{-1}(s)$ and $\Phi_2 = \Phi - \Phi_1$. Then, the following assertions hold:*

(i) *For every positive real number ℓ , there exists a constant $k_\ell > 0$ such that*

$$|\Phi(t, s)| \leq k_\ell, \quad |\Phi_1(t, s)| \leq k_\ell, \quad (t, s) \in E_\ell := \{(t, s) \in \mathbb{R}^2 : |t - s| \leq \ell\}.$$

(ii) $\Phi_1 \in BAA(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$, $\Phi_2 \in BC_0(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$ and $\Phi \in BAAA(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$.

(iii) $n \mapsto \Phi(n+1, n)$, $n \mapsto \int_n^{n+1} \Phi(n+1, u)B(u)du$ and $n \mapsto \int_n^{n+1} \Phi(n+1, u)f(u)du$ are all discrete asymptotically almost automorphic. Thus, the two functions $C(n)$ and $h(n)$ in equation (6) are both asymptotically almost automorphic.

(iv) *The functions*

$$\Phi(t, [t]), \quad \int_{[t]}^t \Phi(t, u)B(u)du, \quad \int_{[t]}^t \Phi(t, u)f(u)du,$$

are all $\mathbb{Z}$ -asymptotically almost automorphic.

Proof. The proof of (i) has been essentially given in [3, Lemma 3.2]. Here, for the reader's convenience, we give a sketch of the proof. For $0 \leq t - s \leq \ell$, by using Gronwall's inequality and

$$\Phi(t, s) = I + \int_s^t A(u)\Phi(u, s)du,$$

where I is the identity matrix in $M_{p \times p}(\mathbb{C})$, one can show that $|\Phi(t, s)| \leq |I|e^{\ell\|A\|}$. For $-\ell \leq t - s \leq 0$, similarly, by using

$$\Phi(t, s) = I + \int_s^t \Phi(t, u)A(u)du,$$

one can also show that $|\Phi(t, s)| \leq |I|e^{\ell\|A\|}$. Analogously, one can prove that $|\Phi_1(t, s)| \leq |I|e^{\ell\|A_1\|}$ for every $(t, s) \in E_\ell$. In addition, by (i) of Lemma 2.4, $\{A_1(t) : t \in \mathbb{R}\} \subset \{\overline{A(t)} : t \in \mathbb{R}\}$, which implies that $\|A_1\| \leq \|A\|$, and thus

$$|\Phi_1(t, s)| \leq |I|e^{\ell\|A\|}, \quad (t, s) \in E_\ell.$$

Next, let us prove (ii). Since $A_1 \in AA(\mathbb{R}, M_{p \times p}(\mathbb{C}))$, for every $\{\bar{s}_n\} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subset \{\bar{s}_n\}$ and a function $\widetilde{A}_1$, such that

$$\lim_{n \rightarrow \infty} A_1(t + s_n) = \widetilde{A}_1(t), \quad \lim_{n \rightarrow \infty} \widetilde{A}_1(t - s_n) = A_1(t).$$

Fix $s, t \in \mathbb{R}$ with $t \geq s$. Similar to (i), we have

$$\Phi_1(t, s) = I + \int_s^t A_1(u)\Phi_1(u, s)du, \quad \widetilde{\Phi}_1(t, s) = I + \int_s^t \widetilde{A}_1(u)\widetilde{\Phi}_1(u, s)du,$$

where $\widetilde{\Phi}_1(t)$ is the fundamental matrix solution of system $x'(t) = \widetilde{A}_1(t)x(t)$ and $\widetilde{\Phi}_1(t, s) = \widetilde{\Phi}_1(t)\widetilde{\Phi}_1^{-1}(s)$. Then, by using $\|\widetilde{A}_1\| \leq \|A_1\| \leq \|A\|$, we get

$$\begin{aligned} & |\Phi_1(t + s_n, s + s_n) - \widetilde{\Phi}_1(t, s)| \\ &= \left| \int_s^t A_1(u + s_n)\Phi_1(u + s_n, s + s_n)du - \int_s^t \widetilde{A}_1(u)\widetilde{\Phi}_1(u, s)du \right| \\ &\leq \int_s^t |A_1(u + s_n) - \widetilde{A}_1(u)| \cdot |\Phi_1(u + s_n, s + s_n)|du + \|\widetilde{A}_1\| \int_s^t |\Phi_1(u + s_n, s + s_n) - \widetilde{\Phi}_1(u, s)|du \\ &\leq k_{t-s} \int_s^t |A_1(u + s_n) - \widetilde{A}_1(u)|du + \|A\| \int_s^t |\Phi_1(u + s_n, s + s_n) - \widetilde{\Phi}_1(u, s)|du, \end{aligned}$$

which yields for $n \rightarrow \infty$

$$|\Phi_1(t + s_n, s + s_n) - \widetilde{\Phi}_1(t, s)| \leq e^{(t-s)\|A\|} \cdot k_{t-s} \int_s^t |A_1(u + s_n) - \widetilde{A}_1(u)|du \rightarrow 0.$$

Analogously to the above proof, one can show that $\lim_{n \rightarrow \infty} \widetilde{\Phi}_1(t - s_n, s - s_n) = \Phi_1(t, s)$ and similarly for the case of $t < s$. This means that $\Phi_1 \in BAA(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$. It remains to prove that $\Phi_2 \in BC_0(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$. We only consider the case of $s \leq t$. By a direct calculation, we have

$$\begin{aligned} |\Phi_2(t + r, s + r)| &= |\Phi(t + r, s + r) - \Phi_1(t + r, s + r)| \\ &= \left| I + \int_{s+r}^{t+r} A(u)\Phi(u, s + r)du - I - \int_{s+r}^{t+r} A_1(u)\Phi_1(u, s + r)du \right| \\ &\leq \int_{s+r}^{t+r} |A_2(u)\Phi(u, s + r)|du + \int_{s+r}^{t+r} |A_1(u)\Phi_2(u, s + r)|du \\ &\leq k_{t-s} \cdot (t - s) \cdot \sup_{u \in [s+r, t+r]} |A_2(u)| + \|A\| \int_s^t |\Phi_2(u + r, s + r)|du, \end{aligned}$$

which means that

$$|\Phi_2(t+r, s+r)| \leq k_{t-s} \cdot (t-s) \cdot \sup_{u \in [s+r, t+r]} |A_2(u)| \cdot e^{\|A\|(t-s)} \rightarrow 0, \quad r \rightarrow \infty.$$

Therefore, $\Phi_2 \in BC_0(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$, and thus $\Phi \in BAAA(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$.

Now, let's come to the proof of (iii). Firstly, by (ii), it is easy to see that

$$\Phi(\cdot + 1, \cdot) \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C})).$$

The proof for $n \mapsto \int_n^{n+1} \Phi(n+1, u)B(u)du$ is similar to that of $n \mapsto \int_n^{n+1} \Phi(n+1, u)f(u)du$. So, we only prove $n \mapsto \int_n^{n+1} \Phi(n+1, u)f(u)du$ belongs to $AAA(\mathbb{Z}, \mathbb{C}^p)$. Observe that

$$\begin{aligned} & \int_n^{n+1} \Phi(n+1, u)f(u)du \\ &= \int_n^{n+1} \Phi_1(n+1, u)f_1(u)du + \int_n^{n+1} \Phi_1(n+1, u)f_2(u)du + \int_n^{n+1} \Phi_2(n+1, u)f(u)du, \end{aligned}$$

where $f = f_1 + f_2$, $f_1 \in \mathbb{Z}AA(\mathbb{R}, \mathbb{C}^p)$ and $f_2 \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{C}^p)$. By [3, Lemma 3.3], $n \mapsto \int_n^{n+1} \Phi_1(n+1, u)f_1(u)du$ belongs to $AA(\mathbb{Z}, \mathbb{C}^p)$. By using Lebesgue dominated convergence theorem and (i), we have

$$\lim_{n \rightarrow \infty} \int_n^{n+1} \Phi_1(n+1, u)f_2(u)du = \lim_{n \rightarrow \infty} \int_0^1 \Phi_1(n+1, u+n)f_2(u+n)du = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_n^{n+1} \Phi_2(n+1, u)f(u)du = \lim_{n \rightarrow \infty} \int_0^1 \Phi_2(n+1, n+u)f(u+n)du = 0.$$

Thus, we conclude that $n \mapsto \int_n^{n+1} \Phi(n+1, u)f(u)du$ is discrete asymptotically almost automorphic.

It remains to prove (iv). It follows from $\Phi \in BAAA(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$ that the functions $\Phi(t, [t])$ is $\mathbb{Z}$ -asymptotically almost automorphic. As in the proof of (iii), let $f = f_1 + f_2$, $f_1 \in \mathbb{Z}AA(\mathbb{R}, \mathbb{C}^p)$ and $f_2 \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{C}^p)$. Then, there holds

$$\int_{[t]}^t \Phi(t, u)f(u)du = \int_{[t]}^t \Phi_1(t, u)f_1(u)du + \int_{[t]}^t \Phi_1(t, u)f_2(u)du + \int_{[t]}^t \Phi_2(t, u)f(u)du.$$

Again by [3, Lemma 3.3], we get

$$t \mapsto \int_{[t]}^t \Phi_1(t, u)f_1(u)du$$

is $\mathbb{Z}$ -almost automorphic. Moreover, by (i), $f_2 \in \mathbb{Z}C_0(\mathbb{R}, \mathbb{C}^p)$, and $\Phi_2 \in BC_0(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$, we conclude

$$\left| \int_{[t]}^t \Phi_1(t, u)f_2(u)du \right| \leq k_1 \cdot \sup_{[t] \leq u \leq t} |f_2(u)| \rightarrow 0, \quad t \rightarrow \infty,$$

and

$$\left| \int_{[t]}^t \Phi_2(t, u)f(u)du \right| \leq \sup_{[t] \leq u \leq t} |\Phi_2(t, u)| \cdot \|f\| \rightarrow 0, \quad t \rightarrow \infty.$$

Indeed, by a similar proof to that of $\Phi_2 \in BC_0(\mathbb{R} \times \mathbb{R}, M_{p \times p}(\mathbb{C}))$ in (ii), one can obtain

$$|\Phi_2(t, u)| \leq k_{t-u} \cdot \sup_{u \leq v \leq t} |A_2(v)| \cdot e^{\|A\|(t-u)} \leq k_1 \sup_{[t] \leq v \leq t} |A_2(v)| \cdot e^{\|A\|}, \quad [t] \leq u \leq t,$$

which yields that

$$\sup_{[t] \leq u \leq t} |\Phi_2(t, u)| \leq k_1 e^{\|A\|} \cdot \sup_{[t] \leq v \leq t} |A_2(v)| \rightarrow 0, \quad t \rightarrow \infty.$$

In conclusion, we know that the function

$$t \mapsto \int_{[t]}^t \Phi(t, u) f(u) du$$

is $\mathbb{Z}$ -asymptotically almost automorphic. Analogously, one can also obtain that the function

$$t \mapsto \int_{[t]}^t \Phi(t, u) B(u) du$$

is $\mathbb{Z}$ -asymptotically almost automorphic. □

Let us recall the notions of exponential dichotomy and Green's function for homogenous difference system

$$y(n+1) = C(n)y(n), \quad n \in \mathbb{Z}, \quad (7)$$

where $C(n) \in M_{p \times p}(\mathbb{C})$ is invertible and $y(n) \in \mathbb{C}^p$, $n \in \mathbb{Z}$. We will also study asymptotically almost automorphy for the Green's function of difference system (7).

Definition 3.2 (cf. [3]). *Let $Y(n)$ be a fundamental matrix of difference system (7). The system (7) is said to have an exponential dichotomy with parameters (α, K, P) if there exist a projection P , which commutes with $Y(n)$, and positive constants $K, \alpha > 0$ such that*

$$|G(n, m)| \leq K e^{-\alpha|n-m|}, \quad n, m \in \mathbb{Z},$$

where

$$G(n, m) := \begin{cases} Y(n) P Y^{-1}(m), & n \geq m, \\ -Y(n)(I - P) Y^{-1}(m), & n < m, \end{cases}$$

is called the Green's function of (7).

Before discussing the asymptotically almost automorphy of Green's function, we first establish the following lemma, which is an extension of [6, Lemma 2.9].

Lemma 3.3. *Let $C \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$ such that every $C(n)$ is invertible and the set $\{C^{-1}(n) : n \in \mathbb{Z}\}$ is bounded. Then, $C^{-1} \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$.*

Proof. Let

$$C(n) = C_1(n) + C_2(n), \quad n \in \mathbb{Z},$$

where $C_1 \in AA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$ and $C_2 \in C_0(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$. By (iii) of Lemma 2.17, we have

$$\{C_1(n) : n \in \mathbb{Z}\} \subset \overline{\{C(n) : n \in \mathbb{Z}\}}.$$

Then, for every $n \in \mathbb{Z}$, there exists a sequence $\{s_k\} \subset \mathbb{Z}$ such that

$$C(s_k) \rightarrow C_1(n), \quad k \rightarrow \infty.$$

Since $C(s_k)$ is invertible, the set $\{C^{-1}(n) : n \in \mathbb{Z}\}$ is bounded, and

$$C_1(n) = C(s_k) + C_1(n) - C(s_k) = C(s_k) \cdot \left(I + C^{-1}(s_k)[C_1(n) - C(s_k)] \right),$$

we conclude that $C_1(n)$ is invertible. Moreover, there holds

$$C^{-1}(s_k) - C^{-1}(n) = C_1^{-1}(n)[C_1(n) - C(s_k)]C^{-1}(s_k),$$

which means that $C^{-1}(s_k) \rightarrow C_1^{-1}(n)$ as $k \rightarrow \infty$, and thus

$$\{C_1^{-1}(n) : n \in \mathbb{Z}\} \subset \overline{\{C^{-1}(n) : n \in \mathbb{Z}\}}.$$

Then, it follows that $\{C_1^{-1}(n) : n \in \mathbb{Z}\}$ is also bounded, and thus, by [6, Lemma 2.9], $n \mapsto C_1^{-1}(n)$ belongs to $AA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$. In addition, by $C_2 \in C_0(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$, we have

$$C^{-1}(n) - C_1^{-1}(n) = C_1^{-1}(n)[C_1(n) - C(n)]C^{-1}(n) = -C_1^{-1}(n)C_2(n)C^{-1}(n) \rightarrow 0, \quad n \rightarrow \infty.$$

Then, we conclude that $n \mapsto C^{-1}(n)$ belongs to $AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$. $\square$

Lemma 3.4. *Let $C \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$ such that every $C(n)$ is invertible and the set $\{C^{-1}(n) : n \in \mathbb{Z}\}$ is bounded. Also, assume that system (7) has an exponential dichotomy with parameters (α, K, P) . Then, the Green's function $G \in BAAA(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$.*

Proof. For every $n, m \in \mathbb{Z}$, we denote $Y(n, m) = Y(n)Y^{-1}(m)$. Then,

$$Y(n, m) = \begin{cases} \prod_{l=m}^{n-1} C(m+n-1-l), & n \geq m, \\ \prod_{l=n}^{m-1} C^{-1}(l), & n < m. \end{cases}$$

Noting that P commutes with $Y(n)$, it suffices to prove that $Y(n, m)$ is Bi -asymptotically almost automorphic.

Let $C(n) = C_1(n) + C_2(n)$, $n \in \mathbb{Z}$, where $C_1 \in AA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$ and $C_2 \in C_0(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$. By Lemma 3.3, $C^{-1}(n)$ is also asymptotically almost automorphic and $C_1^{-1}(n)$ is just the almost automorphic component of $C^{-1}(n)$. Denote

$$Y_1(n, m) = \begin{cases} \prod_{l=m}^{n-1} C_1(m+n-1-l), & n \geq m, \\ \prod_{l=n}^{m-1} C_1^{-1}(l), & n < m. \end{cases}$$

Taking an arbitrary sequence $\{\bar{s}_k\} \subset \mathbb{R}$, there exist a subsequence $\{s_k\}$, and two functions D, E such that

$$\lim_{k \rightarrow \infty} C_1(n + s_k) = D(n), \quad \lim_{k \rightarrow \infty} D(n - s_k) = C_1(n), \quad n \in \mathbb{Z},$$

and

$$\lim_{k \rightarrow \infty} C_1^{-1}(n + s_k) = E(n), \quad \lim_{k \rightarrow \infty} E(n - s_k) = C_1^{-1}(n), \quad n \in \mathbb{Z}.$$

Denote

$$\widetilde{Y}_1(n, m) = \begin{cases} \prod_{l=m}^{n-1} D(m+n-1-l), & n \geq m, \\ \prod_{l=n}^{m-1} E(l), & n < m. \end{cases}$$

Fix $n, m \in \mathbb{Z}$ with $n > m$. We have

$$\begin{aligned} Y_1(n + s_k, m + s_k) &= \prod_{l=m+s_k}^{n+s_k-1} C_1(m+n+2s_k-1-l) \\ &= \prod_{l=m}^{n-1} C_1(m+n-1-l+s_k) \\ &\rightarrow \prod_{l=m}^{n-1} D(m+n-1-l) = \widetilde{Y}_1(n, m), \end{aligned}$$

as $k \rightarrow \infty$. Analogously, one can show that

$$\widetilde{Y}_1(n - s_k, m - s_k) \rightarrow Y_1(n, m), \quad k \rightarrow \infty.$$

The proof for the case of $n < m$ is fully similar to the above proof and the proof for the case of $m = n$ is trivial. This shows that $Y_1 \in BAA(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$.

Moreover, we claim that $[Y - Y_1] \in BC_0(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$. In fact, for $n, m \in \mathbb{Z}$ with $n > m$, we have

$$\begin{aligned} & Y(n+k, m+k) - Y_1(n+k, m+k) \\ &= \prod_{l=m+k}^{n+k-1} C(m+n+2k-1-l) - \prod_{l=m+k}^{n+k-1} C_1(m+n+2k-1-l) \\ &= \prod_{l=m}^{n-1} C(m+n+k-1-l) - \prod_{l=m}^{n-1} C_1(m+n-1-l+k) \rightarrow 0, \quad k \rightarrow \infty, \end{aligned}$$

since $C(n)$ and $C_1(n)$ are both bounded, and as $k \rightarrow \infty$,

$$C(m+n+k-1-l) - C_1(m+n-1-l+k) = C_2(m+n-1-l+k) \rightarrow 0, \quad l = m, m+1, \dots, n-1.$$

This shows that $Y \in BAAA(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$. $\square$

4 AAA solutions

In this section, we discuss the existence of asymptotically almost automorphic solutions for difference system (6) and DEPCA (1).

Theorem 4.1. *Let $C \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$ such that every $C(n)$ is invertible and the set $\{C^{-1}(n) : n \in \mathbb{Z}\}$ is bounded. Also, assume that $h \in AAA(\mathbb{Z}, \mathbb{C}^p)$ and system (7) has an exponential dichotomy with parameters (α, K, P) . Then, system (6) has a unique asymptotically almost automorphic solution given by*

$$y(n) = \sum_{k \in \mathbb{Z}} G(n, k+1)h(k), \quad n \in \mathbb{Z}.$$

Moreover $|y(n)| \leq K(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1} \|h\|$ for all $n \in \mathbb{Z}$.

Proof. By [10, Theorem 5.7], system (6) has a unique bounded solution $y(n)$ given by

$$y(n) = \sum_{k \in \mathbb{Z}} G(n, k+1)h(k), \quad n \in \mathbb{Z}.$$

It follows from Lemma 3.4 that $G \in BAAA(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$. Thus, we have the following decomposition:

$$\begin{aligned} y(n) &= \sum_{k \in \mathbb{Z}} G(n, k+1)h(k) \\ &= \sum_{k \in \mathbb{Z}} G_1(n, k+1)h_1(k) + \sum_{k \in \mathbb{Z}} G_1(n, k+1)h_2(k) + \sum_{k \in \mathbb{Z}} G_2(n, k+1)h(k), \end{aligned}$$

where $G = G_1 + G_2$, $h = h_1 + h_2$, $G_1 \in BAA(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$, $h_1 \in AA(\mathbb{Z}, \mathbb{C}^p)$, $G_2 \in BC_0(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$, $h_2 \in C_0(\mathbb{Z}, \mathbb{C}^p)$.

It follows from [2, Theorem 3.4] (see also [6, Theorem 3.1]) that

$$\sum_{k \in \mathbb{Z}} G_1(\cdot, k+1)h_1(k) \in AA(\mathbb{Z}, \mathbb{C}^p).$$

There holds

$$\sum_{k \in \mathbb{Z}} G_1(n, k+1)h_2(k) = \sum_{k=-\infty}^{n-1} G_1(n, k+1)h_2(k) + \sum_{k=n}^{+\infty} G_1(n, k+1)h_2(k)$$

$$= \sum_{k=-\infty}^0 G_1(n, k+n)h_2(k+n-1) + \sum_{k=1}^{+\infty} G_1(n, k+n)h_2(k+n-1).$$

By using the Lebesgue dominated convergence theorem, $h_2 \in C_0(\mathbb{Z}, \mathbb{C}^p)$, and

$$|G_1(n, k+n)| \leq Ke^{-\alpha|k|}, \quad k, n \in \mathbb{Z},$$

which is deduced from

$$\{G_1(n, k+n) : n \in \mathbb{Z}\} \subset \overline{\{G(n, k+n) : n \in \mathbb{Z}\}} \quad (\text{see (iii) of Lemma 2.17}),$$

we conclude

$$\lim_{n \rightarrow \infty} \sum_{k=-\infty}^0 G_1(n, k+n)h_2(k+n-1) = \lim_{n \rightarrow \infty} \sum_{k=1}^{+\infty} G_1(n, k+n)h_2(k+n-1) = 0,$$

which yields that $\sum_{k \in \mathbb{Z}} G_1(\cdot, k+1)h_2(k) \in C_0(\mathbb{Z}, \mathbb{C}^p)$.

It remains to show that $\sum_{k \in \mathbb{Z}} G_2(\cdot, k+1)h(k) \in C_0(\mathbb{Z}, \mathbb{C}^p)$. Noting that

$$\sum_{k \in \mathbb{Z}} G_2(n, k+1)h(k) = \sum_{k \in \mathbb{Z}} G_2(n, k+n)h(k+n-1),$$

for every $k \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} G_2(n, k+n) = 0$ since $G_2 \in BC_0(\mathbb{Z} \times \mathbb{Z}, M_{p \times p}(\mathbb{C}))$, and

$$|G_2(n, k+n)| = |G(n, k+n) - G_1(n, k+n)| \leq 2Ke^{-\alpha|k|}, \quad k, n \in \mathbb{Z},$$

again by the Lebesgue dominated convergence theorem, we get

$$\lim_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}} G_2(n, k+1)h(k) = 0.$$

This completes the proof. $\square$

Theorem 4.2. Let $A, B \in AAA(\mathbb{R}, M_{p \times p}(\mathbb{C}))$, $f \in AAA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p) \cap \mathcal{Lip}(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$, and

$$\left\{ \left(I + \int_n^{n+1} \Phi(n, u)B(u)du \right)^{-1} \right\}_{n \in \mathbb{Z}}$$

be bounded, where $\Phi(t, s) := \Phi(t)\Phi^{-1}(s)$ and $\Phi(t)$ is a fundamental matrix solution of $x'(t) = A(t)x(t)$. Also, assume that the system $y(n+1) = C(n)y(n)$ has an exponential dichotomy with parameters (α, K, P) , where $C(n) = \Phi(n+1, n) + \int_n^{n+1} \Phi(n+1, u)B(u)du$. Then, there exists a constant $L^* > 0$, such that equation (1) has a unique asymptotically almost automorphic solution provided $L_f < L^*$.

Proof. Taking arbitrary $\psi \in AAA(\mathbb{R}, \mathbb{C}^p)$, consider the following equation

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t, \psi(t), \psi([t])). \quad (8)$$

By the arguments in the beginning of Section 3, we know that y_ψ is a solution of equation (8) if and only if for every $n \in \mathbb{Z}$ and $t \in [n, n+1)$,

$$y_\psi(t) = \left[\Phi(t, n) + \int_n^t \Phi(t, u)B(u)du \right] y_\psi(n) + \int_n^t \Phi(t, u)f(u, \psi(u), \psi([u]))du, \quad (9)$$

and

$$y_\psi(n+1) = C(n)y_\psi(n) + h_\psi(n), \quad (10)$$

where $h_\psi(n) = \int_n^{n+1} \Phi(n+1, u) f(u, \psi(u), \psi([u])) du$, $n \in \mathbb{Z}$.

Since $f \in AAA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p) \cap \mathfrak{Lip}(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p, \mathbb{C}^p)$, by Lemma 2.11, $f(\cdot, \psi(\cdot), \psi([\cdot])) \in ZAAA(\mathbb{R}, \mathbb{C}^p)$. Then, by (iii) of Lemma 3.1, $h_\psi \in AAA(\mathbb{Z}, \mathbb{C}^p)$ and $C \in AAA(\mathbb{Z}, M_{p \times p}(\mathbb{C}))$. Note that for every $n \in \mathbb{Z}$, $I + \int_n^{n+1} \Phi(n, u) B(u) du$ being invertible implies that

$$\begin{aligned} C(n) &= \Phi(n+1, n) + \int_n^{n+1} \Phi(n+1, u) B(u) du \\ &= \Phi(n+1, n) \left[I + \int_n^{n+1} \Phi(n, u) B(u) du \right] \end{aligned}$$

is also invertible. Moreover, we have

$$C^{-1}(n) = \left[I + \int_n^{n+1} \Phi(n, u) B(u) du \right]^{-1} \Phi(n, n+1),$$

which means that $\{C^{-1}(n)\}_{n \in \mathbb{Z}}$ is also bounded. Then, it follows from Theorem 4.1 that equation (10) has a unique asymptotically almost automorphic solution given by

$$y_\psi(n) = \sum_{k \in \mathbb{Z}} G(n, k+1) h_\psi(k), \quad n \in \mathbb{Z}.$$

Then, defining y_ψ by (9), we get a solution $y_\psi(t)$ of equation (8). Also, it is easy to see that $y_\psi(t)$ is the unique solution of equation (8).

Next, let us show that for every $\psi \in AAA(\mathbb{R}, \mathbb{C}^p)$, $y_\psi \in AAA(\mathbb{R}, \mathbb{C}^p)$. Observe that $y_\psi([\cdot]) \in ZAAA(\mathbb{R}, \mathbb{C}^p)$, $f(\cdot, \psi(\cdot), \psi([\cdot])) \in ZAAA(\mathbb{R}, \mathbb{C}^p)$, and

$$y_\psi(t) = \left[\Phi(t, [t]) + \int_{[t]}^t \Phi(t, u) B(u) du \right] y_\psi([t]) + \int_{[t]}^t \Phi(t, u) f(u, \psi(u), \psi([u])) du, \quad t \in \mathbb{R}. \quad (11)$$

It follows from (iv) of Lemma 3.1 that every term on the right-hand side of (11) belongs to $ZAAA(\mathbb{R}, \mathbb{C}^p)$, and thus $y_\psi \in ZAAA(\mathbb{R}, \mathbb{C}^p)$. On the other hand, since y_ψ is continuous on $\mathbb{R}$ and

$$y'_\psi(t) = A(t)y_\psi(t) + B(t)y_\psi([t]) + f(t, \psi(t), \psi([t])), \quad t \in (n, n+1), \quad n \in \mathbb{Z},$$

where every term on the right-hand side of the above equality is bounded on $\mathbb{R}$, we conclude (cf. [2, Lemma 4.1]) that $y_\psi(t)$ is uniformly continuous on $\mathbb{R}$. Then, by (v) of Lemma 2.10, $y_\psi \in AAA(\mathbb{R}, \mathbb{C}^p)$.

Now, let us show that equation (1) has a unique asymptotically almost automorphic solution. Define a mapping $\mathcal{M} : AAA(\mathbb{R}, \mathbb{C}^p) \rightarrow AAA(\mathbb{R}, \mathbb{C}^p)$ by

$$(\mathcal{M}\psi)(t) = y_\psi(t), \quad t \in \mathbb{R}, \quad \psi \in AAA(\mathbb{R}, \mathbb{C}^p).$$

From the above proof, $\mathcal{M}$ is well-defined. For every $\psi_1, \psi_2 \in AAA(\mathbb{R}, \mathbb{C}^p)$ and $n \in \mathbb{Z}$, there holds

$$\begin{aligned} |y_{\psi_1}(n) - y_{\psi_2}(n)| &= \left| \sum_{k \in \mathbb{Z}} G(n, k+1) h_{\psi_1}(k) - \sum_{k \in \mathbb{Z}} G(n, k+1) h_{\psi_2}(k) \right| \\ &\leq \frac{K(1+e^{-\alpha})}{1-e^{-\alpha}} \|h_{\psi_1} - h_{\psi_2}\| \\ &\leq \frac{K(1+e^{-\alpha})}{1-e^{-\alpha}} \cdot \sup_{n \in \mathbb{Z}} \int_n^{n+1} |\Phi(n+1, u)| \cdot |f(u, \psi_1(u), \psi_1(n)) - f(u, \psi_2(u), \psi_2(n))| du \\ &\leq \frac{2k_1 L_f K(1+e^{-\alpha})}{1-e^{-\alpha}} \cdot \|\psi_1 - \psi_2\|, \end{aligned}$$

which yields that for every $n \in \mathbb{Z}$ and $t \in [n, n+1)$, there holds

$$\begin{aligned} |(\mathcal{M}\psi_1)(t) - (\mathcal{M}\psi_2)(t)| &\leq \left| \Phi(t, n) + \int_n^t \Phi(t, u) B(u) du \right| \cdot |y_{\psi_1}(n) - y_{\psi_2}(n)| \\ &\quad + \int_n^t |\Phi(t, u)| \cdot |f(u, \psi_1(u), \psi_1([u])) - f(u, \psi_2(u), \psi_2([u]))| du \\ &\leq k_1(1 + \|B\|) \cdot |y_{\psi_1}(n) - y_{\psi_2}(n)| + 2k_1 L_f \cdot \|\psi_1 - \psi_2\| \\ &\leq 2 \left[\frac{k_1^2 L_f K(1 + e^{-\alpha})(1 + \|B\|)}{1 - e^{-\alpha}} + k_1 L_f \right] \cdot \|\psi_1 - \psi_2\|. \end{aligned}$$

Thus, we have

$$\|\mathcal{M}\psi_1 - \mathcal{M}\psi_2\| \leq \frac{L_f \|\psi_1 - \psi_2\|}{L^*},$$

where $L^* = \frac{1 - e^{-\alpha}}{2k_1^2 K(1 + e^{-\alpha})(1 + \|B\|) + 2k_1(1 - e^{-\alpha})}$. This means that in the case of $L_f < L^*$, $\mathcal{M}$ has a unique fixed point in $AAA(\mathbb{R}, \mathbb{C}^p)$, i.e., equation (1) has a unique asymptotically almost automorphic solution. $\square$

Remark 4.3. It is not difficult to give some sufficient conditions to ensure that the assumptions of Theorem 4.2 hold. In fact, if $\|B\|$ is sufficiently small, then

$$\left(I + \int_n^{n+1} \Phi(n, u) B(u) du \right)^{-1} = \sum_{n=0}^{+\infty} \left(- \int_n^{n+1} \Phi(n, u) B(u) du \right)^n$$

is well-defined and bounded for $n \in \mathbb{Z}$. If, in addition $\sup_{n \in \mathbb{Z}} |\Phi(n+1, n)| < 1$, then

$$\sup_{n \in \mathbb{Z}} |C(n)| \leq \sup_{n \in \mathbb{Z}} |\Phi(n+1, n)| \cdot \sup_{n \in \mathbb{Z}} \left| I + \int_n^{n+1} \Phi(n, u) B(u) du \right| < 1,$$

which means that the system (7) is exponentially stable (i.e., exponential dichotomy with $P = I$).

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