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# Complete convergence for weighted sums of pairwise independent random variables

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**Abstract:** In the present paper, we have established the complete convergence for weighted sums of pairwise independent random variables, from which the rate of convergence of moving average processes is deduced.

**Keywords:** Complete convergence, Pairwise independent random variables, Weighted sums, Moving average processes

**MSC:** 60F15

## 1 Introduction

In this paper we are interested in the complete convergence for weighted sums of pairwise independent random variables. First let us recall some definitions and known results.

### 1.1 Complete convergence

The following concept of complete convergence of a sequence of random variables, which plays an important role in limit theory of probability, was introduced firstly by Hsu and Robbins [1]. A random sequence  $\{X_n, n \geq 1\}$  is said to converge completely to the constant  $C$  (write  $X_n \rightarrow C$  completely) if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

In view of the Borel-Cantelli Lemma, this implies that  $X_n \rightarrow C$  almost surely (a.s.). For the case of i.i.d. random variables, Hsu and Robbins [1] proved that the sequence of arithmetic means of the random variables converges completely to the expected value if the variance of the summands is finite. Somewhat later, Erdős [2] proved the converse. These results are summarized as follows.

*Hsu-Robbins-Erdős strong law.* Let  $\{X_n, n \geq 1\}$  be a sequence of i.i.d. random variables with mean zero and set  $S_n = \sum_{i=1}^n X_i$ , then  $EX_1^2 < \infty$  is equivalent to the condition that

$$\sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty, \quad \text{for all } \varepsilon > 0.$$

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The result of Hsu-Robbins-Erdős strong law is a fundamental theorem in probability theory and was intensively investigated in several directions by many authors in the past decades. One of the most important results is Baum and Katz [3] strong law.

**Baum and Katz strong law.** Let  $\alpha p \geq 1$ ,  $p > 2$  and let  $\{X_n\}$  be a sequence of i.i.d. random variables and  $E|X_1|^p < \infty$ . If  $\frac{1}{2} < \alpha \leq 1$ , assume that  $EX_1 = 0$ . Then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right| > \varepsilon n^{\alpha} \right) < \infty \text{ for all } \varepsilon > 0.$$

Baum and Katz strong law bridges the integrability of summands and the rate of convergence in the Marcinkiewicz-Zygmund strong law of large numbers.

In general, the main tools to prove the complete convergence of some random variables are based on the moment inequality or the exponential inequality. However, for some dependent sequences (such as pairwise independent sequence, pairwise negatively dependent sequence), whether these inequalities hold was not known. Recently, Bai et al. [4] obtained the following excellent result for the maximum partial sums of pairwise independent random variables.

**Theorem 1.1** ([4]). *Let  $1 \leq p < 2$  and let  $\{X_n, n \geq 1\}$  be a sequence of pairwise i.i.d. random variables. Then  $EX_1 = 0$  and  $E|X_1|^p < \infty$  if and only if for all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{p-2} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right| > \varepsilon n \right) < \infty.$$

It is well known that the analysis of weighted sums plays an important role in the statistics, such as jackknife estimate, nonparametric regression function estimate and so on. Many authors considered the complete convergence of the weight sums of random variables. Thrum [5] studied the almost sure convergence of weighted sums of i.i.d. random variables; Li et al. [6] obtained complete convergence of weighted sums without identically distributed assumption. Liang and Su [7] extended the results of Thrum [5] and Li et al. [6], and showed the complete convergence of weighted sums of negatively associated sequence. Beak [8] discussed the almost sure convergence for weighted sums of pairwise independent random variables. Huang et al. [9], Shen et al. [10] studied the complete convergence theorems for weighted sums of  $\phi$ -mixing random variables. Miao et al. [11] established some results of complete convergence for martingales and under some uniform mixing conditions, the sufficient and necessary condition of the convergence of the martingale series was established. For the negatively orthant dependent random variables, Gan and Chen [12] discussed the complete convergence of weight sums, and for some special weighted sums, Chen and Sung [13] gave necessary and sufficient conditions for the complete convergence. Deng et al. [14], Zhao et al. [15] presented some results on complete convergence for weighted sums of random variables satisfying the Rosenthal type inequality. Xue et al. [16], Wang et al. [17], Deng et al. [18] studied the complete convergence for weighted sums of negatively superadditive-dependent random variables. Qiu and Chen [19] obtained the complete convergence for the weighted sums of widely orthant dependent random variables. Wu [20], Jabbari [21], Zhang et al. [22] gave the complete convergence for weighted sums of pairwise negative quadrant dependent random variables. For linearly negative quadrant dependent random variables, Choi et al. [23] established the complete convergence of weight sums. Baek and Park [24], Baek et al. [25] gave the complete convergence of arrays of rowwise negatively dependent random variables, and Qiu et al. [26] derived a general result for the complete convergence.

In the present paper, we shall study the sufficient conditions which make the following complete convergence of weighted sums of pairwise independent random variables hold

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} X_i \right| > r n^{1/p} \right) < \infty \text{ for all } r > 0, \quad (1)$$

where  $\{a_{ni}, i \geq 1, n \geq 1\}$  is an array of constants and  $t$  is some parameter which will be defined in the main results.

## 1.2 Stochastic domination

A sequence  $\{X_n, n \geq 1\}$  of random variables is said to be stochastically dominated by a random variable  $X$  if there exists a positive constant  $C$  such that

$$P(|X_n| > x) \leq CP(|X| > x) \quad (2)$$

for all  $x \geq 0$  and  $n \geq 1$ . This dominated condition means weakly dominated, where weak refers to the fact that domination is distributional. In [27], Gut introduced a weakly mean dominated condition. We say that the random variables  $\{X_n, n \geq 1\}$  are weakly mean dominated by the random variable  $X$ , where  $X$  is possibly defined on a different space if for some  $C > 0$ ,

$$\frac{1}{n} \sum_{k=1}^n P(|X_k| > x) \leq CP(|X| > x) \quad (3)$$

for all  $x \geq 0$  and  $n \geq 1$ . It is clear that if  $X$  dominates the sequence  $\{X_n, n \geq 1\}$  in the weakly dominated sense, then it also dominates the sequence in the weakly mean dominated sense. Furthermore, Gut [27] gave an example to show that the condition (3) is weaker than the above condition (2).

Our main results are stated in Section 2 and the proofs are given in Section 3. Throughout this paper, let  $C$  denote a positive constant, which may take different values whenever it appears in different expressions, and  $I(\cdot)$  stand for the indicator function.

## 2 Main results

In each situation studied, we assume that  $\sum_{i=1}^{\infty} a_{ni} X_i$  is finite a.s., which implies that  $\sum_{i=1}^{\infty} a_{ni} X_i$  converges a.s. If  $t < -1$ , then (1) holds obviously and hence it is of interest only for  $t \geq -1$ .

**Theorem 2.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of random variables which are stochastically dominated by the random variable  $X$  (i.e., the inequality (2) holds) satisfying*

$$E|X|^{p(t+\beta+1)} < \infty,$$

where  $p(t + \beta + 1) > 0$  and  $p > 0$ . Let  $\{a_{ni}, i \geq 1, n \geq 1\}$  be a bounded array of real numbers satisfying

$$\sum_{i=1}^{\infty} |a_{ni}|^q = O(n^\beta) \quad (4)$$

for some  $q < p(t + \beta + 1)$ .

(1) If  $0 < p(t + \beta + 1) < 1$ , then we have

$$\sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{\infty} a_{ni} X_i\right| > rn^{1/p}\right) < \infty \text{ for all } r > 0. \quad (5)$$

(2) If  $1 \leq p(t + \beta + 1) < 2$ ,  $\{X_n, n \geq 1\}$  is a sequence of pairwise independent random variables and  $EX_n = 0$ , then (5) holds.

(3) If  $p(t + \beta + 1) = 2$ ,  $\{X_n, n \geq 1\}$  is a sequence of pairwise independent random variables and  $EX_n = 0$  and assume that the condition (4) is replaced by the following condition

$$\sum_{i=1}^{\infty} |a_{ni}|^2 = O(n^\beta \log^{-\alpha} n) \quad (6)$$

for some  $\alpha > 1$ , then (5) holds.

**Remark 2.2.** It is easy to see that the condition (4) implies

$$\sum_{i=1}^{\infty} |a_{ni}|^{q+\gamma} = O(n^\beta) \text{ for any } \gamma > 0.$$

**Remark 2.3.** Sung [28] considered the same problems for weighted sums of independent random variables. For the case  $p(t + \beta + 1) \geq 2$ , Sung [28] gave the following result: if  $\{X_n, n \geq 1\}$  are independent,  $EX_n = 0$  and

$$\sum_{i=1}^{\infty} |a_{ni}|^2 = O(n^\alpha)$$

for some  $\alpha < 2/p$ , then (5) holds. The method to prove the case  $p(t + \beta + 1) \geq 2$  in [28] is to use the complete convergence theorem for arrays of rowwise independent random variables from Sung et al. [29]. The key tool to prove the complete convergence for arrays of rowwise independent random variables is the Hoffman-Jørgensen inequality (see [30]). But we do not know whether the Hoffman-Jørgensen inequality for pairwise independent random variables holds or not.

**Theorem 2.4.** Let  $\{X_n, n \geq 1\}$  be a sequence of random variables which are stochastically dominated by the random variable  $X$  (i.e., the inequality (2) holds) satisfying

$$E|X|^{p(t+\beta+1)} \log |X| < \infty,$$

where  $p(t + \beta + 1) > 0$  and  $p > 0$ . Let  $\{a_{ni}, i \geq 1, n \geq 1\}$  be a bounded array of real numbers satisfying

$$\sum_{i=1}^{\infty} |a_{ni}|^{p(t+\beta+1)} = O(n^\beta). \quad (7)$$

(1) If  $0 < p(t + \beta + 1) < 1$ , then we have

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} X_i \right| > rn^{1/p} \right) < \infty \text{ for all } r > 0. \quad (8)$$

(2) If  $1 \leq p(t + \beta + 1) < 2$ ,  $\{X_n, n \geq 1\}$  is a sequence of pairwise independent random variables and  $EX_n = 0$ , then (5) holds.

*Proof.* The proof is similar to that of Theorem 2.1, so we omit it.  $\square$

**Corollary 2.5.** Let  $\{X_n, n \geq 1\}$  be a sequence of random variables which are weakly mean dominated by the random variable  $X$  (i.e., the inequality (3) holds) satisfying

$$E|X|^{p(t+2)} < \infty,$$

for some  $0 < p < 2$  and  $1 < p(t + 2) < 2$ .

(1) If  $0 < p(t + 2) < 1$ , then we have

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^n X_i \right| > rn^{1/p} \right) < \infty \text{ for all } r > 0. \quad (9)$$

(2) If  $1 \leq p(t + 2) < 2$ ,  $\{X_n, n \geq 1\}$  is a sequence of pairwise independent random variables and  $EX_n = 0$ , then (9) holds.

**Corollary 2.6.** Let  $\{X_n, -\infty < n < \infty\}$  be a sequence of zero mean pairwise independent random variables which are stochastically dominated by the random variable  $X$  (i.e., the inequality (2) holds) satisfying

$$E|X|^{p(t+2)} < \infty,$$

for some  $0 < p < 2$  and  $1 < p(t+2) < 2$ . Let  $\{a_n, -\infty < n < \infty\}$  be a sequence of real numbers such that

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Set

$$a_{ni} = \sum_{j=i+1}^{i+n} a_j$$

for each  $i$  and  $n$ . Then for any  $r > 0$ ,

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=-\infty}^{\infty} a_{ni} X_i \right| > r n^{1/p} \right) < \infty.$$

### 3 Proofs of main results

In order to prove our main results, we need some preliminary lemmas.

**Lemma 3.1** ([31]). Let  $1 \leq r \leq 2$  and let  $\{X_n, n \geq 1\}$  be a sequence of pairwise independent random variables with  $EX_n = 0$  and  $E|X_n|^r < \infty$  for all  $n \geq 1$ . Then there exists a positive  $C_r$  depending only on  $r$ , such that

$$E \left| \sum_{k=1}^n X_k \right|^r \leq C_r \sum_{k=1}^n E |X_k|^r, \quad \forall n \geq 1.$$

The following lemma is well known and its proof is standard.

**Lemma 3.2.** Let  $\{X_n, n \geq 1\}$  be a sequence of random variables satisfying a weak mean dominating condition with mean dominating random variable  $X$  (i.e., the inequality (3) holds). Let  $p > 0$  and for some  $\alpha > 0$ ,

$$X'_i = X_i I_{\{|X_i| \leq \alpha\}}, \quad X''_i = X_i I_{\{|X_i| > \alpha\}}$$

and

$$X' = X I_{\{|X| \leq \alpha\}}, \quad X'' = X I_{\{|X| > \alpha\}}.$$

Then if  $E|X|^p < \infty$ , we have

- (1)  $n^{-1} \sum_{k=1}^n E|X_k|^p \leq CE|X|^p$ ,
- (2)  $n^{-1} \sum_{k=1}^n E|X'_k|^p \leq C \left( E|X'|^p + \alpha^p P(|X| > \alpha) \right)$ ,
- (3)  $n^{-1} \sum_{k=1}^n E|X''_k|^p \leq CE|X''|^p$ .

**Remark 3.3.** Under the assumptions in Lemma 3.2, if the dominating condition (3) is replaced by the condition (2), then it is easy to see that for all  $k \geq 1$ ,

- (1)  $E|X_k|^p \leq CE|X|^p$ ,
- (2)  $E|X'_k|^p \leq C \left( E|X'|^p + \alpha^p P(|X| > \alpha) \right)$ ,
- (3)  $E|X''_k|^p \leq CE|X''|^p$ .

*Proof of Theorem 2.1.* For  $i \geq 1, n \geq 1$ , let

$$X'_{ni} = X_i I_{\{|X_i| \leq n^{1/p}\}}, \quad X''_{ni} = X_i I_{\{|X_i| > n^{1/p}\}} \quad (10)$$

and

$$X'_n = X I_{\{|X| \leq n^{1/p}\}}, \quad X''_n = X I_{\{|X| > n^{1/p}\}}. \quad (11)$$

(1) Since  $0 < p(t + \beta + 1) < 1$ , we can take a positive constant  $\varepsilon$  such that  $0 < p(t + \beta + 1) + \varepsilon \leq 1$ . Let  $u = p(t + \beta + 1)$ , then from Remark 3.3 we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{\infty} a_{ni} X'_{ni}\right| > rn^{1/p}\right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} E\left|\sum_{i=1}^{\infty} a_{ni} X'_{ni}\right|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} \sum_{i=1}^{\infty} E|a_{ni} X'_{ni}|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{u+\varepsilon} \left(E|X'_n|^{u+\varepsilon} + n^{\frac{u+\varepsilon}{p}} P(|X| > n^{1/p})\right). \end{aligned}$$

From Remark 2.2, it is easy to see that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{u+\varepsilon} n^{\frac{u+\varepsilon}{p}} P(|X| > n^{1/p}) \\ &\leq C \sum_{n=1}^{\infty} n^{t+\beta} P(|X| > n^{1/p}) \\ &\leq C \sum_{n=1}^{\infty} n^{\frac{u}{p}-1} \sum_{k=n}^{\infty} P(k^{u/p} < |X|^u \leq (k+1)^{u/p}) \\ &\leq C \sum_{k=1}^{\infty} k^{u/p} P(k^{u/p} < |X|^u \leq (k+1)^{u/p}) \leq CE|X|^u < \infty \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{u+\varepsilon} E|X'_n|^{u+\varepsilon} &\leq C \sum_{n=1}^{\infty} n^{-1-\frac{\varepsilon}{p}} E|X'_n|^{u+\varepsilon} \\ &\leq C \sum_{k=1}^{\infty} k^{-\frac{\varepsilon}{p}} E|X|^{u+\varepsilon} I_{\{(k-1)^{u/p} < |X|^u \leq k^{u/p}\}} \\ &\leq C \sum_{k=1}^{\infty} E|X|^u I_{\{(k-1)^{u/p} < |X|^u \leq k^{u/p}\}} \leq CE|X|^u < \infty. \end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{\infty} a_{ni} X'_{ni}\right| > rn^{1/p}\right) < \infty. \quad (12)$$

Now we choose  $\varepsilon > 0$  such that  $p(t + \beta + 1) - \varepsilon \geq q$  and  $p(t + \beta + 1) - \varepsilon > 0$ . Let  $u = p(t + \beta + 1)$ , then from Remark 2.2 and Remark 3.3, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{\infty} a_{ni} X''_{ni}\right| > rn^{1/p}\right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u-\varepsilon}{p}} \sum_{i=1}^{\infty} E|a_{ni} X''_{ni}|^{u-\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{-1+\frac{\varepsilon}{p}} E|X''_n|^{u-\varepsilon} \\ &\leq C \sum_{k=1}^{\infty} k^{\frac{\varepsilon}{p}} E|X|^{u-\varepsilon} I_{\{k^{1/p} < |X| \leq (k+1)^{1/p}\}} \\ &\leq CE|X|^u < \infty \end{aligned}$$

which can yield the first result of Theorem 2.1 by combining with (12).

(2) **Case 1:**  $1 < p(t + \beta + 1) < 2$ . We can choose a positive constant  $\varepsilon > 0$  such that  $p(t + \beta + 1) + \varepsilon \leq 2$ . Let  $u = p(t + \beta + 1)$ , then from Lemma 3.1, Remark 2.2 and Remark 3.3, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} (X'_{ni} - EX'_{ni}) \right| > rn^{1/p} \right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u+\varepsilon}{p}} E \left| \sum_{i=1}^{\infty} a_{ni} (X'_{ni} - EX'_{ni}) \right|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u+\varepsilon}{p}} \sum_{i=1}^{\infty} E |a_{ni} (X'_{ni} - EX'_{ni})|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u+\varepsilon}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{u+\varepsilon} E |X'_{ni}|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t+\beta-\frac{u+\varepsilon}{p}} \left( E |X'_n|^{u+\varepsilon} + n^{\frac{u+\varepsilon}{p}} P(|X| > n^{1/p}) \right). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} n^{t+\beta-\frac{u+\varepsilon}{p}} E |X'_n|^{u+\varepsilon} \leq C \sum_{k=1}^{\infty} k^{-\frac{\varepsilon}{p}} E |X|^{u+\varepsilon} I_{\{(k-1)^{1/p} < |X| \leq k^{1/p}\}} \leq CE |X|^u < \infty$$

and

$$\sum_{n=1}^{\infty} n^{t+\beta} P(|X| > n^{1/p}) \leq C \sum_{k=1}^{\infty} k^{u/p} P(k^{u/p} < |X|^u \leq (k+1)^{u/p}) \leq CE |X|^u < \infty$$

we can get

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} (X'_{ni} - EX'_{ni}) \right| > rn^{1/p} \right) < \infty. \quad (13)$$

Now we choose  $\varepsilon > 0$  such that  $p(t + \beta + 1) - \varepsilon \geq q$  and  $p(t + \beta + 1) - \varepsilon \geq 1$ , then from Lemma 3.1, Remark 2.2 and Remark 3.3, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} (X''_{ni} - EX''_{ni}) \right| > rn^{1/p} \right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u-\varepsilon}{p}} E \left| \sum_{i=1}^{\infty} a_{ni} (X''_{ni} - EX''_{ni}) \right|^{u-\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u-\varepsilon}{p}} \sum_{i=1}^{\infty} E |a_{ni} (X''_{ni} - EX''_{ni})|^{u-\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u-\varepsilon}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{u-\varepsilon} E |X''_{ni}|^{u-\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t+\beta-\frac{u-\varepsilon}{p}} E |X''_n|^{u-\varepsilon} \\ &\leq C \sum_{k=1}^{\infty} k^{\frac{\varepsilon}{p}} E |X|^{u-\varepsilon} I_{\{k^{1/p} < |X| \leq (k+1)^{1/p}\}} \\ &\leq CE |X|^u < \infty. \end{aligned} \quad (14)$$

From (13) and (14), we have proved the desired result for the case  $1 < p(t + \beta + 1) < 2$ .

**Case 2:**  $p(t + \beta + 1) = 1$ . Without loss of generality, we assume that  $t \geq -1$  (which implies  $\beta \leq 1/p$ ). From Remark 2.2 and Remark 3.3, we have

$$\begin{aligned} &n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} EX_i I_{\{|X_i| > n^{1/p}\}} \right| \\ &\leq C n^{-1/p} E |X| I_{\{|X| > n^{1/p}\}} \sum_{i=1}^{\infty} |a_{ni}| \\ &\leq C n^{\beta-1/p} E |X| I_{\{|X| > n^{1/p}\}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (15)$$

Hence there exists some positive constant  $N_0$  such that

$$\sum_{n=N_0}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} (X''_{ni} - EX''_{ni}) \right| > rn^{1/p} \right) \leq \sum_{n=N_0}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} X''_{ni} \right| > rn^{1/p}/2 \right) \quad (16)$$

Now we choose  $\varepsilon > 0$  such that  $1 - \varepsilon \geq q$  and  $1 - \varepsilon > 0$ , then we have

$$\begin{aligned} \sum_{n=N_0}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} X''_{ni} \right| > rn^{1/p}/2 \right) &\leq C \sum_{n=N_0}^{\infty} n^{t-\frac{1-\varepsilon}{p}} \sum_{i=1}^{\infty} |a_{ni}|^{1-\varepsilon} E|X''_{ni}|^{1-\varepsilon} \\ &\leq C \sum_{n=N_0}^{\infty} n^{t+\beta-\frac{1-\varepsilon}{p}} E|X''_n|^{1-\varepsilon} \\ &\leq C \sum_{k=N_0}^{\infty} k^{-\frac{\varepsilon}{p}} E|X|^{1-\varepsilon} I_{\{(k-1)^{1/p} < |X| \leq k^{1/p}\}} \\ &\leq CE|X| < \infty. \end{aligned} \quad (17)$$

Furthermore by the similar proof (13), we can obtain

$$\sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} (X'_{ni} - EX'_{ni}) \right| > rn^{1/p} \right) < \infty.$$

Based on the above discussions, the second result of the theorem can be proved.

(3) For the case  $p(t + \beta + 1) = 2$ , from Lemma 3.1 and Remark 3.3, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^{\infty} a_{ni} X_i \right| > rn^{1/p} \right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{2}{p}} E \left| \sum_{i=1}^{\infty} a_{ni} X_i \right|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{2}{p}} \sum_{i=1}^{\infty} |a_{ni}|^2 E|X_i|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{2}{p}+\beta} \log^{-\alpha} n E|X|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{-1} \log^{-\alpha} n E|X|^2 < \infty. \end{aligned} \quad \square$$

*Proof of Corollary 2.5.* The proof is similar to Theorem 2.1, we give only the proof of the first result. For  $i \geq 1, n \geq 1$ , let the random variables  $X'_{ni}, X''_{ni}, X'_n, X''_n$  be defined as (10) and (11). Since  $0 < p(t + 2) < 1$ , we can take a positive constant  $\varepsilon$  such that  $0 < p(t + 2) + \varepsilon \leq 1$ . Let  $u = p(t + 2)$ , then from Lemma 3.2 we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P \left( \left| \sum_{i=1}^n X'_{ni} \right| > rn^{1/p} \right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} E \left| \sum_{i=1}^n X'_{ni} \right|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t-\frac{\varepsilon+u}{p}} \sum_{i=1}^n E|X'_{ni}|^{u+\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t+1-\frac{\varepsilon+u}{p}} \left( E|X'_n|^{u+\varepsilon} + n^{\frac{u+\varepsilon}{p}} P(|X| > n^{1/p}) \right). \end{aligned}$$

It is easy to see that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{t+1} P(|X| > n^{1/p}) &\leq C \sum_{n=1}^{\infty} n^{\frac{u}{p}-1} \sum_{k=n}^{\infty} P(k^{u/p} < |X|^u \leq (k+1)^{u/p}) \\ &\leq C \sum_{k=1}^{\infty} k^{u/p} P(k^{u/p} < |X|^u \leq (k+1)^{u/p}) \leq CE|X|^u < \infty \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} n^{t+1-\frac{\varepsilon+u}{p}} E|X'_n|^{u+\varepsilon} &= \sum_{n=1}^{\infty} n^{-1-\frac{\varepsilon}{p}} E|X'_n|^{u+\varepsilon} \\ &\leq C \sum_{k=1}^{\infty} k^{-\frac{\varepsilon}{p}} E|X|^{u+\varepsilon} I_{\{(k-1)^{u/p} < |X|^u \leq k^{u/p}\}} \\ &\leq C \sum_{k=1}^{\infty} E|X|^u I_{\{(k-1)^{u/p} < |X|^u \leq k^{u/p}\}} \\ &\leq CE|X|^u < \infty. \end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^n X'_{ni}\right| > rn^{1/p}\right) < \infty. \quad (18)$$

Now we choose  $\varepsilon > 0$  such that  $p(t+2) - \varepsilon \geq q$  and  $p(t+2) - \varepsilon > 0$ . Let  $u = p(t+2)$ , then from Lemma 3.2, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^n X''_{ni}\right| > rn^{1/p}\right) &\leq C \sum_{n=1}^{\infty} n^{t-\frac{u-\varepsilon}{p}} \sum_{i=1}^n E|X''_{ni}|^{u-\varepsilon} \\ &\leq C \sum_{n=1}^{\infty} n^{t+1-\frac{u-\varepsilon}{p}} E|X''_n|^{u-\varepsilon} \\ &\leq C \sum_{k=1}^{\infty} k^{\frac{\varepsilon}{p}} E|X|^{u-\varepsilon} I_{\{k^{1/p} < |X| \leq (k+1)^{1/p}\}} \\ &\leq CE|X|^u < \infty \end{aligned}$$

which can yield the first result of Corollary 2.5 by combining with (18).  $\square$

*Proof of Corollary 2.6.* It is easy to check that

$$\sum_{i=-\infty}^{\infty} |a_{ni}| = O(n) \quad \text{and} \quad \sum_{i=-\infty}^{\infty} a_{ni}^2 = O(n).$$

Taking  $\beta = 1$ ,  $q = 1$ , then from (2) of Theorem 2.1, the desired results can be obtained.  $\square$

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors contributed equally to the manuscript, and they read and approved the final manuscript.

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