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Review Article

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A hierarchy in the family of real surjective functions

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Abstract: This expository paper focuses on the study of *extreme surjective* functions in $\mathbb{R}^{\mathbb{R}}$. We present several different types of extreme surjectivity by providing examples and crucial properties. These examples help us to establish a hierarchy within the different classes of surjectivity we deal with. The classes presented here are: everywhere surjective functions, strongly everywhere surjective functions, κ-everywhere surjective functions, perfectly everywhere surjective functions and Jones functions. The algebraic structure of the sets of surjective functions we show here is studied using the concept of lineability. In the final sections of this work we also reveal unexpected connections between the different degrees of extreme surjectivity given above and other interesting sets of functions such as the space of additive mappings, the class of mappings with a dense graph, the class of Darboux functions and the class of Sierpiński-Zygmund functions in $\mathbb{R}^{\mathbb{R}}$.

Keywords: Lineability, Everywhere surjective, Jones function, Sierpiński-Zygmund function

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1 Introduction

At the beginning of the 20th century Lebesgue [1] proved the existence of a mapping $f : [0,1] \to [0,1]$ such that $f(I) = [0,1]$ for every non-degenerate subinterval I of $[0,1]$. Lebesgue's example can be adapted to construct a mapping defined on the whole real line that transforms every non-degenerate interval into $\mathbb{R}$. This exotic property turns out to be shared by a surprisingly large class of functions that we call *everywhere surjective*.

We point out in Section 2 that everywhere surjective functions attain every real value at least $\aleph_0$ many times in every non-degenerate interval. In fact, it is possible to define an everywhere surjective function that attains each real number $\mathfrak{c}$ many times in every non-degenerate interval, where $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$. An example of a function enjoying this refined form of extreme surjectivity will also be given. This example, far from being an isolated case, is just an instance of a very large class of functions called *strongly everywhere surjective*. The notion of strongly everywhere surjectivity does not exhaust all possibilities in the search of extreme surjectivity. Indeed, there are surjective functions satisfying even more restrictive conditions. We also construct a function that attains every real number $\mathfrak{c}$ many times in every perfect set, which is obviously a much stronger form of surjectivity. These

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functions are called *perfectly everywhere surjective*. We can take an even further step forward towards "supreme surjectivity". In 1942, F. B. Jones [2] constructed a function whose graph intersects every closed set in $\mathbb{R}^2$ with uncountable projection on the abscissa axis. The functions that satisfy this latter property are called Jones functions. It is easily seen that a Jones function is perfectly everywhere surjective. The class of Jones functions can be proved to be large from an algebraic point of view too.

In order to formalize what is meant by an "algebraically large set" the notion of *lineability* is commonly used. We say that a subset M of a linear space E is λ-lineable, if $M \cup \{0\}$ contains a linear subspace of E of dimension λ. If $M \cup \{0\}$ contains an infinite dimensional linear space we simply say that M is lineable. In Section 3 we will show, among other important results, that the four classes of surjective functions mentioned above are lineable, actually $2^{\mathfrak{c}}$-lineable. It is important to mention that the study of the lineability of sets of strange functions has become a fruitful field since the term "lineability" was coined in 2005 (see [3]). A thorough description of the most relevant lineability problems and other related topics can be found in the monograph [4] or in the expository paper [5]. The interested reader may also consult the references [6–23].

In Section 4 we establish the connection existing between the classes of extremely surjective functions defined in Section 2 with other classes of interesting functions. We consider the sets of additive (that is, $\mathbb{Q}$-linear) mappings, functions with a dense graph, Darboux functions and Sierpiński-Zygmund functions.

This survey paper is written in such a way that it is accessible to the largest possible audience. For this reason we provide a good account of examples, which are presented in detailed for completeness. We also give full proofs of most of the lineability problems introduced in Section 3. We have included the proofs of several well-known topological results in order to make the paper as inclusive and self-contained as possible. However we have decided to omit the proofs that either are too complex or require complicated techniques of set theory.

We will use the following standard definitions and notations: $\mathbb{R}^{\mathbb{R}}$ stands for the set of all mappings from $\mathbb{R}$ to $\mathbb{R}$. D denotes the subset of $\mathbb{R}^{\mathbb{R}}$ of the Darboux functions, i.e., functions that transform intervals into intervals. S, $\mathcal{C}$ and $\mathcal{I}$ will denote, respectively, the sets of surjective, continuous and injective functions from $\mathbb{R}$ to $\mathbb{R}$. If $C \subset \mathbb{R}^2$, then $\mathrm{dom}(C)$ denotes the projection of C on the abscissa axis. If $f \in \mathbb{R}^{\mathbb{R}}$, we will often denote the graph of f, $\mathrm{graph}(f) := \{(x, f(x)) : x \in \mathbb{R}\}$, simply by f.

2 A few examples of extreme surjective functions

In this section we provide a few examples of surjective functions enjoying the property that they transform every non-degenerate interval into the whole real line. We will see that there is a hierarchy among the functions satisfying this property. Let us see first several examples of everywhere surjective functions.

2.1 Everywhere surjective functions

First recall that a mapping $f : \mathbb{R} \to \mathbb{R}$ is everywhere surjective if it transforms non-degenerate intervals into the whole real line, or equivalently, if $f((a,b)) = \mathbb{R}$, for all $a, b \in \mathbb{R}$ with $a < b$. The set of all everywhere surjective mappings is represented by ES. The construction of one everywhere surjective function is not trivial. The first known example of such a function dates back to Lebesgue and is more than a century old. Here we give several more modern examples. The first of them appears in [24] and is presented below in detail for the sake of completeness.

Example 2.1. *If we define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \lim_{n\to\infty} \tan(n!\pi x) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

then f *satisfies the following properties:*

1. *If* $x \in \mathbb{R}$ *and* $q \in \mathbb{Q}$ *then* $f(x+q) = f(x)$.
2. f *is surjective.*

3. *f is surjective on every non-degenerate interval.*

In order to prove the assertions 1–3 the following remark can be useful:

Remark 2.2. *If $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ then*

$$\lim_{n\to\infty} \frac{\lfloor r(n+1) \rfloor}{n+1} = r,$$

for each $r \in \mathbb{R}$. Indeed, $x - 1 \leq \lfloor x \rfloor \leq x$, $\forall x \in \mathbb{R}$. If we set in the previous inequalities $x = r(n+1)$ for arbitrary $r \in \mathbb{R}$ and $n \in \mathbb{N}$, then $r(n+1) - 1 \leq \lfloor r(n+1) \rfloor \leq r(n+1)$. Dividing by $n+1$ we arrive at

$$r - \frac{1}{n+1} = \frac{r(n+1)-1}{n+1} \leq \frac{\lfloor r(n+1) \rfloor}{n+1} \leq \frac{r(n+1)}{(n+1)} = r.$$

Finally, taking limits we conclude that

$$\lim_{n\to\infty} \frac{\lfloor r(n+1) \rfloor}{n+1} = r.$$

Proof.
(1) Given $x \in \mathbb{R}$ and $q \in \mathbb{Q}$, $\exists r, s \in \mathbb{Z}$ such that $q = \frac{r}{s}$. If $n \geq s$, we have that $n!q = n!\frac{r}{s} \in \mathbb{Z}$. Thus $n!\pi x - n!\pi(x+q)$ is a multiple of π. Therefore $\tan(n!\pi(x+q)) = \tan(n!\pi x)$, $\forall n \geq s$. If the limit does not exist, by definition we have $0 = f(x) = f(x+q)$. Otherwise $\lim_{n\to\infty}(\tan(n!\pi(x+q))) = \lim_{n\to\infty}(\tan(n!\pi x))$, from which we conclude that $f(x+q) = f(x)$.

(2) Given $y \in \mathbb{R}$ we choose $r \in [0, 1)$ such that $\tan(\pi r) = y$. Let $x \in \mathbb{R}$ be given by

$$x = \sum_{n=0}^{\infty} \frac{\lfloor nr \rfloor}{n!}.$$

It remains to show that $f(x) = y$. Let us consider the nth-partial sum of x

$$x_n = \sum_{k=0}^{n} \frac{\lfloor rk \rfloor}{k!}$$

and the remaining terms in x by

$$\epsilon_n = \sum_{k=n+1}^{\infty} \frac{\lfloor rk \rfloor}{k!}.$$

Of course $x = x_n + \epsilon_n$. Notice that $n!x_n \in \mathbb{Z}$, $\forall n$, and hence, by the previous step we have that $\tan(n!\pi x) = \tan(n!\pi\epsilon_n)$, $\forall n$. Therefore

$$n!\epsilon_n = n! \sum_{k=n+1}^{\infty} \frac{\lfloor rk \rfloor}{k!} = \frac{\lfloor r(n+1) \rfloor}{n+1} + n! \sum_{k=n+2}^{\infty} \frac{\lfloor rk \rfloor}{k!}.$$

Since

$$\lim_{n\to\infty} \frac{\lfloor r(n+1) \rfloor}{n+1} = r$$

and

$$\lim_{n\to\infty} n! \sum_{k=n+2}^{\infty} \frac{\lfloor rk \rfloor}{k!} = 0,$$

we conclude that $\lim_{n\to\infty} n!\epsilon_n = r$, from which

$$f(x) = \lim_{n\to\infty} \tan(n!\pi x) = \lim_{n\to\infty} \tan(n!\pi\epsilon_n) = \tan(\pi r) = y.$$

(3) Assume that $a, b, y \in \mathbb{R}$ with $a < b$. By (2) there exists $u \in \mathbb{R}$ such that $f(u) = y$, and by (1) we have that

$f(u) = f(u+q) = y$, $\forall q \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $a < u+q < b$. If $x = u+q$ then $a < x < b$ and

$$\begin{aligned} f(x) &= \lim_{n\to\infty} (\tan(n!\pi(u+q))) \\ &= \lim_{n\to\infty} \frac{\tan(n!\pi u) + \tan(n!\pi q)}{1 - \tan(n!\pi u)\tan(n!\pi q)} \\ &= \lim_{n\to\infty} \tan(n!\pi u) \\ &= f(u) = y. \end{aligned}$$

Observe that $\tan(n!\pi q) \to 0$ as $n \to \infty$. We conclude that $\{f(x) : a < x < b\} = \mathbb{R}$. □

The second construction we present in this section is based on the fact that every interval contains a Cantor like set and that Cantor sets are uncountable. The example is taken from [23] (see also [4], [5] and [15]).

Example 2.3. *We construct a mapping $f : \mathbb{R} \to \mathbb{R}$ as follows: Let $(I_n)_{n\in\mathbb{N}}$ be a sequence containing all the intervals with rational endpoints. Then I_1 contains a Cantor like set, which we denote by C_1. On the other hand, $I_2 \setminus C_1$, contains another Cantor like set, which is denoted by C_2. Now the set $I_3 \setminus (C_1 \cup C_2)$ contains a new Cantor like set, namely C_3. Repeating this process, we construct by induction a sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise disjoint Cantor like sets, such that $I_n \setminus \left(\bigcup_{k=1}^{n-1} C_k\right) \supset C_n$. Since C_n is uncountable, there exists a bijection $\varphi_n : C_n \to \mathbb{R}$, for every $n \in \mathbb{N}$. It is now that we define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} \varphi_n(x) & \text{if } x \in C_n, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, if $I \subset \mathbb{R}$ is a non-degenerate interval, then there exists $k \in \mathbb{N}$ with $I \supset I_k$. By construction of C_n we have that $I_k \supset C_k$ and hence, by definition of f, $f(I) \supset f(I_k) \supset f(C_k) = \varphi_k(C_k) = \mathbb{R}$. Interestingly, the mapping f is null almost everywhere in $\mathbb{R}$.

The third example we provide is based on the fact that there is a partition of $\mathbb{R}$ into $\mathfrak{c}$ many dense sets. This can be achieved by considering the relationship in $\mathbb{R}$ given by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q} \ (x, y \in \mathbb{R}).$$

The equivalence classes have the form $[\alpha] = \alpha + \mathbb{Q}$ and are obviously pairwise disjoint, dense sets in $\mathbb{R}$. Since $[\alpha]$ is countable and $\mathbb{R} = \bigcup_{\alpha\in\mathbb{R}} [\alpha]$, it is obvious that $\mathbb{R}/\sim$ contains $\mathfrak{c}$ elements.

Example 2.4. *Let $\{D_\alpha : \alpha \in \mathbb{R}\}$ be a partition of $\mathbb{R}$ into $\mathfrak{c}$ dense sets. If we define now $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = \alpha$, $\forall x \in D_\alpha$, then f is obviously everywhere surjective.*

Remark 2.5. *The construction of the mapping in Example 2.4 shows clearly that f attains every real number infinitely countably many times in each non-degenerate interval. This property is shared by all functions in* ES. *Indeed, let (a, b) be an interval with $a, b \in \mathbb{R}$ and $a < b$, $y \in \mathbb{R}$ and $f \in$ ES. Suppose (I_n) is a sequence of open, non-empty, pairwise-disjoint intervals in I. For instance we can take*

$$I_n = \left(a + \frac{b-a}{n+1}, a + \frac{b-a}{n}\right),$$

for every $n \in \mathbb{N}$. Since $f \in$ ES, there exists $x_n \in I_n \subset I$ such that $f(x_n) = y$. Since the I_n's are pairwise-disjoint, we have constructed a sequence (x_n) of distinct points in (a, b) such that $f(x_n) = y$.

The next lemma will be very useful throughout the paper. For instance, if we apply it to the family

$$\{(a, b) \times \{y\} : a, b, y \in \mathbb{R} \text{ and } a < b\},$$

we obtain again an example of a function in ES. We recall that if $A \subset \mathbb{R}^2$ then $\mathrm{dom}(A)$ denotes the projection of A over the abscissa axis.

Lemma 2.6. *Let $\{A_\alpha\}_{\alpha<\mathfrak{c}}$ be a family of subsets in $\mathbb{R}^2$ such that* $\operatorname{card}(\operatorname{dom}(A_\alpha)) = \mathfrak{c}$*, for each $\alpha < \mathfrak{c}$. Then there exists a function $f \in \mathbb{R}^{\mathbb{R}}$ such that $f \cap A_\alpha \neq \varnothing$.*

Proof. We proceed to construct the function by transfinite induction. Let $\beta < \mathfrak{c}$. Assume we have already constructed $f(x_\lambda)$ for $\lambda < \beta$. We choose $x_\beta \in \operatorname{dom}(A_\beta) \setminus \{x_\lambda : \lambda < \beta\}$. Observe that $\operatorname{dom}(A_\beta) \setminus \{x_\lambda : \lambda < \beta\} \neq \varnothing$ since $\operatorname{card}(\operatorname{dom}(A_\beta)) = \mathfrak{c}$ and $\beta < \mathfrak{c}$. Now, as $x_\beta \in \operatorname{dom}(A_\beta)$, there exists $y_\beta \in \mathbb{R}$ such that $(x_\beta, y_\beta) \in A_\beta$. Define $f(x_\beta) = y_\beta$. This construction defines f on a set of cardinality $\mathfrak{c}$ of $\mathbb{R}$, namely, on $\{x_\alpha : \alpha < \mathfrak{c}\}$. If we set $f(x) = 0$ for $x \notin \{x_\alpha : \alpha < \mathfrak{c}\}$, then $f \cap A_\alpha \neq \varnothing$ for each $\alpha < \mathfrak{c}$. □

Among the functions in ES there are some that are yet *more surjective* since they are able to attain every real number uncountably many times in each non-degenerate interval of $\mathbb{R}$. We introduce these functions in the next subsection.

2.2 Strongly everywhere surjective functions

Recall that $f : \mathbb{R} \to \mathbb{R}$ is strongly everywhere surjective if f attains every real number $\mathfrak{c}$ many times in each non-degenerate interval of $\mathbb{R}$. The set of all the strongly everywhere surjective functions is denoted by SES. Obviously, we have that SES $\subset$ ES. Let us check first that SES is non-empty.

Recall that the Cantor set is homeomorphic to $\{0,1\}^{\mathbb{N}}$, which, in its turn is homeomorphic to

$$\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} = \bigcup_{\alpha\in\{0,1\}^{\mathbb{N}}} \big(\{0,1\}^{\mathbb{N}} \times \{\alpha\}\big).$$

Since the $\{0,1\}^{\mathbb{N}} \times \{\alpha\}$'s are pairwise disjoint sets homeomrphic to the Cantor set, we have the following:

Lemma 2.7. *Let C be a Cantor-like set (i.e., homeomorphic to the Cantor set) in $[a,b]$ and $a, b \in \mathbb{R}$ with $a < b$. Then there is a family $\{C_\lambda : \lambda \in \mathbb{R}\}$ of Cantor-like subsets of C such that $C = \bigcup_{\lambda\in\mathbb{R}} C_\lambda$ and $C_\lambda \cap C_\mu = \varnothing$ for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$.*

An example of a mapping in SES can be constructed using Lemma 2.7 by adapting Example 2.3. The example is taken from [15] (see also [4] and [5]).

Example 2.8. *In Example 2.3 we had a sequence $\big(C_n\big)_{n\in\mathbb{N}}$ of pairwise disjoint, Cantor like sets such that $I_n \setminus \big(\bigcup_{k=1}^{n-1} C_k\big) \supset C_n$ for every $n \in \mathbb{N}$. Now, according to Lemma 2.7, for each $n \in \mathbb{N}$ there is a partition $\{C_n^i : i \in \mathbb{R}\}$ of C_n consisting of Cantor-like sets. Since the C_n^i's are uncountable, for each $n \in \mathbb{R}$ and $i \in \mathbb{R}$ there exists a bijection $\varphi_n^i : C_n^i \to \mathbb{R}$. Finally, define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} \varphi_n^i(x) & \text{if } x \in C_n^i, \\ 0 & \text{otherwise.} \end{cases}$$

It only remains to show that f is strongly everywhere surjective. Indeed, take $I \subset \mathbb{R}$ a non-degenerate interval. Then there exists $k \in \mathbb{N}$ with $I \supset I_k$. For this k we have $f(I) \supset f(I_k) \supset f(C_k^i) = \varphi_k^i(C_k^i) = \mathbb{R}$. Also, f attains obviously every real number $\mathfrak{c}$ times in I.

Remark 2.9. *Notice that the function constructed in Example 2.4 attains every real number only countably many times in every interval, and therefore it is* ES *but not* SES*. Hence*

$$\text{SES} \subsetneq \text{ES}\,.$$

In the next section we will see that in the case where the Continuum Hypothesis is not assumed, there is a hierarchy of degrees of surjectivity between the classes ES and SES.

2.3 Everywhere κ-surjective functions

If κ is a cardinal number such that $\aleph_0 \leq \kappa \leq \mathfrak{c}$, we say that a function $f \in \mathbb{R}^{\mathbb{R}}$ is *everywhere κ-surjective* if for every $y \in \mathbb{R}$, f attains y at least κ times in every non-degenerate interval. We denote by ES_κ the set consisting of all the everywhere κ-surjective functions in $\mathbb{R}^{\mathbb{R}}$.

Observe that

- $\mathrm{ES}_{\aleph_0} = \mathrm{ES}$.
- $\mathrm{ES}_{\mathfrak{c}} = \mathrm{SES}$.
- If $\aleph_0 \leq \lambda \leq \kappa \leq \mathfrak{c}$ then $\mathrm{ES}_\kappa \subset \mathrm{ES}_\lambda$.

Given κ such that $\aleph_0 \leq \kappa < \mathfrak{c}$, we construct in the following example an everywhere κ-surjective function that is not everywhere κ^+-surjective. This shows that

$$\mathrm{ES}_\kappa \subsetneq \mathrm{ES}_\lambda,$$

for all κ, λ with $\aleph_0 \leq \lambda < \kappa \leq \mathfrak{c}$.

Example 2.10. *Let κ be a cardinal number with $\aleph_0 \leq \kappa \leq \mathfrak{c}$ and consider $\{D_\alpha : \alpha \in \mathbb{R}\}$ the partition of $\mathbb{R}$ into $\mathfrak{c}$ many dense, countable sets constructed in the comments preceding Example 2.4. Now consider a partition $\{\kappa_\beta : \beta \in \mathbb{R}\}$ of $\mathbb{R}$ into $\mathfrak{c}$ many sets κ_β of cardinality κ and set $D'_\beta = \bigcup_{\alpha \in \kappa_\beta} D_\alpha$ for every $\beta \in \mathbb{R}$. Since $\aleph_0 \leq \kappa$, the D'_β's have cardinality κ. Therefore $\{D'_\beta : \beta \in \mathbb{R}\}$ is a partition of $\mathbb{R}$ into $\mathfrak{c}$ many κ-dense sets. If we define $f(x) = \beta$ for all $x \in D'_\beta$ for all $\beta \in \mathbb{R}$, f is obviously everywhere κ-surjective. Also, observe that f attains every real number exactly κ times in every non-degenerate interval, which shows, additionally, that f cannot be everywhere λ-surjective for every $\lambda > \kappa$.*

The functions in SES might seem sufficiently *special* or *pathological*, however it is possible to construct even more *surprising* functions in the class SES, as we will see in the next two sections.

2.4 Perfectly everywhere surjective functions

Observe that in the definition of strong everywhere surjectivity we can restrict ourselves without loss of generality to closed, non-degenerate intervals. In other words, a function $f : \mathbb{R} \to \mathbb{R}$ is strongly everywhere surjective if and only if f attains every real number $\mathfrak{c}$ times in every non-degenerate, closed interval. Now, a non-degenerate, closed interval is a simple example of perfect set. We recall that $P \subset \mathbb{R}$ is perfect if $P' = P$. The question that arises now is whether a strongly everywhere surjective function attains each real number $\mathfrak{c}$ times in every perfect set. The answer to this question is no. Indeed, we just need to consider the function defined in Example 2.8, which is SES. However f attains every real number only once in each Cantor set C_n^i, which is perfect. From now on, we will say that $f : \mathbb{R} \to \mathbb{R}$ is *perfectly everywhere surjective* if f is surjective on every perfect set. The set of all perfectly everywhere surjective mappings is denoted by PES. We will see later that $f \in \mathrm{PES}$ if and only if f attains every real number $\mathfrak{c}$ times in every perfect set $P \subset \mathbb{R}$. This shows that the elements of PES represent a stronger form of surjectivity than the elements of SES. The example of a PES function we provide here is taken from [15]. In its construction we will need the following well-known fact, whose proof is given for completeness.

Lemma 2.11. *If P is perfect, then* $\mathrm{card}(P) = \mathfrak{c}$.

Proof. Without loss of generality, we can assume that P is bounded. Then, since P is closed, there exist $\alpha = \min P$ and $\beta = \max P$. Then $P \subseteq [\alpha, \beta]$. If m is the middle point of $[\alpha, \beta]$, we define

$$P(0) := \begin{cases} [\alpha, m] \cap P & \text{if } m \in ([\alpha, m] \cap P)', \\ [\alpha, m) \cap P & \text{if } m \notin ([\alpha, m] \cap P)', \end{cases}$$

and

$$P(1) := \begin{cases} [m,\beta] \cap P & \text{if } m \in ([m,\beta] \cap P)', \\ (m,\beta] \cap P & \text{if } m \notin ([m,\beta] \cap P)', \end{cases}$$

Then $P(0)$ and $P(1)$ are perfect, infinite sets. If we repeat the same process in $P(0)$ and $P(1)$ we obtain perfect subsets of P, $P(0,0)$, $P(0,1)$ on the one hand, and $P(1,0)$, $P(1,1)$ on the other. This process defines $P(\alpha_1,\dots,\alpha_n)$ for all $n \in \mathbb{N}$ and for every choice of zeros and ones $\alpha_1,\dots,\alpha_n$. Let us consider a sequence $\alpha = (\alpha_n) \in \{0,1\}^{\mathbb{N}}$. Using Cantor's Theorem there exists $x(\alpha) \in P$ such that

$$\bigcap_{k=1}^{\infty} P(\alpha_1,\alpha_2,\dots,\alpha_k) = \{x(\alpha)\}.$$

Let us consider now two different sequences in $\{0,1\}^{\mathbb{N}}$, $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$, and assume n_0 is a natural number such that $\alpha_{n_0} \neq \beta_{n_0}$. Then

$$P(\alpha_1,\dots,\alpha_{n_0}) \cap P(\beta_1,\dots,\beta_{n_0}) = \varnothing,$$

from which $x(\alpha) \neq x(\beta)$. Since $\operatorname{card}(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}$, the proof is finished. □

Example 2.12. *Consider the family*

$$\{P \times \{y\} : \ P \subset \mathbb{R} \text{ is perfect and } y \in \mathbb{R}\},$$

whose cardinality is $\mathfrak{c}$ *because perfect sets are closed and there are only* $\mathfrak{c}$ *many closed sets. Notice also that* $\operatorname{card}(\operatorname{dom}(P \times \{y\})) = \operatorname{card}(P) = \mathfrak{c}$. *Applying now Lemma 2.6 to this family we obtain a function* $f \in$ PES.

Remark 2.13. *Notice that* PES $\subsetneq$ SES. *Actually the function constructed in Example 2.8 is in* SES $\setminus$ PES*, as pointed out at the beginning of this section.*

There is an even stronger form of surjectivity than perfectly everywhere surjective functions that will be studied in the next section.

2.5 Jones functions

In 1942, F. B. Jones [2] found an example of a function in $\mathbb{R}^{\mathbb{R}}$ such that for any closed subset $C \subset \mathbb{R}^2$ with uncountable projection over the abscissa axis, $f \cap C \neq \varnothing$. A function satisfying this property is called a Jones function. The set of all Jones functions is denoted by J. (Notice that, since $\operatorname{dom}(C)$ is σ-compact, then uncountable is equivalent to cardinality $\mathfrak{c}$ in the previous definition.)

Example 2.14. *In order to obtain a function* $f \in$ J*, we just need to apply Lemma 2.6 to the family*

$$\{C \subset \mathbb{R}^2 : \ C \text{ is closed and } \operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}\}.$$

Remark 2.15. *Observe that if* $f \in$ J*, then* $f \in$ PES *since* $P \times \{y\}$ *is closed in* $\mathbb{R}^2$ *for all perfect set* $P \subset \mathbb{R}$. *Therefore* J $\subset$ PES.

Consider the function f *constructed in the proof of Lemma 2.6 for the family*

$$\{(P \setminus \{y\}) \times \{y\} : \ P \subset \mathbb{R} \text{ is perfect and } y \in \mathbb{R}\}.$$

Then $f \cap C = \varnothing$ *where* C *is the closed set* $\{(x,x) : x \geq 1\}$. *Hence* $f \in$ PES *but* $f \notin$ J*, and therefore*

$$\text{J} \subsetneq \text{PES}.$$

3 Algebraic size of sets of surjective functions

In this section we discuss the algebraic size of the sets ES, ES_κ, SES, PES and J from the lineability viewpoint.

In order to prove that ES, SES and PES are $2^\mathfrak{c}$-lineable, the following result will be crucial. We reproduce the original proof for completeness:

Lemma 3.1 (Aron et al. [3]). *There exists a vector subspace V_0 of $\mathbb{R}^\mathbb{R}$ whose dimension is $2^\mathfrak{c}$ such that every non-null element of V_0 is surjective. In other words,* S *is $2^\mathfrak{c}$-lineable.*

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}^\mathbb{N}$ be a bijection that transforms $(0, 1)$ into the set of sequences whose first element is 0. For each $A \subset \mathbb{R}$, we define

$$H_A : \mathbb{R}^\mathbb{N} \longrightarrow \mathbb{R}$$

by

$$H_A(y, x_1, x_2, x_3, \dots) = y \cdot \prod_{i=1}^{\infty} I_A(x_i),$$

Where I_A is the characteristic function of A. We have the following:

(a) The family $\{H_A : A \subset \mathbb{R}, A \neq \varnothing\}$ is linearly independent. In order to prove it, let us consider m different subsets $C_1, C_2, \cdots, C_m$, of $\mathbb{R}$ and m non-null numbers $\lambda_1, \lambda_2, \dots, \lambda_m$. Assume that

$$\sum_{j=1}^{m} \lambda_j H_{C_j} \equiv 0.$$

Since the C_j's are different, there exists $k \in \{1, 2, \dots, m\}$ and x_j such that $x_j \in C_k \setminus C_j$ for each $j \neq k$. In order to see the latter, assume that for every $k \in \{1, \dots, m\}$, there exists $j \neq k$ such that $C_k \setminus C_j = \varnothing$. This would be equivalent to saying that for all $k \in \{1, \dots, m\}$ there exists $j \neq k$ with $C_k \subset C_j$. Renaming the sets if necessary, we would have:

$$C_1 \subset C_2 \subset \ldots \subset C_m \subset C_\alpha,$$

where $\alpha \in \{1, ..., m-1\}$. This would imply that at least two sets coincide, which is a contradiction.
Now, we can set, without loss of generality, that $k = m$. let

$$\overline{x} = (1, x_1, x_2, \dots, x_{m-2}, x_{m-1}, x_{m-1}, x_{m-1}, \dots).$$

We have that

$$\begin{aligned} 0 = \sum_{j=1}^{m} \lambda_j H_{C_j}(\overline{x}) &= 1 \cdot \sum_{j=1}^{m} \left[\lambda_j \cdot \prod_{i=1}^{\infty} I_{C_j}(x_i) \right] \\ &= \left[\lambda_1 \cdot \prod_{i=1}^{\infty} I_{C_1}(x_i) \right] + \left[\lambda_2 \cdot \prod_{i=1}^{\infty} I_{C_2}(x_i) \right] + \cdots \\ &\quad \cdots + \left[\lambda_{m-1} \cdot \prod_{i=1}^{\infty} I_{C_{m-1}}(x_i) \right] + \left[\lambda_m \cdot \prod_{i=1}^{\infty} I_{C_m}(x_i) \right] \\ &= 0 + 0 + \cdots + 0 + \lambda_m = \lambda_m. \end{aligned}$$

(b) Since the λ_k's were not null, we have reached a contradiction, and therefore the family $\{H_A : A \subset \mathbb{R}, A \neq \varnothing\}$ is linearly independent.

(c) Observe that H_A is surjective for every $A \subset \mathbb{R}$ since, for every $s \in \mathbb{R}$, we have that $H_A(s, a, a, a, \dots) = s$, where $a \in A$.

(d) In order to see that $h \in \Gamma = \text{span}\{H_A : A \subset \mathbb{R}, A \neq \varnothing\}$, $h \neq 0$ is surjective, we can proceed as in part (a) above.

(e) It is clear that $\dim(\Gamma) = 2^c$ since $\operatorname{card}(\{A : A \subset \mathbb{R}, A \neq \varnothing\}) = 2^{\mathfrak{c}}$.

The space we are looking for is

$$V_0 = \operatorname{span}\{H_A \circ \varphi : A \subseteq \mathbb{R}\}. \qquad \square$$

Remark 3.2. *In connection with Lemma 3.1, the reader may find of interest the fact that* $\mathrm{S} \cap \mathcal{C}$ *is* $\mathfrak{c}$*-lineable. To see this we just need to realize that the span of* $\{e^{rx} - e^{-rx} : r \in (0, \infty)\}$ *is a* $\mathfrak{c}$*-dimensional space contained in* $(\mathrm{S} \cap \mathcal{C}) \cup \{0\}$.

Theorem 3.3. *The sets* ES*,* SES *and* PES *are* $2^{\mathfrak{c}}$*-lineable.*

Proof. Let us choose $f_1 \in \mathrm{ES}$, $f_2 \in \mathrm{SES}$ and $f_3 \in \mathrm{PES}$. Then the spaces

$$E_k := \{f \circ f_k : f \in V_0\},$$

for $k = 1, 2, 3$, where V_0 is as in Lemma 3.1, satisfy $E_1 \subset \mathrm{ES} \cup \{0\}$, $E_2 \subset \mathrm{SES} \cup \{0\}$ and $E_3 \subset \mathrm{PES} \cup \{0\}$, and have cardinality $2^{\mathfrak{c}}$. $\square$

It turns out that J is $2^{\mathfrak{c}}$-lineable too. This is proved in Theorem 3.6 below. Since J is a subset of all the other classes of surjective functions introduced in Section 2, Theorem 3.6 also proves that ES, SES and PES are $2^{\mathfrak{c}}$-lineable. From this viewpoint Theorem 3.3 (and hence Lemma 3.1 too) would be unnecessary. We have decided to include Theorem 3.3 because its proof is accessible to a much larger audience.

The proof of the $2^{\mathfrak{c}}$-lineability is based on a couple of topological results about Bernstein sets. We recall that $B \subseteq \mathbb{R}$ is a Bernstein set if for every perfect set $P \subset \mathbb{R}$, we have that $B \cap P \neq \varnothing$ and $(\mathbb{R} \setminus B) \cap P \neq \varnothing$.

Lemma 3.4. *There exists a family* $\{B_\alpha : \alpha < \mathfrak{c}\}$ *of pairwise disjoint, Bernstein subsets of* $\mathbb{R}$ *such that*

$$\mathbb{R} = \bigcup_{\alpha < \mathfrak{c}} B_\alpha.$$

Proof. It suffices to find in $\mathbb{R}$ $\mathfrak{c}$ many pairwise disjoint sets in $\mathbb{R}$, B_α, $\alpha < \mathfrak{c}$, such that B_α is perfectly dense, i.e., $B_\alpha \cap P \neq \varnothing$ for every perfect set $P \subset \mathbb{R}$. Indeed, if $\alpha < \beta < \mathfrak{c}$ we have also $B_\beta \cap P \neq \varnothing$, so $(\mathbb{R} \setminus B_\alpha) \cap P \neq \varnothing$, and hence B_α is a Bernstein set.

In principle there is no need to assume that

$$\bigcup_{\alpha < \mathfrak{c}} B_\alpha = \mathbb{R}.$$

In order to see the latter, suppose we have already constructed a family $\{B_\alpha : \alpha < \mathfrak{c}\}$ of pairwise disjoint Bernstein sets and enumerate

$$\mathbb{R} \setminus \bigcup_{\alpha < \mathfrak{c}} B_\alpha = \{z_\alpha : \alpha < \kappa\},$$

where $\kappa \leq \mathfrak{c}$. If we set

$$B_\alpha^* := \begin{cases} B_\alpha \cup \{z_\alpha\} & \text{if } \alpha < \kappa, \\ B_\alpha & \text{if } \kappa \leq \alpha < \mathfrak{c}, \end{cases}$$

then these new sets are also pairwise disjoint, Bersntein sets and their union is $\mathbb{R}$.

Let us enumerate the perfect sets of $\mathbb{R}$ as $\{P_\beta : \beta < \mathfrak{c}\}$. We just need to construct by transfinite induction a double sequence $(x_{\alpha\beta})_{\alpha,\beta<\mathfrak{c}}$ of different elements in $\mathbb{R}$ in such a way that $x_{\alpha\beta} \in P_\beta$ for all $\alpha, \beta < \mathfrak{c}$ because in that case the sets $B_\alpha = \{x_{\alpha\beta} : \beta < \mathfrak{c}\}$ satisfy what we need.

Suppose that in the step γ of the induction we have constructed the elements $x_{\alpha\beta}$, where $\alpha, \beta < \gamma$. Since the cardinality of the constructed elements is $\gamma^2 < \mathfrak{c}$, we can choose $2\gamma + 1$ additional elements, namely, $x_{\alpha\gamma} \in P_\gamma$ with $\alpha < \gamma$ and $x_{\lambda\beta} \in P_\beta$ with $\beta \leq \gamma$. Therefore we have constructed $\{x_{\alpha\beta} : \alpha, \beta \leq \gamma\}$. $\square$

Lemma 3.5. *Let B be a Bernstein set. There exists a Jones function f such that for all $g \in \mathbb{R}^{\mathbb{R}}$ such that $f|_B \equiv g|_B$ then g is a Jones function.*

Proof. It is enough to apply Lemma 2.6 to the family

$$\{(B \times \mathbb{R}) \cap C : C \subset \mathbb{R}^2 \text{ is closed and } \operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}\}.$$

To show that this family satisfies the hipothesis of Lemma 2.6 it suffices to prove that $\operatorname{card}(B \cap \operatorname{dom}(C)) = \mathfrak{c}$ because $\operatorname{dom}((B \times \mathbb{R}) \cap C) = B \cap \operatorname{dom}(C)$. Indeed, $\operatorname{dom}(C)$ is a σ-compact set of cardinality $\mathfrak{c}$ and therefore at least one of the compact sets that form the union must have cardinality $\mathfrak{c}$. Hence that compact is the union of a perfect set and a countable set (see [25]) and so it must contain a perfect set. On the other hand, any perfect set contains a Cantor-like set C^*. Taking into account Lemma 2.7 it is straightforward that $\operatorname{card}(B \cap C^*) = \mathfrak{c}$. □

The $2^{\mathfrak{c}}$-lineability of J is proved in [14]. We reproduce below the author's proof for completeness.

Theorem 3.6 (Gámez-Merino, [14])**.** *The set* J *is* $2^{\mathfrak{c}}$*-lineable.*

Proof. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ as in Lemma 3.4. For each $\alpha < \mathfrak{c}$ let f_α be a function in J such that every $g \in \mathbb{R}^{\mathbb{R}}$ with $f_\alpha|_{B_\alpha} \equiv g|_{B_\alpha}$ satisfies that $g \in$ J (see Lemma 3.5). We can also assume that $f_\alpha|_{\mathbb{R} \setminus B_\alpha} \equiv 0$. Now consider the set

$$V := \left\{ \sum_{\alpha < \mathfrak{c}} \varphi(\alpha) f_\alpha : \varphi \in \mathbb{R}^{\mathfrak{c}} \right\}.$$

Observe that V is clearly a linear space and that every non-null element of V is in J by Lemma 3.5 because if $\varphi(\beta) \neq 0$ for some $\beta < \mathfrak{c}$ then $\sum_{\alpha<\mathfrak{c}} \varphi(\alpha) f_\alpha$ coincides with f_β in B_β. Also, V is isomorphic to $\mathbb{R}^{\mathfrak{c}}$, whose cardinality is $2^{\mathfrak{c}}$, which concludes the proof. □

Remark 3.7. *As mentioned above it is interesting to observe that the space defined in Theorem 3.6 also proves that the other classes of surjective functions introduced in Section 2, namely* ES*,* ES_κ*,* SES *and* PES *are also* $2^{\mathfrak{c}}$*-lineable since* J *is a subset of them.*

Another fact that reveals that the size of J (and hence the size of ES, ES_κ, SES and PES too) is enormous, is shown by the following result, whose proof can be deduced from the fact that the *additivity* of J is bigger than 2 (see [16] for details). However, we give below our own proof:

Theorem 3.8. *For every $f \in \mathbb{R}^{\mathbb{R}}$ there exist $g, h \in$ J such that $f = g + h$.*

Proof. For $f \in \mathbb{R}^{\mathbb{R}}$, let us consider the family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where

$$\mathcal{F}_1 := \{C \subset \mathbb{R}^2 : C \text{ is closed and } \operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}\},$$
$$\mathcal{F}_2 := \left\{\{(x, y + f(x)) : (x, y) \in C\} : C \subset \mathbb{R}^2 \text{ is closed and } \operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}\right\}.$$

Let $g \in \mathbb{R}^{\mathbb{R}}$ be the function constructed in Lemma 2.6 for the family $\mathcal{F}$. Since $g \cap C \neq \varnothing$ for all closed $C \subset \mathbb{R}^2$ with $\operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}$ we have that $g \in$ J. We also have that $(g - f) \cap C \neq \varnothing$ for all closed $C \subset \mathbb{R}^2$ with $\operatorname{card}(\operatorname{dom}(C)) = \mathfrak{c}$, which implies that $g - f \in$ J, and hence $h := f - g \in$ J. □

In the rest of this section we present a series of results showing what is known nowadays about the algebraic size of the sets S \ ES, $\mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa$ with $\aleph_0 \leq \lambda < \kappa \leq \mathfrak{c}$, SES \ PES and PES \ J. Among the above problems we know the optimal solution to only two of them:

Theorem 3.9 (Gámez-Merino et al. [15, Theorem 2.7])**.** *The sets* S \ ES *and* SES \ PES *are* $2^{\mathfrak{c}}$*-lineable, and this is optimal.*

For the rest of the cases we only have partial and probably not optimal answers.

Theorem 3.10 (Bartoszewicz et al. [8, Theorem 3.12])**.** *If $\aleph_0 \leq \lambda < \kappa \leq \mathfrak{c}$ then* $\mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa$ *is* 2^λ*-lineable.*

Remark 3.11. *Although it is not explicitly shown in [8], it can be deduced that* $\mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa$ *is* 2^μ*-lineable for every* $\mu < \kappa$ *since, in that case,* $\mathrm{ES}_\mu \setminus \mathrm{ES}_\kappa \subset \mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa$ *and, by Theorem 3.10 is* 2^μ*-lineable.*

The case $\lambda = \aleph_0$ *and* $\kappa = \mathfrak{c}$ *is explicitly studied in [11, Theorem 2.14].*

Also, it can be proved that the result given in Theorem 3.10 is not optimal in general. If we admit Martin's Axiom, $2^\mu = \mathfrak{c}$ *for all* $\aleph_0 \leq \mu < \mathfrak{c}$*. However we have the following result:*

Theorem 3.12 (Ciesielski et al. [11, Corollary 2.15])**.** *The set* $\mathrm{ES} \setminus \mathrm{SES}$ *is* $\mathfrak{c}^+$*-lineable.*

The estimate given in Theorem 3.12 implies that $\mathrm{ES} \setminus \mathrm{SES}$ is $2^\mathfrak{c}$-lineable under CH (Continuum Hypothesis). Whether or not $\mathrm{ES} \setminus \mathrm{SES}$ is $2^\mathfrak{c}$-lineable in ZFC (Zermelo-Fraenkel Theory with Axiom of Choice) is still an open question.

We do not know much about the size of the set $\mathrm{PES} \setminus \mathrm{J}$. We do not even know whether this family is lineable or not.

4 Relationship between extremely surjective functions and other classes

In this section we will study the relationship between the class ES and other families of interesting functions like, for instance, the class of additive mappings (or equivalently, $\mathbb{Q}$-linear), the class of function in $\mathbb{R}^\mathbb{R}$ with dense graph in $\mathbb{R}^2$, the class of Darboux functions and the set of Sierpińiski-Zygmund functions. We will deal in the first place with additive mappings and functions with a dense graph.

4.1 Everywhere surjective functions, additive mappings and functions with a dense graph

Recall that $f \in \mathbb{R}^\mathbb{R}$ is addtive if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. It is easy to prove that a function is $\mathbb{Q}$-linear if and only if it is additive. We denote the sets of additive mappings and the set of functions with a dense graph, respectively by Add and DG.

The classes DG and Add are related to ES as follows:

(a) $\mathrm{ES} \subset \mathrm{DG}$, which is obvious, and

(b) $\mathrm{ES} \cap \mathrm{Add} = \mathrm{Add} \cap (\mathrm{S} \setminus \mathcal{I})$. Recall that S and $\mathcal{I}$ denote, respectively, the surjective and injective elements of $\mathbb{R}^\mathbb{R}$.

In order to see (b), we reproduce the argument used in [20]. Observe first that $f : \mathbb{R} \to \mathbb{R}$ is in ES if and only if $f^{-1}(t)$ is dense for all $t \in \mathbb{R}$. Also, any 1-dimensional $\mathbb{Q}$-subspace of $\mathbb{R}$ is dense, and therefore any proper $\mathbb{Q}$-subspace of $\mathbb{R}$ is dense too. Since $\mathrm{ES} \cap \mathrm{Add} \subset \mathrm{Add} \cap (\mathrm{S} \setminus \mathcal{I})$ is trivially true, assume that $f \in \mathrm{Add} \cap (\mathrm{S} \setminus \mathcal{I})$. Since f is surjective, for every $t \in \mathbb{R}$ there exists $x \in \mathbb{R}$ with $f(x) = t$. Notice that $f^{-1}(t) = x + \ker(f)$. Also $\ker(f)$ is dense. Indeed, since f is not injective, $\ker(f) \neq \{0\}$, and hence the $\mathbb{Q}$-subspace $f^{-1}(0) = \ker(f)$ is dense. We conclude that $f^{-1}(t)$ is dense for all t, or in other words, $f \in \mathrm{ES} \cap \mathrm{Add}$.

It is easy to prove that $\mathrm{ES} \cap \mathrm{Add} \neq \varnothing$. Indeed, if $H = \{h_i : i < \mathfrak{c}\}$ is a Hamel basis, we just need to define f on H such that f is surjective and not injective. Extending f to $\mathbb{R}$ by linearity we obtain an additive mapping in $\mathrm{ES} \cap \mathrm{Add}$. In fact we have a much stronger result whose original proof, for completeness, is given below:

Theorem 4.1 (García-Pacheco et al. [20])**.** *The set* $\mathrm{ES} \cap \mathrm{Add}$ *is* $2^\mathfrak{c}$*-lineable.*

Proof. Consider a Hamel basis I of $\mathbb{R}$ regarded as a $\mathbb{Q}$-linear space and let $\Phi : I \to \mathbb{R}$ be bijective. Define

$$W = \{g \circ \Phi : g \in V_0\},$$

where V_0 is a $2^\mathfrak{c}$-dimensional space of surjective functions (except for the zero function). Clearly $\mathrm{card}(W) = 2^\mathfrak{c}$ and each non-null element $f : I \to \mathbb{R}$ of W is a surjective function that can be extended by linearity, uniquely, to

a $\mathbb{Q}$-linear mapping $\bar{f} : \mathbb{R} \to \mathbb{R}$. A moment's thought reveals that $U = \{\bar{f} : f \in W\}$ is in fact a $2^{\mathfrak{c}}$-dimensional space contained in $(\mathrm{ES} \cap \mathrm{Add}) \cup \{0\}$. □

It is interesting to observe that the non-null elements of the space V_0 introduced in Lemma 3.1 are not $\mathbb{Q}$-linear. Indeed, using the terminology of the proof of Lemma 3.1, for each $f \in V_0$ and $x \in (0,1)$, we have $f(x) = (H_A \circ \varphi)(x) = H_A(0, x_1, x_2, ...) = 0 \cdot \prod_{i=1}^{\infty} I_A(x_i) = 0$ from which f is neither injective nor lies in ES. Hence f cannot be $\mathbb{Q}$-linear. It is still possible to prove that $\mathrm{ES} \setminus \mathrm{Add}$ is not only empty, but also algebraically large. We give the proof for completeness.

Theorem 4.2 (García-Pacheco et al., [20])**.** *The set* $\mathrm{ES} \setminus \mathrm{Add}$ *is* $2^{\mathfrak{c}}$*-lineable.*

Proof. Choose $f \in \mathrm{ES} \cap \mathrm{Add}$ and define $W = \{g \circ f : g \in V_0\}$ with V_0 as in Lemma 3.1. It is easily seen that W is a $2^{\mathfrak{c}}$-dimensional space (isomorphic to V_0) whose non-null elements are in ES.

On the other hand, if $g \in V_0 \setminus \{0\}$, since g is not additive, there exist $x, y \in \mathbb{R}$ such that

$$g(x+y) \neq g(x) + g(y).$$

Let $a, b \in \mathbb{R}$ be such that $f(a) = x$ and $f(b) = y$. Then

$$\begin{aligned}(g \circ f)(a+b) &= g(f(a) + f(b)) \\ &= g(x+y) \\ &\neq g(x) + g(y) \\ &= (g \circ f)(a) + (g \circ f)(b),\end{aligned}$$

which shows that $g \circ f$ is not additive. Hence $W \subset (\mathrm{ES} \setminus \mathrm{Add}) \cup \{0\}$ and the proof is finished. □

Next we study the lineability of the set $\mathrm{DG} \cap \mathrm{Add} \setminus \mathrm{ES}$. We reproduce the original proof for completeness.

Theorem 4.3 (García-Pacheco et al., [20])**.** *The set* $\mathrm{DG} \cap \mathrm{Add} \setminus \mathrm{ES}$ *is* $2^{\mathfrak{c}}$*-lineable.*

Proof. Let I be a Hamel basis of $\mathbb{R}$ regarded as a $\mathbb{Q}$-linear space. Fix $i \in I$ and consider a bijection $\phi : I \to I \setminus \{i\}$. It is straightforward to prove that ϕ can be extended by linearity to an injective $\mathbb{Q}$-linear mapping $\Phi : \mathbb{R} \to \mathbb{R}$.

Observe that $i \notin \Phi(\mathbb{R})$. If there existed $\alpha_1, ..., \alpha_k \in \mathbb{Q}$ and $i_1, ..., i_k \in I$ such that $\Phi(\alpha_1 i_1 + ... + \alpha_k i_k) = i$, then $\alpha_1 \phi(i_1) + ... + \alpha_k \phi(i_k) - i = 0$. The latter contradicts the fact that $\{\Phi(i_1), \ldots, \Phi(i_k), i\}$ is linearly independent.

It can also be proved that $\overline{\Phi(\mathbb{R})} = \mathbb{R}$. Indeed, choose $\epsilon > 0$ and $p \in \mathbb{R} \setminus \Phi(\mathbb{R})$, and consider $j \in I \setminus \{i\}$ and $\alpha \in \mathbb{Q}$ such that $|\alpha j - p| < \epsilon$. Since there is $s \in I$ with $\phi(s) = j$, we have that $|\Phi(\alpha s) - p| = |\alpha \phi(s) - p| = |\alpha j - p| < \epsilon$.

Define now

$$U = \{\Phi \circ g : g \in W\},$$

where W is any $2^{\mathfrak{c}}$-dimensional linear space such that $W \subseteq (\mathrm{ES} \cap \mathrm{Add}) \cup \{0\}$ (see Theorem 4.1). It is clear that U is a $2^{\mathfrak{c}}$-dimensional linear space and that every non-null element of U is $\mathbb{Q}$-linear and not surjective. Also, f maps every non-degenerate interval to $\Phi(\mathbb{R})$, which completes the proof. □

To finish this section we have included a result on the elements of $\mathrm{Add} \setminus \mathrm{DG}$.

Proposition 4.4. *A function* $f \in \mathrm{Add}$ *is discontinuous if and only if* $f \in \mathrm{DG}$.

Proof. Choose $f \in \mathrm{Add}$. If $f \in \mathrm{DG}$, then f is obviously discontinuous. If we assume now that f is not continuous, then f cannot be homogeneous, and hence there does not exist $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. If we take $x_1 \neq 0$, there is $x_2 \neq 0$ such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}.$$

The vectors $v_1 = (x_1, f(x_1))$ and $v_2 = (x_2, f(x_2))$ are clearly linearly independent and therefore they generate $\mathbb{R}$. If $q_1, q_2 \in \mathbb{Q}$, we can approximate $q_1 v_1 + q_2 v_2$ to any vector v since $\mathbb{Q}$ is dense in $\mathbb{R}^2$. Therefore

$$\begin{aligned} q_1 v_1 + q_2 v_2 &= q_1(x_1, f(x_1)) + q_2(x_2, f(x_2)) \\ &= (q_1 x_1 + q_2 x_2, q_1 f(x_1) + q_2 f(x_2)) \\ &= (q_1 x_1 + q_2 x_2, f(q_1 x_1 + q_2 x_2)). \end{aligned}$$

Then

$$\overline{\text{graph}(f)} = \overline{\{(x, y) : x = q_1 x_1 + q_2 x_2, y = f(x); q_1, q_2 \in \mathbb{Q}\}} = \mathbb{R}^2.$$

In other words $f \in \text{DG}$. □

The next section is devoted to the study of the linear structure of the set of the Daboux functions in $\mathbb{R}^{\mathbb{R}}$.

4.2 Darboux functions

We recall that $f \in \mathbb{R}^{\mathbb{R}}$ is Darboux if it transforms intervals into intervals and that D represents the set of Darboux functions. Obviously $\text{ES} \subset \text{D}$ and therefore D is $2^{\mathfrak{c}}$-lineable. We also have the following interesting results:

Theorem 4.5. *The set* $\text{D} \setminus \text{ES}$ *is* $2^{\mathfrak{c}}$*-lineable.*

Proof. Let V be a $2^{\mathfrak{c}}$-dimensional space in $\text{ES} \cup \{0\}$. For each $f \in V$ let us define

$$f^{\sharp}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(\log x) & \text{if } x > 0. \end{cases}$$

If we consider the $2^{\mathfrak{c}}$-dimensional linear space $W = \{f^{\sharp} : f \in V\}$, then it is plain that $W \subset (\text{D} \setminus \text{ES}) \cup \{0\}$. □

Theorem 4.6. *The set* $\text{S} \setminus \text{D}$ *is* $2^{\mathfrak{c}}$*-lineable.*

Proof. Observe that the space V generated by the characteristic functions of subsets of $(-\infty, 0]$ has cardinality $2^{\mathfrak{c}}$ and hence it is $2^{\mathfrak{c}}$-dimensional. Let $B_1 = \{e_\alpha : \alpha < 2^{\mathfrak{c}}\}$ be a basis for V. Now let us consider a basis $B_2 = \{f_\alpha : \alpha < 2^{\mathfrak{c}}\}$ of V_0, where V_0 is as in Lemma 3.1. If for each $\alpha < 2^{\mathfrak{c}}$ we define

$$g_\alpha(x) = \begin{cases} e_\alpha(x) & \text{if } x \leq 0, \\ f_\alpha(\log x) & \text{if } x > 0, \end{cases}$$

then the span of $\{g_\alpha : \alpha < 2^{\mathfrak{c}}\}$ is a $2^{\mathfrak{c}}$-dimensional space contained in $(\text{S} \setminus \text{D}) \cup \{0\}$. □

4.3 Sierpiński-Zygmund functions

The construction of a Sierpiński-Zygmund function is motivated by the following result:

Theorem 4.7 (Blumberg [26])**.** *For every* $f \in \mathbb{R}^{\mathbb{R}}$ *there exists a dense set* $Z \subset \mathbb{R}$ *such that* $f|_Z$ *is continuous.*

The set Z provided in Blumberg's proof turns out to be countable. Sierpiński and Zygmund asked whether or not an uncountable set could be found satisfying Theorem 4.7. This led them in 1923 ([27]; see also [28, pp. 165,166]) to the construction of an instance of what nowadays it is known as a Sierpiński-Zygmund function. We recall that $f \in \mathbb{R}^{\mathbb{R}}$ is Sierpiński-Zygmund if for every $Z \subset \mathbb{R}$ with cardinality $\mathfrak{c}$, the restriction $f|_Z$ is not continuous. We denote the set of Sierpiński-Zygmund functions by SZ.

If CH holds, the restriction of a Sierpiński-Zygmund function to any uncountable set cannot be continuous. The Continuum Hypothesis is necessary in this setting. Shinoda proved in [29] that if Martin's Axiom and the negation

of CH hold, and $\aleph_0 < \kappa < \mathfrak{c}$ then for every $f \in \mathbb{R}^{\mathbb{R}}$ there exists a set $Z \subset \mathbb{R}$ of cardinality κ such that $f|_Z$ is continuous.

It is interesting to observe that Sierpiński-Zygmund's example satisfies a stronger condition, namely $f|_Z$ is not Borel for all $Z \subset \mathbb{R}$ with $\text{card}(Z) = \mathfrak{c}$, which is a stronger condition than that of the definition of Sierpiński-Zygmund function. This motivates the following definition

$$\text{SZ}(\mathcal{B}\text{or}) := \left\{ f \in \mathbb{R}^{\mathbb{R}} : \forall Z \subset \mathbb{R} \text{ with cardinality } \mathfrak{c}, \text{ the restriction } f|_Z \text{ is not Borel} \right\}.$$

Obviously $\text{SZ}(\mathcal{B}\text{or}) \subset \text{SZ}$. The question is whether or not $\text{SZ}(\mathcal{B}\text{or}) = \text{SZ}$ is undecidable under the usual set theoretic settings. However, if $\text{dec}(\mathcal{B}\text{or}, \mathcal{C})$ denotes the minimal cardinal κ such that for every Borel function $f : X \to \mathbb{R}$ there is a partition $(X_\alpha)_{\alpha<\kappa}$ of X with $f|_{X_\alpha}$ continuous for all $\alpha < \kappa$, then it can be proved that:

Theorem 4.8 (Bartoszewicz et al., [8, Theorem 4.4])**.** *For the sets* $\text{SZ}(\mathcal{B}\text{or})$ *and* SZ *we have:*

1. *If* $\mathfrak{c}$ *is a successor cardinal and* $\text{dec}(\mathcal{B}\text{or}, \mathcal{C}) = \mathfrak{c}$*, then* $\text{SZ} \neq \text{SZ}(\mathcal{B}\text{or})$.
2. *If* $\mathfrak{c}$ *is a regular cardinal and* $\text{dec}(\mathcal{B}\text{or}, \mathcal{C}) < \mathfrak{c}$*, then* $\text{SZ} = \text{SZ}(\mathcal{B}\text{or})$.

Another interesting question to be considered is that the standard axioms of set theory (like ZFC) do not guarantee the existence of Sierpiński-Zygmund functions that are surjective or Darboux. However, the following can be proved assuming stronger hypothesis:

Theorem 4.9 (Ciesielski et al., [11])**.** *If* $\text{cov}(\mathcal{M}) = \mathfrak{c}$*, i.e., the union of less than continuum many meager sets does not cover* $\mathbb{R}$*, then* $\text{SZ} \cap \text{ES}$ *is* $\mathfrak{c}^+$*-lineable.*

Observe that Martin's Axiom (see [30]) implies the condition $\text{cov}(\mathcal{M}) = \mathfrak{c}$. However, assuming different set of theoretic hypotheses, it is possible to prove that SZ and ES are even disjoint:

Theorem 4.10 (Balcernak et al., [31])**.** *Under the CPA, Covering Property Axiom (see [32] for details), we have that* $\text{SZ} \cap (D \cup \text{S}) = \varnothing$ *(hence* $\text{SZ} \cap \text{ES} = \varnothing$*).*

5 Conclusions and open questions

The diagram in Figure 1 shows how some of the classes introduced in this paper are related to each other.

Fig. 1. Relationship between some of the classes mentioned in the paper where $A \to B$ means $A \subsetneq B$. Observe that here $\aleph_0 < \kappa < \mathfrak{c}$

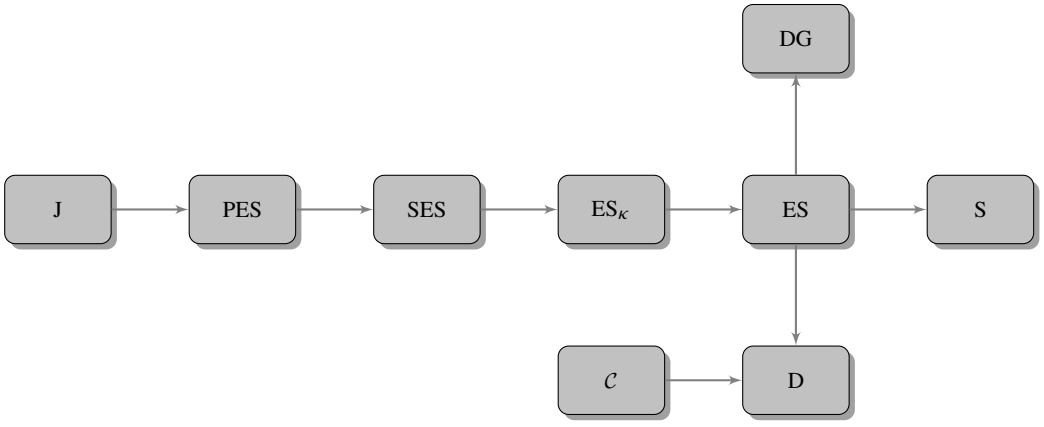

Table 1 summarizes all the results presented in Sections 3 and 4. Observe that there are still three open questions:

1. We know that ES \ SES is $\mathfrak{c}^+$-lineable (see Theorem 3.12). However, we do not know whether $\mathfrak{c}^+$ is optimal or not.
2. The optimal lineability of $\mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa$ with $\aleph_0 \leq \lambda < \kappa \leq \mathfrak{c}$ is not known.
3. Nothing is known about the lineability of the set PES \ J.

Table 1. Summary of the most important lineability results related to surjective functions

Set	Lineability	Reference
J	$2^\mathfrak{c}$	[14]
PES	$2^\mathfrak{c}$	[15]
SES	$2^\mathfrak{c}$	[15]
$\mathrm{ES}_\kappa \quad (\aleph_0 \leq \kappa \leq \mathfrak{c})$	$2^\mathfrak{c}$	[15]
ES	$2^\mathfrak{c}$	[3]
$\mathrm{PES} \setminus J$	?	—
SES \ PES	$2^\mathfrak{c}$	[15]
ES \ SES	$\geq \mathfrak{c}^+$	[15]
$\mathrm{ES}_\lambda \setminus \mathrm{ES}_\kappa \quad (\aleph_0 \leq \lambda < \kappa \leq \mathfrak{c})$	$\geq 2^\mu \quad (\mu < \kappa)$	[8]
S \ ES	$2^\mathfrak{c}$	[15]
D \ ES	$2^\mathfrak{c}$	Theorem 4.5
S \ D	$2^\mathfrak{c}$	Theorem 4.6
ES ∩ Add	$2^\mathfrak{c}$	[20]
$\mathrm{DG} \cap \mathrm{Add} \setminus ES$	$2^\mathfrak{c}$	[20]
ES \ Add	$2^\mathfrak{c}$	[20]
D	$2^\mathfrak{c}$	[3]
DG \ (ES ∪ Add)	$2^\mathfrak{c}$	[20]

Besides lineability, another important tool used to measure the algebraic size of a family of functions is the notion of algebrability and strong algebrability.

(a) We say that a family $\mathcal{F} \subset \mathbb{K}^\mathbb{K}$ (i.e., the algebra of all the functions $f : \mathbb{K} \to \mathbb{K}$) is κ-algebrable if $\mathcal{F} \cup \{0\}$ contains a κ-generated subalgebra $\mathcal{A}$ of $\mathbb{K}^\mathbb{K}$, i.e., the minimal cardinality of the system of generators of $\mathcal{A}$ is κ.

(b) We say that a family $\mathcal{F} \subset \mathbb{K}^\mathbb{K}$ is strongly κ-algebrable if $\mathcal{F} \cup \{0\}$ contains a κ-generated subalgebra $\mathcal{A}$ of $\mathbb{K}^\mathbb{K}$ isomorphic to a free algebra.

Notice that it is not possible to construct an algebra of real surjective functions since f^2 is never surjective if $f \in \mathbb{R}^\mathbb{R}$. However, there are a few nice results in the literature about algebras of surjective functions in $\mathbb{C}^\mathbb{C}$. We refer to [8] for a complete account of results on algebrability of complex surjective functions. For instance, in [8] it is proved that the family of complex Jones functions is strongly $2^\mathfrak{c}$-algebrable. Here a complex Jones function stands for a mapping $f \in \mathbb{C}^\mathbb{C}$ such that for every closed set $C \subset \mathbb{C}^2$ with uncountable projection on the first coordinate, we have that C meets the graph of f.

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