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Growth functions for some uniformly amenable groups

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Abstract: We present a simple constructive proof of the fact that every abelian discrete group is uniformly amenable. We improve the growth function obtained earlier and find the optimal growth function in a particular case. We also compute a growth function for some non-abelian uniformly amenable group.

Keywords: Abelian groups, Amenable groups, Uniformly amenable groups

MSC: 20F65, 20K21, 22D40

1 Introduction

A discrete group G (group G with the discrete topology) is amenable if and only if for every finite subset $A \subset G$ and every $t > 0$, there exists a non-empty finite subset $U \subset G$ such that $|AU| \le (1+t)|U|$ where $|U|$ denotes the cardinality of U.

A group G is uniformly amenable if the cardinality of the set U in the definition of amenability depends only on t and on the cardinality of A and not on the particular finite set A. More precisely (see [1]).

Definition 1.1. *A group G is uniformly amenable if there exists a* growth *function $q : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{N}$ such that for every finite subset $A \subset G$ and every $t > 0$,*
(*) *there exists a non-empty finite subset $U \subset G$, $|U| \le q(t, |A|)$ and $|AU| \le (1+t)|U|$.*

The condition in Definition 1.1 is called the *uniform Følner condition*. There is another approach to uniform amenability due to H. Kesten. It was originally introduced while considering so called symmetric random walks on groups, which are a kind of Markov chains with G as the state space (see [2]). However, Kesten's approach can be reformulated in a combinatorial manner involving only the group structure without referring to probabilistic issues.

Let A be a finite symmetric subset of G (that means $A^{-1} = A$). By $m_{2n}(A)$ we denote the number of all (ordered) $2n$-tuples $(a_1, \ldots, a_{2n})$ with $a_i \in A$, $i = 1, \ldots, 2n$ such that the product $a_1 \ldots a_{2n}$ equals e (the identity element of the group). The group G is uniformly amenable if and only if for any natural number k and for any A with $|A| = k$,

$$(m_{2n}(A))^{\frac{1}{2n}} \rightrightarrows k,$$

where the convergence is uniform for all symmetric subsets with the same cardinality k. This condition is called the *uniform Kesten condition*. The equivalence between the uniform Følner and Kesten conditions was proved by Wysoczański in [10] using ultrapowers of groups.

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There is an interesting relationship between the growth function q and values of m_{2n}, namely, according to Kaimanovich, Vershik ([3] and [4]), for a uniformly amenable group G the following inequality holds for all symmetric subsets A with cardinality k

$$m_{2n}(A) \geq k^{2n} \sup_{0<t<\frac{1}{4},\, 2\sqrt{t}<h<1} (1-h)^n \frac{1-\frac{4t}{h^2}}{q(t,k+1)}.$$

If additionally A does not contain e and elements of order two, the estimation can be refined to

$$m_{2n}(A) \geq k^{2n} \sup_{0<t<\frac{1}{4},\, 2\sqrt{t}<h<1} (1-h)^n \frac{1-\frac{4t}{h^2}}{q(t,\frac{k}{2}+1)}.$$

Uniformly amenable groups are obviously amenable. The group S_∞ of permutations of $\mathbb{N}$ which move only a finite number of elements is an example of a group which is amenable but not uniformly. It is known ([5]) that the class of uniformly amenable groups is closed under extensions and taking subgroups.

More recently some results on uniformly amenable groups were obtained in [6].

Uniformly amenable groups are related to invariant uniform approximation in Banach spaces. We recall briefly this connection in order to motivate our results.

Let G be a compact abelian group, let Γ be the discrete abelian group of the characters of G. Let $X = L^1(G)$ be the Banach space of complex functions on G absolutely integrable functions with respect to Haar measure on G. For $g \in X$ and $\gamma \in \Gamma$ let $\hat{g}(\gamma) = \int g(x)\gamma(x)dx$. We treat $\hat{g}$ as a functional on Γ. We say that X has an invariant uniform approximation property (inv. ubap) if there exists a uniform bound $\beta : \mathbb{R}\times\mathbb{N} \to \mathbb{N}$, such that for every $\epsilon > 0$ and every finite subset $M \subset \Gamma$ there exists a function $g \in X$ with $\|g\|_1 \leq (1+\epsilon)$ and $|supp(\hat{g})| \leq \beta(\epsilon, |M|)$.

In [7], Lemma 2.1 it was proven that if Γ has a growth function $q(t,k)$ then X has a uniform bound $\beta(\epsilon,k) \leq q(\epsilon,k)^2$. It was also proven that every Γ has a growth function $q(t,k) = (k/t)^k$ and therefore X has a uniform bound $\beta(\epsilon,k) \leq (k/\epsilon)^{2k}$.

J. Bourgain gave a direct proof in [8] that X has inv. ubap but his proof is difficult to understand and the result is described in terms of entropy numbers.

P. Wojtaszczyk interpreted in [9] (page 209, Theorem 13) the result of Bourgain as a uniform bound $\beta(\epsilon,k) \leq (c/\epsilon)^{2k}$ where c is a constant depending on the group G. He also gave a proof of this fact using functional analysis.

In the present paper we give an elementary simple proof that every discrete abelian group has a growth function $q(t,k+1) = \binom{n+k-1}{k}$, where $t = k/n$. By the Stirling formula this is smaller than $(2e/t)^k$. By Lemma 2.1 in [7] it gives a universal bound $\beta(\epsilon,k) = (2e/\epsilon)^{2(k-1)}$ with a small explicit constant $c = 2e$ independant of the group G.

We prove in Lemma 2.5 that our $q(t,k+1)$ is the minimal growth function if $k = 2$. We also present an example of a group M of matrices with arbitrary ring coefficients and prove that it is uniformly amenable and we find a growth function for this group. It follows that the group G given in [10] is uniformly amenable because G is a subgroup of M.

2 Abelian groups

It is known that every discrete abelian group is uniformly amenable. In this section we shall give a very simple proof of this fact.

We denote by C_n^k the binomial coefficient $C_n^k = \binom{n}{k}$.

We start with a lemma.

Lemma 2.1. *If a group G satisfies the condition (*) for every finite subset A containing e, then it satisfies (*) for every finite subset A with the same growth function $q(t,k)$.*

Proof. Let $A = \{a_1, a_2, \dots, a_k\} \subset G$ and let $b_i = a_1^{-1}a_i$ for $i = 1, 2, \dots, k$. Let $B = \{b_1, b_2, \dots, b_k\}$. Suppose that U is a finite subset of G such that $|BU| \le (1+t)|U|$. Then $a_1 BU = AU$ has the same cardinality as BU so $|AU| \le (1+t)|U|$. □

We now pass to the abelian groups. We denote the composition law by $+$ and we denote the neutral element by 0.

Lemma 2.2. *Let G be an abelian group and let $A = \{0, a_1, \dots, a_k\} \subset G$. Let $U = \{\Sigma_{i=1}^k n_i a_i : n_i \in \{0, 1, \dots, n-1\}, \Sigma n_i \le n-1\}$. Then $|(A+U) \setminus U| \le (\frac{k}{n})|U|$.*

Proof. Consider all elements in $(A+U) \setminus U$. Every such element can be represented by a sum $w = \Sigma n_s a_s$ where $0 \le n_s \le n$ and $\Sigma_{s=1}^k n_s = n\}$. Consider such sums representing all distinct elements of $(A+U) \setminus U$.

$(A+U) \setminus U = \{w_i = \Sigma_{s=1}^k n_{i,s} a_s : \Sigma_{s=1}^k n_{i,s} = n \ for \ i = 1, 2, \dots, d\}$.

We fix $j \in \{1, 2, \dots, k\}$. We shall prove that the elements $\{u_{i,m} = w_i - m a_j : i = 1, 2, \dots, d, \ m = 1, \dots, n_{i,j}\}$ are distinct elements of U (if $n_{i,j} = 0$ there are no elements $u_{i,m}$.)

Since $n_{i,j} \ge 1$ and $1 \le m \le n_{i,j}$ the coefficients of $u_{i,m}$ are non-negative and their sum is at most $n-1$ so the elements $u_{i,m}$ have the form of the elements in U and they belong to U. Suppose that $u_{i,m} = u_{s,r}$ for distinct pairs $(i,m) \ne (s,r)$. That means $w_i - m a_j = w_s - r a_j$. If $m = r$ then $w_i = w_s$, which contradicts the assumption that $w_i \ne w_s$. If $m \ne r$ we may assume $0 < m - r < n_{i,j}$. Then

$w_s = w_s - r a_j + r a_j = w_i - m a_j + r a_j = w_i - (m-r)a_j \in U$ because $0 < m - r < m \le n_{i,j}$ so $w_i - (m-r)a_j$ has the form of the elements in U. But this contradicts the assumptions that $w_s \notin U$. So we have at least $\Sigma_{i=1}^d n_{i,j}$ distinct elements in U.

This is true for $j = 1, 2, \dots, k$. On the other hand $\Sigma_{j=1}^k (\Sigma_{i=1}^d n_{i,j}) = \Sigma_{i=1}^d (\Sigma_{j=1}^k n_{i,j}) = dn$. Therefore there exists j such that $|U| \ge \Sigma_{i=1}^d n_{i,j} \ge \frac{dn}{k}$. Since $|(A+U) \setminus U| = d$ we have $|(A+U) \setminus U| \le (\frac{k}{n})|U|$. □

Theorem 2.3. *Every abelian group is uniformly amenable with the growth function which satisfies $q(\frac{k}{n}, k+1) = C_{n+k-1}^k$ for natural k and n.*

Proof. Let k and $t = \frac{k}{n}$ be given. Consider a subset A of G with $k+1$ elements. By Lemma 2.1 we may assume that $A = \{0, a_1, \dots, a_k\}$. Let U be as in Lemma 2.2 for the set A. A sequence of the coefficients $(n_1, n_2, \dots, n_k)$ of a point in U represents an integer point of a simplex cut off from the non-negative cone of the Euclidean space $\mathbb{R}^k$ (e.g. the non-negative quarter of the plane or the non-negative octant of the space) by the hyperplane $\Sigma_{i=1}^k n_i = n-1$. There are C_{n+k-1}^k integer points in this simplex (an easy induction argument) and therefore there are at most C_{n+k-1}^k elements in U. By Lemma 2.2 we have $|(A+U) \setminus U| \le \frac{k}{n}|U|$ hence $|A+U| \le (1+\frac{k}{n})|U|$ as required. □

Remark 2.4. *For the other values of t we can choose the smallest n such that $\frac{k}{n} < t$ and let $q(t, k+1) = q(\frac{k}{n}, k+1)$. We believe that the value of $q(t, k+1)$ in the theorem is the smallest possible. We have not enough evidence to make it a conjecture. We can prove it for $k = 2$.*

Lemma 2.5. *For every abelian group G (when G has elements of the infinite order or of an arbitrarily high order) the minimal growth function $q(t,k)$ has values $q(\frac{2}{n}, 3) = C_{n+1}^2$ (and for every uniformly amenable group G which has elements of the infinite order or of an arbitrarily high order every growth function satisfies $q(\frac{2}{n}, 3) \ge C_{n+1}^2$).*

Proof. We know by Theorem 2.3 that if G is abelian, there exists a growth function with these values. Let us fix $k = 2$ and n. We choose an element $g \in G$ of a sufficiently high order (at least of order $2n^4$). We choose $a_1 = g, a_2 = n^2 g$ and $A = \{0, a_1, a_2\}$. Suppose U is a finite subset of G such that $|A+U| \le (1+\frac{2}{n})|U|$. Since $0 \in A$ we have $U \subset (A+U)$ and therefore $|(A+U) \setminus U| \le \frac{2}{n}|U|$. We shall prove that $|(A+U) \setminus U| \ge n+1$ and therefore $|U| \ge C_{n+1}^2 = \frac{n(n+1)}{2}$.

We partition the group G into cosets with respect to the cyclic subgroup $H = \langle g \rangle$ generated by the element g (the left cosets Hb if G is not abelian) and we partition U into the non-empty pieces U_i contained in different cosets. Then $A + U_i$ is contained in the same coset as U_i. If for every U_i we have $|A + U_i)| > (1+\frac{2}{n})|U_i|$, then $|(A+U)| > (1+\frac{2}{n})|U|$ which contradicts our assumptions. Therefore for some U_i we have $|(A+U_i)| \le (1+\frac{2}{n})|U_i|$.

We shall prove that this implies $|U_i| \geq C^2_{n+1}$ and therefore $|U| \geq C^2_{n+1}$. So we may restrict the argument to one coset and we may assume that $G =< g >$.

We need to prove that $|U| \geq \frac{n(n+1)}{2}$ so we may assume that $|U| \leq n^2 - 1$. If g is of a finite order, let $ord(g) = r \geq 2n^4$. Every element of U can be written as $u = mg = n_1 a_1 + n_2 a_2$ where $0 \leq n_1 \leq n^2 - 1$ and if $ord(g) = r$ then $0 \leq n_2$ and $m = n_1 + n^2 n_2 \leq r - 1$. So every element of U can be identified with a point (n_1, n_2) in the plane and different pairs (n_1, n_2) correspond to different elements of $U \subset G$. If $u = mg$ then the first coordinate of u is the remainder of m modulo n^2.

When we translate U in G we get a different correspondence of the elements of U with the points of the plane and different set of pairs (n_1, n_2) corresponds to the elements of U.

If $u = (n_1, n_2) \in U$ then $a_1 + u = (n_1 + 1, n_2)$ and $a_2 + u = (n_1, n_2 + 1)$.

Claim. *After a suitable translation of U we may assume that U is not a union of two subsets U_1 and U_2 in the plane in a distance more than $\sqrt{2}$ (we say that U is* connected*).*

Proof. If subsets U_1 and U_2 in the plane are in a distance more than $\sqrt{2}$ from each other, then sets $A + U_1$ and $A + U_2$ are disjoint as subsets of the plane but some points in the plane coincide as elements of G. We shall prove that sets $A + U_1$ and $A + U_2$ are disjoint as subsets of G.

Since $|U| < n^2$ not every remainder modulo n^2 is equal to the first coordinate of some point in U. If G is infinite we first translate U so that the first coordinate of every point in U is different from $n^2 - 1$. Suppose $U = U_1 \cup U_2$ as in the claim. Two distinct points in the plane (n_1, n_2) and (p_1, p_2) represent the same element of G if $p_1 - n_1 = n^2(n_2 - p_2)$, but this does not happen for the points of the sets $A + U_1$ and $A + U_2$.

If $ord(g) = r$ the same claim requires an additional translation of U. We first translate U so that $0 \in U$. Since $|U| < n^2$ and $r \geq 2n^4$ there must exist a gap of length at least $2n^2$ between some pair of consecutive values m_1, m_2 such that $m_1 g, m_2 g \in U$ and $m_2 - m_1 \geq 2n^2$. After another translation we may assume that the gap is at the end so $mg \notin U$ for $m > r - 2n^2$. After another suitable translation by less than $n^2 - 1$ we may assume that the first coordinate of every point in U is different from $n^2 - 1$ and $mg \notin U$ for $m \geq r - n^2$. Suppose $U = U_1 \cup U_2$ as in the claim. Now two distinct points in the plane (n_1, n_2) and (p_1, p_2) represent the same element of G if either $p_1 - n_1 = n^2(n_2 - p_2)$ or $p_1 + n^2 p_2 - n_1 - n^2 n_2$ is a multiple of r, but none of these may happen for the points of the sets $A + U_1$ and $A + U_2$.

Since $A + U_1$ and $A + U_2$ are disjoint and $|(A + U) \setminus U| \leq \frac{2}{n}|U|$ we must have $|(A + U_i) \setminus U_i| \leq \frac{2}{n}|U_i|$ for $i = 1$ or for $i = 2$ and we may restrict the discussion to this smaller set U_i and prove that $|U_i| \geq C^2_{n+1}$. So we may assume that U is connected. □

We assume that U is connected. We make another translation, if necessary, to get $min\{n_1 : (\exists n_2)\ (n_1, n_2) \in U\} = 0$ and $min\{n_2 : (\exists n_1)\ (n_1, n_2) \in U\} = 0$. This does not change the shape of U, it is still connected.

Let (a, b) be a point in U such that $a + b$ is maximal. Since U is connected every vertical line $n_1 = c$ with $0 < c < a$ meets U and every horizontal line $n_2 = d$ with $0 < d < b$ meets U.

If $u_1 = (n_1, n_2) \in U$ is the rightmost element of U in its row then $u_1 + a_1 = (n_1 + 1, n_2) \in (A + U) \setminus U$. We call $u_1 + a_1$ a *horizontal boundary point* of U. If $u_2 = (m_1, m_2) \in U$ is the top element of U in its column then $u_2 + a_2 = (m_1, m_2 + 1) \in (A + U) \setminus U$. We call $u_2 + a_2$ a *vertical boundary* point of U.

Observe that if $u = (n_1, n_2) \in U$ then, by the connectivity of U, there are at least $n_1 + 1$ vertical boundary points in $(A + U) \setminus U$ and at least $n_2 + 1$ horizontal boundary points in $(A + U) \setminus U$. We need only $n + 1$ points therefore we may assume that $n_1 < n$ and $n_2 < n$. If $a_1 + u$ or $a_2 + u$ is a boundary point, then its coordinates (m_1, m_2) still satisfy $m_1 < n^2$ and $m_1 + n^2 m_2$ is smaller than the order of g. It follows that distinct pairs of coordinates define distinct elements of G also for the boundary points.

We shall prove that each horizontal boundary point $(n_1 + 1, n_2)$ with $n_2 \leq b$ is distinct from any vertical boundary point $(m_1, m_2 + 1)$ with $m_1 \leq a$.

Suppose $(n_1 + 1, n_2) = (m_1, m_2 + 1)$ is a horizontal and a vertical boundary point. If $n_2 = b$ then $n_1 = a$ and $m_1 = n_1 + 1 = a + 1$. If $m_1 = a$ then $m_2 = b$ and $n_2 = m_2 + 1 = b + 1$. If $n_2 < b$ and $m_1 < a$ then there are no points of U on the vertical line above this point and on the horizontal line to the right of this point and the top right corner $U_1 = \{(c_1, c_2) \in U : c_1 > m_1, c_2 > n_2\}$ splits off, U is not connected.

This means that there are at least $a + b + 2$ points in $(A + U) \setminus U$.

Let $d = |(A+U)\setminus U| \geq a+b+2$. The set U lies in the triangle $\{(n_1, n_2) : 0 \leq n_1,\ 0 \leq n_2,\ n_1+n_2 \leq a+b\}$, so $\frac{(a+b+1)(a+b+2)|}{2} \geq |U|$ and by our assumptions $|U| \geq \frac{n}{2}d \geq \frac{n(a+b+2)}{2}$. It follows that $a+b+1 \geq n$ and $|(A+U)\setminus U| = d \geq a+b+2 \geq n+1$. □

3 Some linear uniformly amenable groups

Let R be a commutative ring with a unit 1.

We consider the following group of matrices over R

$$M = \{A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in R\}.$$

An element of M can be identified with a triple (x, y, z) of elements of R. Then the composition (the product) of the triples is defined by $(x_1, y_1, z_1)(x, y, z) = (x_1 + x, y_1 + y, z_1 + z + x_1 y)$.

Later we shall also perform the addition of the triples as elements of $R \oplus R \oplus R$.

The elements $(0, 0, z)$ lie in the center of M and form an abelian normal subgroup C of M. The quotient M/C is abelian so the group M is solvable and it is known that solvable groups are uniformly amenable. We want to find a growth function $q(t, k)$ for this group, which will also give a direct proof of the fact, that M is uniformly amenable.

This example was inspired by a paper [10] by J. Wysoczański. He considered a group G which is a direct sum of groups M over all prime fields $F_p = \mathbb{Z}/p\mathbb{Z}$. If we take for the ring R the direct sum of the fields F_p and we adjoin a unit - the element with each coordinate equal to 1 in the corresponding field we get a ring R_1 with a unit. The group G is a subgroup of the group M over R_1.

We now consider the group M over an arbitrary commutative ring R with a unit. Let $A = \{a_0, a_1, \ldots, a_{k-1}\} \subset M$ where $a_i = (-x_i, -y_i, -z_i)$ for $i = 0, 1, \ldots, k-1$.

The minus sign is for the convenience of the subsequent description where the negative coefficients play a special role.

We want to estimate the number $|AU|$. By Lemma 2.1 we may assume that $a_0 = (0, 0, 0)$ is the unit of the group M. For a fixed $n \in \mathbb{N}$ we define $U_n \subset M$,

$$U_n = \{u = (u_1, u_2, u_3) : u_1 = \Sigma_i t_i x_i, u_2 = \Sigma_i t_i y_i, u_3 = \Sigma_i t_i z_i + \Sigma_{i,j} t_{i,j} x_i y_j\}$$

where $i, j \in \{1, 2, \ldots, k-1\}$, $t_i \in \{0, 1, \ldots, n-1\}$, $t_{i,j} \in \{0, 1, \ldots, n^2-1\}$.

An element of U_n is determined by $(k-1)^2 + (k-1)$ integer coefficients. The set J of indices of these coefficients consists of integers and pairs of integers $J = \{i : 1 \leq i \leq k-1, (i, j) : 1 \leq i, j \leq k-1\}$.

We shall denote an element $u \in U$ by $u = (t_\alpha)_{\alpha \in J}$. Then, by the definition of the composition law in M, $w = a_s u$ has a similar form, $w = (f_\alpha)$, where $f_s = t_s - 1$, $f_{s,j} = t_{s,j} - t_j$ and $f_\alpha = t_\alpha$ for other $\alpha \in J$. In particular f_α may be negative if $\alpha = s$ or $\alpha = (s, j)$ and $f_s \geq -1$ and $f_{s,j} \geq 1-n$.

Lemma 3.1. *For any $s \in \{1, 2, \ldots, k-1\}$ we have $|a_s U_n \setminus U_n| < (\frac{2^k}{n})|U_n|$.*

Proof. We fix an integer $s \in \{1, 2, \ldots, k-1\}$. We consider elements of $a_s U_n \setminus U_n$ and for each such element we choose one representative $w = (f_\alpha)$. For any non-empty subset of indices $I \subset \{s, (s, 1), (s, 2), \ldots, (s, k-1)\}$ we consider the set B_I of all chosen representatives $w = (f_\alpha)$ of the elements in $a_s U_n \setminus U_n$ which have the negative coefficients f_α exactly for $\alpha \in I$. Suppose that $B_I = \{w_1, w_2, \ldots, w_d\}$.

Let $e_I = (b_1, b_2, b_3) \in R \oplus R \oplus R$ where $b_1 = x_s, b_2 = y_s, b_3 = z_s + n\Sigma_{(s,j)\in I} x_s y_j$ if $s \in I$ and $b_1 = 0, b_2 = 0, b_3 = n\Sigma_{(s,j)\in I} x_s y_j$ if $s \notin I$.

We shall prove that the elements $w_i + me_I,\ i = 1, 2, \ldots, d,\ m = 1, 2, \ldots n$ belong to U_n and are distinct. Only the coefficients $f_\alpha,\ \alpha \in I$ of w change. If $s \in I$ then $f_s = -1$ and it increases by m when we add me_I where $1 \leq m \leq n$ so f_s falls into the proper range for the elements in U_n. If $(s, j) \in I$ then $1 - n \leq f_{s,j} \leq -1$ and $f_{s,j}$ increases by mn so it also falls into the proper range for the elements in U_n.

Suppose that $w_i + me_I = w_j + re_I$ for different pairs (i, m) and (j, r). If $m = r$ then $w_i = w_j$ which contradicts the assumptions. If $m \neq r$ we may assume that $m > r$. Then $w_j = w_j + re_I - re_I = w_i + (m-r)e_I \in U_n$ which contradicts our assumptions. Therefore $|U_n| \geq nd$ and $|B_I| \leq \frac{1}{n}|U_n|$. There are $(2^k - 1)$ non-empty subsets of $\{s, (s,1), (s,2), \ldots, (s, k-1)\}$ and each element of $a_s U_n \setminus U_n$ lies in one of them so $|a_s U_n \setminus U_n| < (\frac{2^k}{n})|U_n|$. □

Corollary 3.2. *For every commutative ring R with unit the group M is uniformly amenable with a growth function*

$$q(\epsilon, k) = (1 + \frac{(k-1)(2^k)}{\epsilon})^{2(k-1)^2+k-1}$$

Proof. For a given k and ϵ choose $n = [\frac{(k-1)2^k}{\epsilon}] + 1$. Then $\frac{(k-1)2^k}{\epsilon} < n$ and $\frac{(k-1)2^k}{n} < \epsilon$. Let $A = \{a_0 = (0,0,0), a_1, \ldots, a_{k-1}\} \subset M$. We choose the set U_n as in the previous lemma. Then $|U_n| \leq n^{2(k-1)^2+k-1} \leq (1 + \frac{(k-1)2^k}{\epsilon})^{2(k-1)^2+k-1} = q(\epsilon, k)$. It suffices to prove that $|AU_n| < (1+\epsilon)|U_n|$. We have $a_0 U_n = U_n$ and by the previous lemma $|AU_n| \leq |U_n| + (k-1)(\frac{2^k}{n})|U_n| < (1+\epsilon)|U_n|$. □

Remark 3.3. *Many of the sets B_I in the proof of the previous lemma may have very small contribution to the "boundary" $AU_n \setminus U_n$. In particular if different sets of coefficients (t_α) represent different elements of U_n then for n large $|AU_n \setminus U_n|$ is of order $\frac{k^2}{n}|U_n|$. Indeed if a set I in the proof of Lemma 3.1 has only one element then $|B_I|$ is equal about $n^{2(k-1)^2+k-2}$. There are k such sets for each $s = 1, 2, \ldots, k-1$. If I has more than one element then $|B_I|$ is at least n times smaller therefore for n large $|AU_n \setminus U_n|$ is equal about $k^2 n^{2(k-1)^2+k-2}$. Moreover $|U_n| = n^{2(k-1)^2+k-1}$ therefore $|AU_n \setminus U_n|$ is equal about $\frac{k^2}{n}|U_n|$. In this case (when n is large and a_i's are "independent") it suffices to have $n > \frac{k^2}{\epsilon}$ and $|U_n| = (\frac{k^2}{\epsilon})^{2(k-1)^2+k-1}$. Probably this is the correct size of $q(\epsilon, k)$ but we cannot prove it.*

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