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Evaluation of the convolution sum involving the sum of divisors function for 22, 44 and 52

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Abstract: The convolution sum, $\sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ \alpha l + \beta m = n}} \sigma(l)\sigma(m)$, where $\alpha\beta = 22, 44, 52$, is evaluated for all natural numbers n . Modular forms are used to achieve these evaluations. Since the modular space of level 22 is contained in that of level 44, we almost completely use the basis elements of the modular space of level 44 to carry out the evaluation of the convolution sums for $\alpha\beta = 22$. We then use these convolution sums to determine formulae for the number of representations of a positive integer by the octonary quadratic forms $a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)$, where $(a, b) = (1, 11), (1, 13)$.

Keywords: Sums of Divisors function, Convolution Sums, Dedekind eta function, Modular Forms, Eisenstein Series, Cusp Forms, Octonary quadratic Forms, Number of Representations

MSC: 11A25, 11E20, 11E25, 11F11, 11F20, 11F27

1 Introduction

Let in the sequel $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers, respectively.

Suppose that $k, n \in \mathbb{N}$. Then the sum of positive divisors of n to the power of k , $\sigma_k(n)$, is defined by

$$\sigma_k(n) = \sum_{0 < d | n} d^k. \quad (1)$$

We write $\sigma(n)$ as a synonym for $\sigma_1(n)$. For $m \notin \mathbb{N}$ we set $\sigma_k(m) = 0$.

Suppose now that $\alpha, \beta \in \mathbb{N}$ are such that $\alpha \leq \beta$. Then the convolution sum, $W_{(\alpha, \beta)}(n)$, is defined as follows:

$$W_{(\alpha, \beta)}(n) = \sum_{\substack{(l, m) \in \mathbb{N}_0^2 \\ \alpha l + \beta m = n}} \sigma(l)\sigma(m). \quad (2)$$

We write $W_\beta(n)$ as a synonym for $W_{(1, \beta)}(n)$. Given $\alpha, \beta \in \mathbb{N}$, if for all $(l, m) \in \mathbb{N}_0^2$ it holds that $\alpha l + \beta m \neq n$ then we set $W_{(\alpha, \beta)}(n) = 0$.

For those convolution sums $W_{(\alpha, \beta)}(n)$ that have so far been evaluated, the levels $\alpha\beta$ are given in Table 1.

We discuss the evaluation of the convolution sums of level $\alpha\beta = 22, 44$ and $\alpha\beta = 52$, i.e., $(\alpha, \beta) = (1, 22), (2, 11), (1, 44), (4, 11), (1, 52), (4, 13)$. Convolution sums of these levels have not been evaluated yet as one can notice from Table 1.

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Table 1. Known convolution sums $W_{(\alpha,\beta)}(n)$

Level $\alpha\beta$	Authors	References
1	M. Besge, J. W. L. Glaisher, S. Ramanujan	[1–3]
2, 3, 4	J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams	[4]
5, 7	M. Lemire & K. S. Williams, S. Cooper & P. C. Toh	[5, 6]
6	S. Alaca & K. S. Williams	[7]
8, 9	K. S. Williams	[8, 9]
10, 11, 13, 14	E. Royer	[10]
12, 16, 18, 24	A. Alaca & S. Alaca & K. S. Williams	[11–14]
15	B. Ramakrishnan & B. Sahu	[15]
20, 10	S. Cooper & D. Ye	[16]
23	H. H. Chan & S. Cooper	[17]
25	E. X. W. Xia & X. L. Tian & O. X. M. Yao	[18]
27, 32	S. Alaca & Y. Kesicioğlu	[19]
36	D. Ye	[20]
14, 26, 28, 30	E. Ntjenjem	[21]

As an application, convolution sums are used to determine explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms

$$a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2), \quad (3)$$

and

$$c(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2) + d(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2), \quad (4)$$

respectively, where $a, b, c, d \in \mathbb{N}$.

So far known explicit formulae for the number of representations of n by the octonary form Equation 3 are referenced in Table 2.

Table 2. Known representations of n by the form Equation 3

(a, b)	Authors	References
(1,2)	K. S. Williams	[8]
(1,4)	A. Alaca & S. Alaca & K. S. Williams	[12]
(1,5)	S. Cooper & D. Ye	[16]
(1,6)	B. Ramakrishnan & B. Sahu	[15]
(1,8)	S. Alaca & Y. Kesicioğlu	[19]
(1,7)	E. Ntjenjem	[21]

We determine formulae for the number of representations of a positive integer n by the octonary quadratic form Equation 3 for which $(a, b) = (1, 11), (1, 13)$. These formulae for the number of representations are also new according to Table 2.

This paper is organized in the following way. In Section 2 we discuss modular forms, briefly define eta functions and convolution sums, and prove the generalization of the extraction of the convolution sum. Our main results on the evaluation of the convolution sums are discussed in Section 3. The determination of formulae for the number of representations of a positive integer n is discussed in Section 4.

Software for symbolic scientific computation is used to obtain the results of this paper. This software comprises the open source software packages *GiNaC*, *Maxima*, *REDUCE*, *SAGE* and the commercial software package *MAPLE*.

2 Modular forms and convolution sums

Let $\mathbb{H}$ be the upper half-plane, that is $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, and let $G = \text{SL}_2(\mathbb{R})$ be the group of 2×2 -matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ hold. Let furthermore $\Gamma = \text{SL}_2(\mathbb{Z})$ be the full modular group which is a subgroup of $\text{SL}_2(\mathbb{R})$. Let $N \in \mathbb{N}$. Then

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

is a subgroup of G and is called the *principal congruence subgroup of level N* . A subgroup H of G is called a *congruence subgroup of level N* if it contains $\Gamma(N)$.

Relevant for our purposes is the following congruence subgroup:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let $k, N \in \mathbb{N}$ and let $\Gamma' \subseteq \Gamma$ be a congruence subgroup of level $N \in \mathbb{N}$. Let $k \in \mathbb{Z}$, $\gamma \in \text{SL}_2(\mathbb{Z})$ and $f : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$. We denote by $f^{[\gamma]_k}$ the function whose value at z is $(cz + d)^{-k} f(\gamma(z))$, i.e., $f^{[\gamma]_k}(z) = (cz + d)^{-k} f(\gamma(z))$. The following definition is based on the textbook by N. Koblitz [22, p. 108].

Definition 2.1. Let $N \in \mathbb{N}$, $k \in \mathbb{Z}$, f be a meromorphic function on $\mathbb{H}$ and $\Gamma' \subset \Gamma$ a congruence subgroup of level N .

- (a) f is called a modular function of weight k for Γ' if
 - (a1) for all $\gamma \in \Gamma'$ it holds that $f^{[\gamma]_k} = f$.
 - (a2) for any $\delta \in \Gamma$ it holds that $f^{[\delta]_k}(z)$ can be expressed in the form $\sum_{n \in \mathbb{Z}} a_n e^{\frac{2\pi i z n}{N}}$, wherein $a_n \neq 0$ for finitely many $n \in \mathbb{Z}$ such that $n < 0$.
- (b) f is called a modular form of weight k for Γ' if
 - (b1) f is a modular function of weight k for Γ' ,
 - (b2) f is holomorphic on $\mathbb{H}$,
 - (b3) for all $\delta \in \Gamma$ and for all $n \in \mathbb{Z}$ such that $n < 0$ it holds that $a_n = 0$.
- (c) f is called a cusp form of weight k for Γ' if
 - (c1) f is a modular form of weight k for Γ' ,
 - (c2) for all $\delta \in \Gamma$ it holds that $a_0 = 0$.

For $k, N \in \mathbb{N}$, let $\mathfrak{M}_k(\Gamma_0(N))$ be the space of modular forms of weight k for $\Gamma_0(N)$, $\mathfrak{S}_k(\Gamma_0(N))$ be the subspace of cusp forms of weight k for $\Gamma_0(N)$, and $\mathfrak{E}_k(\Gamma_0(N))$ be the subspace of Eisenstein forms of weight k for $\Gamma_0(N)$. Then the decomposition of the space of modular forms as a direct sum of the space generated by the Eisenstein series and the space of cusp forms, i.e., $\mathfrak{M}_k(\Gamma_0(N)) = \mathfrak{E}_k(\Gamma_0(N)) \oplus \mathfrak{S}_k(\Gamma_0(N))$, is well-known; see for example W. A. Stein's book (online version) [23, p. 81].

As noted in Section 5.3 of W. A. Stein's book [23, p. 86] if the primitive Dirichlet characters are trivial and $2 \leq k$ is even, then $E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, where B_k are the Bernoulli numbers.

For the purpose of this paper we only consider trivial Dirichlet characters and $2 \leq k$ even. Theorems 5.8 and 5.9 in Section 5.3 of [23, p. 86] also hold for this special case.

2.1 Eta functions

The Dedekind eta function, $\eta(z)$, is defined on the upper half-plane $\mathbb{H}$ by $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$. We set $q = e^{2\pi i z}$. Then $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q)$, where $F(q) = \prod_{n=1}^{\infty} (1 - q^n)$.

M. Newman [24, 25] systematically used the Dedekind eta function to construct modular forms for $\Gamma_0(N)$. M. Newman determined when a function $f(z)$ is a modular form for $\Gamma_0(N)$ by providing conditions (i)–(iv) in the following theorem. G. Ligozat [26] determined the order of vanishing of an eta function at the cusps of $\Gamma_0(N)$, which is condition (v) or (v') in Theorem 2.2.

The following theorem is proved in L. J. P. Kilford's book [27, p. 99] and G. Köhler's book [28, p. 37]; we will apply that theorem to determine eta quotients, $f(z)$, which belong to $\mathfrak{M}_k(\Gamma_0(N))$, and especially those eta quotients which are in $\mathfrak{S}_k(\Gamma_0(N))$.

Theorem 2.2 (M. Newman and G. Ligozat). *Let $N \in \mathbb{N}$, $D(N)$ be the set of all positive divisors of N , $\delta \in D(N)$ and $r_\delta \in \mathbb{Z}$. Let furthermore $f(z) = \prod_{\delta \in D(N)} \eta^{r_\delta}(\delta z)$ be an η -quotient. If the following five conditions are satisfied*

- (i) $\sum_{\delta \in D(N)} \delta r_\delta \equiv 0 \pmod{24}$,
 - (ii) $\sum_{\delta \in D(N)} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,
 - (iii) $\prod_{\delta \in D(N)} \delta^{r_\delta}$ is a square in $\mathbb{Q}$,
 - (iv) $0 < \sum_{\delta \in D(N)} r_\delta \equiv 0 \pmod{4}$
 - (v) for each $d \in D(N)$ it holds that $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta} r_\delta \geq 0$,
- then $f(z) \in \mathfrak{M}_k(\Gamma_0(N))$, where $k = \frac{1}{2} \sum_{\delta \in D(N)} r_\delta$.

Moreover, the η -quotient $f(z)$ belongs to $\mathfrak{S}_k(\Gamma_0(N))$ if (v) is replaced by

- (v') for each $d \in D(N)$ it holds that $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta} r_\delta > 0$.

2.2 Convolution sums $W_{(\alpha, \beta)}(n)$

Recall that given $\alpha, \beta \in \mathbb{N}$ such that $\alpha \leq \beta$, the convolution sum is defined by Equation 2.

As observed by A. Alaca et al. [11], we can assume that $\gcd(\alpha, \beta) = 1$. Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then the Eisenstein series $L(q)$ and $M(q)$ are defined as follows:

$$L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \quad (5)$$

$$M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n. \quad (6)$$

The following two relevant results are essential for the sequel of this work and are a generalization of the extraction of the convolution sum using Eisenstein forms of weight 4 for all pairs $(\alpha, \beta) \in \mathbb{N}^2$. Their proofs are given by E. Ntjenjem [21].

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{N}$. Then*

$$(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathfrak{M}_4(\Gamma_0(\alpha\beta)).$$

Theorem 2.4. *Let $\alpha, \beta \in \mathbb{N}$ be such that α and β are relatively prime and $\alpha < \beta$. Then*

$$\begin{aligned} (\alpha L(q^\alpha) - \beta L(q^\beta))^2 = & (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left(240 \alpha^2 \sigma_3\left(\frac{n}{\alpha}\right) + 240 \beta^2 \sigma_3\left(\frac{n}{\beta}\right) \right. \\ & + 48 \alpha (\beta - 6n) \sigma\left(\frac{n}{\alpha}\right) + 48 \beta (\alpha - 6n) \sigma\left(\frac{n}{\beta}\right) \\ & \left. - 1152 \alpha \beta W_{(\alpha, \beta)}(n) \right) q^n. \end{aligned} \quad (7)$$

3 Evaluation of the convolution sums $W_{(\alpha,\beta)}(n)$, where $\alpha\beta = 22, 44, 52$

In this section, we give explicit formulae for the convolution sums $W_{(1,22)}(n)$, $W_{(2,11)}(n)$, $W_{(1,44)}(n)$, $W_{(4,11)}(n)$, $W_{(1,52)}(n)$ and $W_{(4,13)}(n)$.

3.1 Bases for $\mathfrak{E}_4(\Gamma_0(\alpha\beta))$ and $\mathfrak{S}_4(\Gamma_0(\alpha\beta))$ with $\alpha\beta = 44, 52$

We observe the following inclusion relations

$$\mathfrak{M}_4(\Gamma_0(11)) \subset \mathfrak{M}_4(\Gamma_0(22)) \subset \mathfrak{M}_4(\Gamma_0(44)) \quad (8)$$

$$\mathfrak{M}_4(\Gamma_0(13)) \subset \mathfrak{M}_4(\Gamma_0(26)) \subset \mathfrak{M}_4(\Gamma_0(52)). \quad (9)$$

Therefore, it suffices to correspondingly determine the basis of the spaces $\mathfrak{M}_4(\Gamma_0(44))$ and $\mathfrak{M}_4(\Gamma_0(52))$, respectively.

We use the dimension formulae for the space of Eisenstein forms and the space of cusp forms in T. Miyake's book [29, Thrm 2.5.2, p. 60] or W. A. Stein's book [23, Prop. 6.1, p. 91] to deduce that $\dim(\mathfrak{E}_4(\Gamma_0(44))) = \dim(\mathfrak{E}_4(\Gamma_0(52))) = 6$, $\dim(\mathfrak{S}_4(\Gamma_0(44))) = 15$ and $\dim(\mathfrak{S}_4(\Gamma_0(52))) = 18$.

Let $D(44) = \{1, 2, 4, 11, 22, 44\}$ and $D(52) = \{1, 2, 4, 13, 26, 52\}$ be the sets of all positive divisors of 44 and 52, respectively.

Theorem 3.1.

- (a) The sets $\mathcal{B}_{E,44} = \{M(q^t) \mid t \in D(44)\}$ and $\mathcal{B}_{E,52} = \{M(q^t) \mid t \in D(52)\}$ are bases of $\mathfrak{E}_4(\Gamma_0(44))$ and $\mathfrak{E}_4(\Gamma_0(52))$, respectively.
- (b) Let $1 \leq i \leq 15$ and $1 \leq j \leq 18$ be positive integers.
 Let $\delta_1 \in D(44)$ and $(r(i, \delta_1))_{i, \delta_1}$ be the Table 3 of the powers of $\eta(\delta_1 z)$.
 Let $\delta_2 \in D(52)$ and $(r(j, \delta_2))_{j, \delta_2}$ be the Table 4 of the powers of $\eta(\delta_2 z)$.
 Let furthermore $A_i(q) = \prod_{\delta_1 \in D(44)} \eta^{r(i, \delta_1)}(\delta_1 z)$ and $B_j(q) = \prod_{\delta_2 \in D(52)} \eta^{r(j, \delta_2)}(\delta_2 z)$ be selected elements of $\mathfrak{S}_4(\Gamma_0(44))$ and $\mathfrak{S}_4(\Gamma_0(52))$, respectively.
 Then the sets $\mathcal{B}_{S,44} = \{A_i(q) \mid 1 \leq i \leq 15\}$ and $\mathcal{B}_{S,52} = \{B_j(q) \mid 1 \leq j \leq 18\}$ are bases of $\mathfrak{S}_4(\Gamma_0(44))$ and $\mathfrak{S}_4(\Gamma_0(52))$, respectively.
- (c) The sets $\mathcal{B}_{M,44} = \mathcal{B}_{E,44} \cup \mathcal{B}_{S,44}$ and $\mathcal{B}_{M,52} = \mathcal{B}_{E,52} \cup \mathcal{B}_{S,52}$ constitute bases of $\mathfrak{M}_4(\Gamma_0(44))$ and $\mathfrak{M}_4(\Gamma_0(52))$, respectively.

For $1 \leq i \leq 15$ and $1 \leq j \leq 18$ let in the sequel $A_i(q)$ be expressed in the form $\sum_{n=1}^{\infty} a_i(n)q^n$ and $B_j(q)$ be expressed in the form $\sum_{n=1}^{\infty} b_j(n)q^n$.

Proof. We give the proof for the case $\alpha\beta = 44$. The case $\alpha\beta = 52$ is proved similarly.

(a) By Theorem 5.8 in Section 5.3 of W. A. Stein [23, p. 86] $M(q^t)$ is in $\mathfrak{M}_4(\Gamma_0(t))$ for each t which is an element of $D(44)$. Since $\mathfrak{E}_4(\Gamma_0(44))$ has a finite dimension, it suffices to show that $M(q^t)$ with $t \in D(44)$ are linearly independent. Suppose that $x_t \in \mathbb{C}$ with $t \in D(44)$. We prove this by induction on the elements of the set $D(44)$ which is assumed to be ascendantly ordered.

The case $t = 1 \in D(44)$ is obvious since comparing the coefficients of q^t on both sides of the equation $x_t M(q^t) = 0$ clearly gives $x_t = 0$.

Suppose now that the cardinality of the set $D(44)$ is greater than 1 and that $M(q^t)$ are linearly independent for all $t|44$ and $t \leq t_1$ for a given t_1 with $1 < t_1 < 44$. Let C be the proper non-empty subset of $D(44)$ which contains all positive divisors of 44 less than or equal to t_1 . Note that all positive divisors of t_1 constitute a subset of C . Let

us consider the non-empty subset $C \cup \{t'\}$ of $D(44)$, wherein t' is the next ascendant element of $D(44)$ which is greater than t_1 the greatest element of the set C . Then

$$\sum_{t \in C \cup \{t'\}} x_t M(q^t) = \sum_{t \in C} x_t M(q^t) + x_{t'} M(q^{t'}) = 0.$$

By the induction hypothesis it holds that $x_t = 0$ for all $t \in C$. So, we obtain from the above equation that $x_{t'} = 0$ when we compare the coefficient of $q^{t'}$ on both sides of the equation.

Hence, the solution is $x_t = 0$ for all t such that t is a positive divisor of 44. Therefore, the set $\mathcal{B}_{E,44}$ is linearly independent. Hence, the set $\mathcal{B}_{E,44}$ is a basis of $\mathfrak{E}_4(\Gamma_0(44))$.

(b) The $A_i(q)$ with $1 \leq i \leq 15$ are obtained from an exhaustive search using Theorem 2.2 (i) – (v'). Hence, each $A_i(q)$ is an element of the space $\mathfrak{S}_4(\Gamma_0(44))$.

Since the dimension of $\mathfrak{S}_4(\Gamma_0(44))$ is 15, it suffices to show that the set $\{A_i(q) \mid 1 \leq i \leq 15\}$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ and $\sum_{i=1}^{15} x_i A_i(q) = 0$. Then

$$\sum_{i=1}^{15} x_i A_i(q) = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{15} x_i a_i(n) \right) q^n = 0$$

which gives the following homogeneous system of linear equations

$$\sum_{i=1}^{15} a_i(n) x_i = 0, \quad 1 \leq n \leq 15. \quad (10)$$

A simple computation using software for symbolic scientific computation shows that the determinant of the matrix of this homogeneous system of linear equations is non-zero. So, $x_i = 0$ for all $1 \leq i \leq 15$. Hence, the set $\{A_i(q) \mid 1 \leq i \leq 15\}$ is linearly independent and therefore a basis of $\mathfrak{S}_4(\Gamma_0(44))$.

(c) Since $\mathfrak{M}_4(\Gamma_0(44)) = \mathfrak{E}_4(\Gamma_0(44)) \oplus \mathfrak{S}_4(\Gamma_0(44))$, the result follows from (a) and (b). $\square$

According to Equation 8 the basis elements $A_i(q)$, where $1 \leq i \leq 5$, are contained in $\mathfrak{S}_4(\Gamma_0(22))$. The basis element $A_2(q)$ is the only element of the space $\mathfrak{S}_4(\Gamma_0(11))$ that we are able to generate with the help of Theorem 2.2. Even though the basis element $A_{14}(q)$ looks like an element of $\mathfrak{S}_4(\Gamma_0(22))$, it cannot be generated at level 22 using Theorem 2.2.

To evaluate the convolution sums $W_{(1,22)}(n)$ and $W_{(2,11)}(n)$, we determine two additional basis elements of $\mathfrak{S}_4(\Gamma_0(22))$ which are

$$A'_6(q) = \frac{\eta(2z)\eta^3(11z)\eta^5(22z)}{\eta(z)} = \sum_{n=1}^{\infty} a'_6(n)q^n,$$

$$A'_7(q) = \frac{\eta^9(2z)\eta^7(11z)}{\eta^5(z)\eta^3(22z)} = \sum_{n=1}^{\infty} a'_7(n)q^n.$$

Due to Equation 9 the basis elements $B_j(q)$, where $1 \leq j \leq 7$ and $j = 15, 17$, belong to $\mathfrak{S}_4(\Gamma_0(26))$. We are unable to generate any elements of the space $\mathfrak{S}_4(\Gamma_0(13))$ using Theorem 2.2. We note that $B_{2j}(q) = B_j(q^2)$, where $4 \leq j \leq 7$, $B_{16}(q) = B_{15}(q^2)$ and $B_{18}(q) = B_{17}(q^2)$. Therefore, one can easily replicate the evaluation of the convolution sums $W_{(1,26)}(n)$ and $W_{(2,13)}(n)$ shown by E. Ntienjem [21].

3.2 Evaluation of $W_{(\alpha,\beta)}(n)$ where $\alpha\beta = 22, 44, 52$

Lemma 3.2. *We have*

$$(L(q^2) - 22L(q^{22}))^2 = 441 + \sum_{n=1}^{\infty} \left(-\frac{3312}{61} \sigma_3(n) + \frac{12672}{61} \sigma_3\left(\frac{n}{2}\right) \right)$$

$$\begin{aligned}
& -\frac{110880}{61}\sigma_3\left(\frac{n}{11}\right) + \frac{6557760}{61}\sigma_3\left(\frac{n}{22}\right) + \frac{12096}{61}a_1(n) + \frac{45792}{61}a_2(n) \\
& + \frac{19872}{61}a_3(n) + \frac{73728}{61}a_4(n) - \frac{50688}{61}a_5(n) + 22176a'_6(n) + 864a'_7(n) \Big) q^n, \quad (11)
\end{aligned}$$

$$\begin{aligned}
(2L(q^2) - 11L(q^{11}))^2 &= 81 + \sum_{n=1}^{\infty} \left(\frac{15840}{61}\sigma_3(n) + \frac{37440}{61}\sigma_3\left(\frac{n}{2}\right) \right. \\
& + \frac{1626768}{61}\sigma_3\left(\frac{n}{11}\right) - \frac{494208}{61}\sigma_3\left(\frac{n}{22}\right) + \frac{36864}{61}a_1(n) + \frac{357408}{61}a_2(n) \\
& + \frac{1160352}{61}a_3(n) + \frac{1539072}{61}a_4(n) + \frac{834048}{61}a_5(n) - 22176a'_6(n) - 864a'_7(n) \Big) q^n, \quad (12)
\end{aligned}$$

$$\begin{aligned}
(L(q) - 44L(q^{44}))^2 &= 1849 + \sum_{n=1}^{\infty} \left(\frac{124464}{61}\sigma_3(n) - \frac{577662336}{40565}\sigma_3\left(\frac{n}{2}\right) \right. \\
& + \frac{68986368}{5795}\sigma_3\left(\frac{n}{4}\right) - \frac{174240}{61}\sigma_3\left(\frac{n}{11}\right) + \frac{62064288}{5795}\sigma_3\left(\frac{n}{22}\right) + \frac{2525690112}{5795}\sigma_3\left(\frac{n}{44}\right) \\
& + \frac{1440}{61}a_1(n) - \frac{82927872}{5795}a_2(n) - \frac{887345568}{5795}a_3(n) - \frac{1676429568}{5795}a_4(n) \\
& - \frac{2804007168}{5795}a_5(n) + \frac{3753380736}{5795}a_6(n) - \frac{13356288}{19}a_7(n) + \frac{4226609664}{5795}a_8(n) \\
& - \frac{633600}{19}a_9(n) - \frac{527332608}{1159}a_{10}(n) + \frac{7679232}{19}a_{11}(n) \\
& - \frac{15231744}{95}a_{12}(n) - \frac{131079168}{95}a_{13}(n) + \frac{317952}{19}a_{14}(n) - \frac{12595968}{95}a_{15}(n) \Big) q^n, \quad (13)
\end{aligned}$$

$$\begin{aligned}
(4L(q^4) - 11L(q^{11}))^2 &= 49 + \sum_{n=1}^{\infty} \left(-\frac{110880}{61}\sigma_3(n) + \frac{80121888}{5795}\sigma_3\left(\frac{n}{2}\right) \right. \\
& - \frac{48338688}{5795}\sigma_3\left(\frac{n}{4}\right) + \frac{1817904}{61}\sigma_3\left(\frac{n}{11}\right) - \frac{98480448}{5795}\sigma_3\left(\frac{n}{22}\right) - \frac{27320832}{5795}\sigma_3\left(\frac{n}{44}\right) \\
& + \frac{110880}{61}a_1(n) + \frac{174857472}{5795}a_2(n) + \frac{1169427168}{5795}a_3(n) + \frac{2114189568}{5795}a_4(n) \\
& + \frac{3025513728}{5795}a_5(n) - \frac{3511080576}{5795}a_6(n) + \frac{13318272}{19}a_7(n) - \frac{3641762304}{5795}a_8(n) \\
& + \frac{633600}{19}a_9(n) + \frac{663913728}{1159}a_{10}(n) - \frac{7679232}{19}a_{11}(n) + \frac{15231744}{95}a_{12}(n) \\
& + \frac{131079168}{95}a_{13}(n) - \frac{317952}{19}a_{14}(n) + \frac{12595968}{95}a_{15}(n) \Big) q^n, \quad (14)
\end{aligned}$$

$$\begin{aligned}
(L(q) - 52L(q^{52}))^2 &= 2601 + \sum_{n=1}^{\infty} \left(\frac{6109008}{1243}\sigma_3(n) - \frac{456504084816}{6064597}\sigma_3\left(\frac{n}{2}\right) \right. \\
& + \frac{254592}{41}\sigma_3\left(\frac{n}{4}\right) - \frac{7361952}{1243}\sigma_3\left(\frac{n}{13}\right) - \frac{4829528827344}{6064597}\sigma_3\left(\frac{n}{26}\right) \\
& + \frac{434738304}{41}\sigma_3\left(\frac{n}{52}\right) - \frac{3066144}{1243}b_1(n) + \frac{498157179048}{6064597}b_2(n) \\
& + \frac{927327070704}{6064597}b_3(n) - \frac{442577500560}{6064597}b_4(n) - \frac{8530413669648}{6064597}b_5(n) \\
& - \frac{10161699732288}{6064597}b_6(n) - \frac{10388366352}{1243}b_7(n) + \frac{1040832}{41}b_8(n) \\
& + 7488b_9(n) + \frac{329100929664}{147917}b_{10}(n) + 27456b_{11}(n) \Big) q^n
\end{aligned}$$

$$\begin{aligned} & - \frac{15249288510144}{6064597} b_{12}(n) + 17472 b_{13}(n) + \frac{47009664}{41} b_{14}(n) \\ & - \frac{25166713896}{551327} b_{15}(n) + \frac{4167031826880}{6064597} b_{16}(n) - \frac{126425023920}{6064597} b_{17}(n) \\ & + \frac{868608}{41} b_{18}(n) \Big) q^n, \quad (15) \end{aligned}$$

$$\begin{aligned} (4L(q^4) - 13L(q^{13}))^2 = & 81 + \sum_{n=1}^{\infty} \left(\frac{3066144}{1243} \sigma_3(n) - \frac{240061230672}{6064597} \sigma_3\left(\frac{n}{2}\right) \right. \\ & + \frac{139392}{41} \sigma_3\left(\frac{n}{4}\right) + \frac{45798672}{1243} \sigma_3\left(\frac{n}{13}\right) - \frac{53922031824}{6064597} \sigma_3\left(\frac{n}{26}\right) \\ & + \frac{20290176}{41} \sigma_3\left(\frac{n}{52}\right) - \frac{3066144}{1243} b_1(n) + \frac{212735819880}{6064597} b_2(n) \\ & + \frac{251848851024}{6064597} b_3(n) - \frac{400561037808}{6064597} b_4(n) - \frac{5152459820400}{6064597} b_5(n) \\ & - \frac{5408748312192}{6064597} b_6(n) - \frac{5489355312}{1243} b_7(n) + \frac{150336}{41} b_8(n) \\ & - 7488 b_9(n) + \frac{151016538432}{147917} b_{10}(n) - 27456 b_{11}(n) \\ & - \frac{8224832431680}{6064597} b_{12}(n) - 17472 b_{13}(n) - \frac{544896}{41} b_{14}(n) \\ & - \frac{11115614088}{551327} b_{15}(n) + \frac{2056953609600}{6064597} b_{16}(n) - \frac{64745693328}{6064597} b_{17}(n) \\ & \left. - \frac{2304}{41} b_{18}(n) \right) q^n. \quad (16) \end{aligned}$$

Proof. We just prove the case $(4L(q^4) - 11L(q^{11}))^2$. The other cases are proved similarly.

It follows from Lemma 2.3 that $(4L(q^4) - 11L(q^{11}))^2 \in \mathfrak{M}_4(\Gamma_0(44))$. Hence, by Theorem 3.1 (c), there exist $X_\delta, Y_j \in \mathbb{C}$, $1 \leq j \leq 15$ and $\delta \in D(44)$, such that

$$\begin{aligned} (4L(q^4) - 11L(q^{11}))^2 = & \sum_{\delta \in D(44)} X_\delta M(q^\delta) + \sum_{j=1}^{15} Y_j A_j(q) \\ = & \sum_{\delta \in D(44)} X_\delta + \sum_{n=1}^{\infty} \left(240 \sum_{\delta \in D(44)} \sigma_3\left(\frac{n}{\delta}\right) X_\delta + \sum_{j=1}^{ms} a_j(n) Y_j \right) q^n. \end{aligned} \quad (17)$$

We equate the right hand side of Equation 17 with that of Equation 7 when setting $(\alpha, \beta) = (4, 11)$ to obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left(240 \sum_{\delta \in D(44)} \sigma_3\left(\frac{n}{\delta}\right) X_\delta + \sum_{j=1}^{15} a_j(n) Y_j \right) q^n = & \sum_{n=1}^{\infty} \left(3840 \sigma_3\left(\frac{n}{4}\right) + 29040 \sigma_3\left(\frac{n}{11}\right) \right. \\ & \left. + 192(11 - 6n) \sigma\left(\frac{n}{4}\right) + 528(4 - 6n) \sigma\left(\frac{n}{11}\right) - 50688 W_{(4,11)}(n) \right) q^n. \end{aligned}$$

We now take the coefficients of q^n for which n is in

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 22, 44\}.$$

This results in a system of linear equations whose unique solution determines the values of the unknown X_δ for all $\delta \in D(44)$ and the values of the unknown Y_j for all $1 \leq j \leq 15$. Hence, we obtain the stated result. $\square$

Our main result of this section is as follows.

Theorem 3.3. *Let n be a positive integer. Then*

$$W_{(1,22)}(n) = \frac{17}{1464} \sigma_3(n) - \frac{1}{122} \sigma_3\left(\frac{n}{2}\right) + \frac{35}{488} \sigma_3\left(\frac{n}{11}\right) + \frac{125}{366} \sigma_3\left(\frac{n}{22}\right)$$

$$\begin{aligned}
& + \left(\frac{1}{24} - \frac{1}{88}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{22}\right) - \frac{21}{2684}a_1(n) - \frac{159}{5368}a_2(n) \\
& - \frac{69}{5368}a_3(n) - \frac{32}{671}a_4(n) + \frac{2}{61}a_5(n) - \frac{7}{8}a'_6(n) - \frac{3}{88}a'_7(n),
\end{aligned} \tag{18}$$

$$\begin{aligned}
W_{(2,11)}(n) = & -\frac{5}{488}\sigma_3(n) + \frac{5}{366}\sigma_3\left(\frac{n}{2}\right) + \frac{137}{1464}\sigma_3\left(\frac{n}{11}\right) + \frac{39}{122}\sigma_3\left(\frac{n}{22}\right) \\
& + \left(\frac{1}{24} - \frac{1}{44}n\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma\left(\frac{n}{11}\right) - \frac{16}{671}a_1(n) - \frac{1241}{5368}a_2(n) \\
& - \frac{4029}{5368}a_3(n) - \frac{668}{671}a_4(n) - \frac{362}{671}a_5(n) + \frac{7}{8}a'_6(n) + \frac{3}{88}a'_7(n),
\end{aligned} \tag{19}$$

$$\begin{aligned}
W_{(1,44)}(n) = & -\frac{13}{366}\sigma_3(n) + \frac{501443}{1784860}\sigma_3\left(\frac{n}{2}\right) - \frac{1361}{5795}\sigma_3\left(\frac{n}{4}\right) + \frac{55}{976}\sigma_3\left(\frac{n}{11}\right) \\
& - \frac{19591}{92720}\sigma_3\left(\frac{n}{22}\right) + \frac{9878}{17385}\sigma_3\left(\frac{n}{44}\right) + \left(\frac{1}{24} - \frac{1}{176}n\right)\sigma(n) \\
& + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{44}\right) - \frac{5}{10736}a_1(n) + \frac{35993}{127490}a_2(n) \\
& + \frac{3081061}{1019920}a_3(n) + \frac{66147}{11590}a_4(n) + \frac{1217017}{127490}a_5(n) - \frac{3258143}{254980}a_6(n) \\
& + \frac{527}{38}a_7(n) - \frac{917233}{63745}a_8(n) + \frac{25}{38}a_9(n) + \frac{20807}{2318}a_{10}(n) \\
& - \frac{303}{38}a_{11}(n) + \frac{601}{190}a_{12}(n) + \frac{2586}{95}a_{13}(n) - \frac{69}{209}a_{14}(n) \\
& + \frac{497}{190}a_{15}(n),
\end{aligned} \tag{20}$$

$$\begin{aligned}
W_{(4,11)}(n) = & \frac{35}{976}\sigma_3(n) - \frac{25291}{92720}\sigma_3\left(\frac{n}{2}\right) + \frac{4178}{17385}\sigma_3\left(\frac{n}{4}\right) - \frac{11}{732}\sigma_3\left(\frac{n}{11}\right) \\
& + \frac{15543}{46360}\sigma_3\left(\frac{n}{22}\right) + \frac{539}{5795}\sigma_3\left(\frac{n}{44}\right) + \left(\frac{1}{24} - \frac{1}{44}n\right)\sigma\left(\frac{n}{4}\right) \\
& + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma\left(\frac{n}{11}\right) - \frac{35}{976}a_1(n) - \frac{75893}{127490}a_2(n) \\
& - \frac{4060511}{1019920}a_3(n) - \frac{917617}{127490}a_4(n) - \frac{1313157}{127490}a_5(n) + \frac{3047813}{254980}a_6(n) \\
& - \frac{1051}{76}a_7(n) + \frac{790313}{63745}a_8(n) - \frac{25}{38}a_9(n) - \frac{288157}{25498}a_{10}(n) \\
& + \frac{303}{38}a_{11}(n) - \frac{601}{190}a_{12}(n) - \frac{2586}{95}a_{13}(n) + \frac{69}{209}a_{14}(n) \\
& - \frac{497}{190}a_{15}(n),
\end{aligned} \tag{21}$$

$$\begin{aligned}
W_{(1,52)}(n) = & -\frac{97}{1243}\sigma_3(n) + \frac{731577059}{582201312}\sigma_3\left(\frac{n}{2}\right) - \frac{17}{164}\sigma_3\left(\frac{n}{4}\right) + \frac{5899}{59664}\sigma_3\left(\frac{n}{13}\right) \\
& + \frac{7739629531}{582201312}\sigma_3\left(\frac{n}{26}\right) - \frac{81757}{492}\sigma_3\left(\frac{n}{52}\right) + \left(\frac{1}{24} - \frac{1}{208}n\right)\sigma(n) \\
& + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{52}\right) + \frac{31939}{775632}b_1(n) - \frac{6918849709}{5045744704}b_2(n) \\
& - \frac{19319313973}{7568617056}b_3(n) + \frac{236419605}{194067104}b_4(n) + \frac{4556844909}{194067104}b_5(n) \\
& + \frac{1357064601}{48516776}b_6(n) + \frac{5549341}{39776}b_7(n) - \frac{139}{328}b_8(n) \\
& - \frac{1}{8}b_9(n) - \frac{65925667}{1775004}b_{10}(n) - \frac{11}{24}b_{11}(n) + \frac{2036496863}{48516776}b_{12}(n) \\
& - \frac{7}{24}b_{13}(n) - \frac{3139}{164}b_{14}(n) + \frac{349537693}{458704064}b_{15}(n)
\end{aligned}$$

$$-\frac{556494635}{48516776} b_{16}(n) + \frac{67534735}{194067104} b_{17}(n) - \frac{29}{82} b_{18}(n), \quad (22)$$

$$\begin{aligned} W_{(4,13)}(n) = & -\frac{31939}{775632} \sigma_3(n) + \frac{5001275639}{7568617056} \sigma_3\left(\frac{n}{2}\right) + \frac{47}{6396} \sigma_3\left(\frac{n}{4}\right) + \frac{24049}{387816} \sigma_3\left(\frac{n}{13}\right) \\ & + \frac{1123375663}{7568617056} \sigma_3\left(\frac{n}{26}\right) - \frac{17613}{2132} \sigma_3\left(\frac{n}{52}\right) + \left(\frac{1}{24} - \frac{1}{52}n\right) \sigma\left(\frac{n}{4}\right) \\ & + \left(\frac{1}{24} - \frac{1}{16}n\right) \sigma\left(\frac{n}{13}\right) + \frac{31939}{775632} b_1(n) - \frac{2954664165}{5045744704} b_2(n) \\ & - \frac{5246851063}{7568617056} b_3(n) + \frac{8345021621}{7568617056} b_4(n) + \frac{35780970975}{2522872352} b_5(n) \\ & + \frac{4695094021}{315359044} b_6(n) + \frac{38120523}{517088} b_7(n) - \frac{261}{4264} b_8(n) \\ & + \frac{1}{8} b_9(n) - \frac{786544471}{46150104} b_{10}(n) + \frac{11}{24} b_{11}(n) + \frac{42837668915}{1892154264} b_{12}(n) \\ & + \frac{7}{24} b_{13}(n) + \frac{473}{2132} b_{14}(n) + \frac{154383529}{458704064} b_{15}(n) \\ & - \frac{5356650025}{946077132} b_{16}(n) + \frac{1348868611}{7568617056} b_{17}(n) + \frac{1}{1066} b_{18}(n). \end{aligned} \quad (23)$$

Proof. We prove the case $W_{(4,13)}(n)$ as the other cases are proved similarly.

We equate the right hand side of Equation 16 with that of Equation 7 when setting $(\alpha, \beta) = (4, 13)$, namely

$$\begin{aligned} \sum_{n=1}^{\infty} & \left(3840 \sigma_3\left(\frac{n}{4}\right) + 40560 \sigma_3\left(\frac{n}{13}\right) + 192(13-6n) \sigma\left(\frac{n}{4}\right) \right. \\ & \left. + 624(4-6n) \sigma\left(\frac{n}{13}\right) - 59904 W_{(4,13)}(n) \right) q^n = \\ & \sum_{n=1}^{\infty} \left(\frac{3066144}{1243} \sigma_3(n) - \frac{240061230672}{6064597} \sigma_3\left(\frac{n}{2}\right) \right. \\ & + \frac{139392}{41} \sigma_3\left(\frac{n}{4}\right) + \frac{45798672}{1243} \sigma_3\left(\frac{n}{13}\right) - \frac{53922031824}{6064597} \sigma_3\left(\frac{n}{26}\right) + \frac{20290176}{41} \sigma_3\left(\frac{n}{52}\right) \\ & - \frac{3066144}{1243} b_1(n) + \frac{212735819880}{6064597} b_2(n) + \frac{251848851024}{6064597} b_3(n) \\ & - \frac{400561037808}{6064597} b_4(n) - \frac{5152459820400}{6064597} b_5(n) - \frac{5408748312192}{6064597} b_6(n) \\ & - \frac{5489355312}{1243} b_7(n) + \frac{150336}{41} b_8(n) - 7488 b_9(n) + \frac{151016538432}{147917} b_{10}(n) \\ & - 27456 b_{11}(n) - \frac{8224832431680}{6064597} b_{12}(n) - 17472 b_{13}(n) - \frac{544896}{41} b_{14}(n) \\ & - \frac{11115614088}{551327} b_{15}(n) + \frac{2056953609600}{6064597} b_{16}(n) - \frac{64745693328}{6064597} b_{17}(n) \\ & \left. - \frac{2304}{41} b_{18}(n) \right) q^n. \end{aligned}$$

We then solve for $W_{(4,13)}(n)$ to obtain the stated result. $\square$

4 Number of representations of a positive integer n by the quaternary quadratic form using $W_{(\alpha,\beta)}(n)$ when $\alpha\beta = 44, 52$

Let $n \in \mathbb{N}_0$ and the number of representations of n by the quaternary quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ be denoted by $r_4(n)$. That means,

$$r_4(n) = \text{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid m = x_1^2 + x_2^2 + x_3^2 + x_4^2\}).$$

We set $r_4(0) = 1$. For all $n \in \mathbb{N}$, the following Jacobi's identity is proved in K. S. Williams' book [30, Thrm 9.5, p. 83]

$$r_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right). \quad (24)$$

Let furthermore the number of representations of n by the octonary quadratic form

$$a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

be denoted by $N_{(a,b)}(n)$. That means,

$$N_{(a,b)}(n) = \text{card}(\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)\}).$$

We infer the following result:

Theorem 4.1. *Let $n \in \mathbb{N}$ and $(a, b) = (1, 11), (1, 13)$. Then*

$$\begin{aligned} N_{(1,11)}(n) &= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{11}\right) - 32\sigma\left(\frac{n}{44}\right) \\ &\quad + 64W_{(1,11)}(n) + 1024W_{(1,11)}\left(\frac{n}{4}\right) - 256\left(W_{(4,11)}(n) + W_{(1,44)}(n)\right), \\ N_{(1,13)}(n) &= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{13}\right) - 32\sigma\left(\frac{n}{52}\right) \\ &\quad + 64W_{(1,13)}(n) + 1024W_{(1,13)}\left(\frac{n}{4}\right) - 256\left(W_{(4,13)}(n) + W_{(1,52)}(n)\right). \end{aligned}$$

Proof. We only prove $N_{(1,11)}(n)$ since that for $N_{(1,13)}(n)$ is done similarly.

It holds that

$$N_{(1,11)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+11m=n}} r_4(l)r_4(m) = r_4(n)r_4(0) + r_4(0)r_4\left(\frac{n}{11}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} r_4(l)r_4(m).$$

We make use of Equation 24 to derive

$$N_{(1,11)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{11}\right) - 32\sigma\left(\frac{n}{52}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} (8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)).$$

We observe that

$$(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)) = 64\sigma(l)\sigma(m) - 256\sigma\left(\frac{l}{4}\right)\sigma(m) - 256\sigma(l)\sigma\left(\frac{m}{4}\right) + 1024\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right).$$

The evaluation of

$$W_{(1,11)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} \sigma(l)\sigma(m)$$

is shown by E. Royer [10, Thrm 1.3]. We map l to $4l$ to infer

$$W_{(4,11)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} \sigma\left(\frac{l}{4}\right)\sigma(m) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 4l+11m=n}} \sigma(l)\sigma(m).$$

The evaluation of $W_{(4,11)}(n)$ is given in Equation 21. We next map m to $4m$ to conclude

$$W_{(1,44)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} \sigma(l)\sigma\left(\frac{m}{4}\right) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+44m=n}} \sigma(l)\sigma(m).$$

The evaluation of $W_{(1,44)}(n)$ is provided by Equation 20. We simultaneously map l to $4l$ and m to $4m$ to deduce

$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=n}} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+11m=\frac{n}{4}}} \sigma(l)\sigma(m) = W_{(1,11)}\left(\frac{n}{4}\right).$$

Again, E. Royer [10, Thrm 1.3] has proved the evaluation of $W_{(1,11)}(n)$.

We then put these evaluations together to obtain the stated result for $N_{(1,11)}(n)$. □

Tables

Table 3. Exponents of η -functions being basis elements of $\mathfrak{S}_4(\Gamma_0(44))$

	1	2	4	11	22	44
1	6	-2	0	6	-2	0
2	4	0	0	4	0	0
3	2	2	0	2	2	0
4	0	4	0	0	4	0
5	-2	6	0	-2	6	0
6	0	2	2	0	2	2
7	0	-3	5	0	5	1
8	0	0	4	0	0	4
9	3	0	1	-1	0	5
10	0	-2	6	0	-2	6
11	1	-3	4	-3	5	4
12	2	0	0	2	-4	8
13	0	2	0	0	-2	8
14	-3	9	0	1	1	0
15	0	0	2	0	-4	10

Table 4. Exponents of η -functions being basis elements of $\mathfrak{S}_4(\Gamma_0(52))$

	1	2	4	13	26	52
1	1	5	0	3	-1	0
2	3	3	0	1	1	0
3	1	3	0	3	1	0
4	3	1	0	1	3	0
5	1	1	0	3	3	0
6	3	-1	0	1	5	0
7	1	-1	0	3	5	0
8	0	3	1	0	1	3
9	2	1	1	-2	3	3
10	0	1	1	0	3	3
11	2	-1	1	-2	5	3
12	0	3	-1	0	1	5
13	2	1	-1	-2	3	5
14	0	1	-1	0	3	5
15	-1	5	0	5	-1	0
16	0	-1	5	0	5	-1
17	7	-3	0	-3	7	0
18	0	7	-3	0	-3	7

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