

Xiaoliang Xie and Wen Zhang*

Hopf bifurcations in a three-species food chain system with multiple delays

DOI 10.1515/math-2017-0039

Received November 2, 2016; accepted February 7, 2017.

Abstract: This paper is concerned with a three-species Lotka-Volterra food chain system with multiple delays. By linearizing the system at the positive equilibrium and analyzing the associated characteristic equation, the stability of the positive equilibrium and existence of Hopf bifurcations are investigated. Furthermore, the direction of bifurcations and the stability of bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem for functional differential equations. Finally, some numerical simulations are carried out for illustrating the theoretical results.

Keywords: Three-species food chain, Delay, Hopf bifurcation; Center manifold, Periodic solutions

MSC: 34C23, 34C25

1 Introduction

The study on the dynamics of predator-prey system is one of the dominant subjects in ecology and mathematical ecology due to its universal existence and importance. As to our knowledge, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. Thus, time delays of one type or another have been incorporated into mathematical models of population dynamics due to maturation time, capturing time or other reasons. In the last decades, many authors have explored the dynamics of systems with time delay and many interesting results have been obtained [1-13].

However, there may be more species in a habitat and they can construct a food chain. Therefore, it is more realistic to consider a multiple-species predator-prey system. Recently, Baek and Lee [14] proposed the following three-species food chain model with the Lotka-Volterra functional response

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a - bx(t) - cy(t)), \\ \frac{dy(t)}{dt} = y(t)(-d_1 + c_1x(t) - e_1z(t)), \\ \frac{dz(t)}{dt} = z(t)(-d_2 + e_2y(t)), \end{cases} \quad (1)$$

where $x(t)$, $y(t)$, $z(t)$ denote the population densities of the lowest-level prey, mid-level predator and top-level predator at time t , respectively. The constant $a > 0$ is called the intrinsic growth rate of the prey species; $b > 0$ measures the intraspecific competition of the prey; $c_1 > 0$ and $e_2 > 0$ represent the conversion rates of the lowest-level prey to the mid-level predator and the mid-level predator to the top-level predator, respectively; $c_1 > 0$ and

Xiaoliang Xie: School of Mathematics and Statistics, Hunan University of Commerce, Changsha, 410205, Hunan, China and Key Laboratory of Hunan Province for Mobile Business Intelligence, Hunan University of Commerce, Changsha 410205, Hunan, China, E-mail: xiexiaoliangmath@163.com

***Corresponding Author: Wen Zhang:** School of Mathematics and Statistics, Hunan University of Commerce, Changsha, 410205 Hunan, China, E-mail: zwmath2011@163.com

$e_1 > 0$ measure the hunting of the mid-level predator to the lowest-level prey and the top-level predator to the mid-level predator; $d_1 > 0$ and $d_2 > 0$ denote the death rates of the mid-level and top-level predator, respectively.

In system (1), the gestation periods and maturation time of some species are all omitted. However, some species need to take time to have the ability to reproduce and capture food. Based on this fact, by incorporating time delays into the maturation time, Cui and Yan considered the following delayed differential system in [15]

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a - bx(t) - cy(t)), \\ \frac{dy(t)}{dt} = y(t)(-d_1 + c_1x(t) - e_1z(t - \tau_1)), \\ \frac{dz(t)}{dt} = z(t)(-d_2 + e_2y(t - \tau_2)), \end{cases} \tag{2}$$

where $\tau_1 \geq 0$ denotes the time from birth to having the ability to predate for top-level predator and $\tau_2 \geq 0$ represents the maturation time that the mid-level predator can be served as food for the top-level predator, respectively. Delayed models similar to (2) have been investigated widely by many authors and lots of interesting results have been obtained. For more details see [16-18].

In system (2), the author just considered the time for the top-level predator to have the ability to predate and the maturation time for the mid-level predator to be served as food for the top-level predator. But the time for the mid-level predator to have the ability to predate and the maturation time for the lowest-level prey to be served as food for the mid-level predator are omitted. Based on this fact, by incorporating delays into the above terms, we consider a more complicated system with multiple delay

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a - bx(t) - cy(t - \tau_1)), \\ \frac{dy(t)}{dt} = y(t)(-d_1 + c_1x(t - \tau_2 - \tau_3) - e_1z(t - \tau_2)), \\ \frac{dz(t)}{dt} = z(t)(-d_2 + e_2y(t - \tau_1 - \tau_3)), \end{cases} \tag{3}$$

where $\tau_1 \geq 0$ and $\tau_2 \geq 0$ denote the time from birth to having the ability to predate for the mid-level predator and the top-level predator. For a special case, we assume that $\tau_2 + \tau_3$ and $\tau_1 + \tau_3$ represent the maturation time for the lowest-level prey to be served as food for the mid-level predator and the maturation time for the mid-level predator to be served as food for the top-level predator, respectively. We address the question how the time delays that we incorporated affect the dynamical properties of the system (3). So the aim of this paper is to study the dynamical behaviors of the system (3), for which we investigate the stability and Hopf bifurcation of a three-species food chain system with multiple delays. We would like to mention that the bifurcation in a predator-prey system with a single or multiple delays had been investigated by many researchers [19-22]. However, to the best of our knowledge, few results for system (3) have been obtained. Therefore, the research of this case is worth considering.

2 Stability of positive equilibrium and existence of local Hopf bifurcations

For convenience, we introduce new variables $x_1(t) = x(t)$, $y_1(t) = y(t - \tau_1)$, $z_1(t) = z(t - \tau_1 - \tau_2)$ and assume that $\tau = \tau_1 + \tau_2 + \tau_3$, so that system (3) can be written as the following system with a single delay:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)(a - bx_1(t) - cy_1(t)), \\ \frac{dy_1(t)}{dt} = y_1(t)(-d_1 + c_1x_1(t - \tau) - e_1z(t)), \\ \frac{dz_1(t)}{dt} = z_1(t)(-d_2 + e_2y_1(t - \tau)). \end{cases} \tag{4}$$

It is easy to see that the system (3) has a unique positive equilibrium $E^* : (x_0^*, y_0^*, z_0^*)$ provided that the condition

$$(H) \quad ae_2c_1 - d_2cc_1 - d_1be_2 > 0$$

holds, where $x_0^* = \frac{ae_2 - d_2c}{be_2}$, $y_0^* = \frac{d_2}{e_2}$, $z_0^* = \frac{ae_2c_1 - d_2cc_1 - d_1be_2}{be_1e_2}$.

Let $\tilde{u}_1(t) = x_1(t) - x_0^*$, $\tilde{u}_2(t) = y_1(t) - y_0^*$, $\tilde{u}_3(t) = z_1(t) - z_0^*$, and use relations $a - bx_0^* - cy_0^* = 0$, $-d_1 + c_1x_0^* - e_1z_0^* = 0$, $-d_2 + e_2y_0^* = 0$, then system (4) can be rewritten as the following equivalent system

$$\begin{cases} \frac{d\tilde{u}_1(t)}{dt} = (\tilde{u}_1(t) + x_0^*)(-b\tilde{u}_1(t) - c\tilde{u}_2(t)), \\ \frac{d\tilde{u}_2(t)}{dt} = (\tilde{u}_2(t) + y_0^*)(c_1\tilde{u}_1(t - \tau) - e_1\tilde{u}_3(t)), \\ \frac{d\tilde{u}_3(t)}{dt} = (\tilde{u}_3(t) + z_0^*)(e_2\tilde{u}_2(t - \tau)). \end{cases} \tag{5}$$

To study the stability of the equilibrium E^* , it is sufficient to study the stability of the origin for system (5). The linearized system of system (5) at origin is

$$\begin{cases} \frac{d\tilde{u}_1(t)}{dt} = -bx_0^*\tilde{u}_1(t) - cx_0^*\tilde{u}_2(t), \\ \frac{d\tilde{u}_2(t)}{dt} = c_1y_0^*\tilde{u}_1(t - \tau) - e_1y_0^*\tilde{u}_3(t), \\ \frac{d\tilde{u}_3(t)}{dt} = e_2z_0^*\tilde{u}_2(t - \tau). \end{cases} \tag{6}$$

The characteristic equation of system (6) is

$$\lambda^3 + Ae^{-\lambda\tau}\lambda + B\lambda^2 + Ce^{-\lambda\tau} + De^{-\lambda\tau}\lambda = 0, \tag{7}$$

where $A = e_1e_2y_0^*z_0^* > 0$, $B = bx_0^* > 0$, $C = be_1e_2x_0^*y_0^*z_0^* > 0$, $D = cc_1x_0^*y_0^* > 0$.

Next, we will investigate the distribution of roots of Eq.(7). Obviously, $\lambda = 0$ is not a root of Eq.(7). When $\tau = 0$, the characteristic equation becomes

$$\lambda^3 + B\lambda^2 + (A + D)\lambda + C = 0. \tag{8}$$

It can be seen that $B > 0$, $B(A + D) - C > 0$, and $C > 0$. Therefore, it follows from the Routh-Hurwitz criteria that all roots of (8) have negative real parts and thus the zero equilibrium of system (5) is asymptotically stable when $\tau = 0$.

Now, we examine when the characteristic equation has pairs of purely imaginary roots. For $\tau > 0$, if $i\omega(\omega > 0)$ is a root of Eq.(7), then ω should satisfy the following equations

$$\begin{aligned} (A + D)\omega\sin(\omega\tau) + C\cos(\omega\tau) &= B\omega^2, \\ C\sin(\omega\tau) - (A + D)\omega\cos(\omega\tau) &= -\omega^3. \end{aligned} \tag{9}$$

Thus

$$\sin(\omega\tau) = \frac{(AB + BD - C)\omega^3}{(A + D)^2\omega^2 + C^2}, \quad \cos(\omega\tau) = \frac{(A + D)\omega^4 + BC\omega^2}{(A + D)^2\omega^2 + C^2}.$$

Squaring and adding both the equations of (9), we have

$$(A + D)^2\omega^8 + [(AB + BD)^2 + C^2]\omega^6 + [B^2C^2 - (A + D)^4]\omega^4 - 2C^2(A + D)^2\omega^2 - C^4 = 0. \tag{10}$$

Let $z = \omega^2$, $a_1 = \frac{(AB+BD)^2+C^2}{(A+D)^2}$, $a_2 = \frac{B^2C^2-(A+D)^4}{(A+D)^2}$, $a_3 = -2C^2$, $a_4 = \frac{-C^4}{(A+D)^2}$, then Eq.(10) becomes

$$h(z) := z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0.$$

Thus

$$\frac{dh(z)}{dz} = 4z^3 + 3a_1z^2 + 2a_2z + a_3 := 4f(z),$$

where

$$f(z) = z^3 + \frac{3}{4}a_1z^2 + \frac{1}{2}a_2z + \frac{1}{4}a_3.$$

Let $m = \frac{8a_2-3a_1^2}{16}$, $n = \frac{a_1^3-4a_1a_2+8a_3}{32}$, $D_0 = \frac{n^2}{4} + \frac{m^3}{27}$. Similar to discussion in [24], assume that $D_0 > 0$, then from the Cardano's formula for the third-degree algebra equation we know that the equation $f(z) = 0$ had only one real root z_1^* . If $D_0 = 0$, then the equation $f(z) = 0$ has three real roots z_1, z_2 and z_3 (where $z_2 = z_3$), and in this case we define z_2^* by $\max\{z_1, z_2\}$. If $D_0 < 0$, we know that the equation $f(z) = 0$ has three different real roots denoted by s_1^*, s_2^* and s_3^* . In this case, we define $z_3^* = \max\{s_1^*, s_2^*, s_3^*\}$. According to Lemma 2.2 in [24], without loss of generality, we can suppose that Eq.(10) has four positive real roots, denoted by z_1, z_2, z_3, z_4 , respectively. Then Eq.(9) should also have four positive real roots $\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}, \omega_3 = \sqrt{z_3}, \omega_4 = \sqrt{z_4}$. Define

$$\tau_k^j = \frac{1}{\omega_k} [\arctan(\frac{(AB + BD - C)\omega_k}{(A + D)\omega_k^2 + BC}) + j\pi], \quad k = 1, 2, 3, 4. \quad j = 0, 1, 2, \dots \tag{11}$$

Then (τ_k^j, ω_k) are solutions of Eq.(7) and $\lambda = \pm i\omega_k$ are a pair of purely imaginary roots of Eq.(7) with $\tau = \tau_k^j$. Define $\tau^0 = \tau_{k_0}^0 = \min_{1 \leq k \leq 4} \{\tau_k^0\}$, $\omega_0 = \omega_{k_0}$, where $k_0 \in \{1, 2, 3, 4\}$. Then τ^0 is the first value of τ such that Eq.(7) has purely imaginary roots. In the following discussions, for the sake of convenience, we denoted τ_k^j by τ^j ($j = 0, 1, 2, \dots$) for fixed $k \in \{1, 2, 3, 4\}$. Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the root of Eq.(7) near $\tau = \tau^j$ satisfying $\alpha(\tau^j) = 0$, $\omega(\tau^j) = \omega_0$ ($j = 0, 1, 2, \dots$).

Lemma 2.1. $\frac{dRe\lambda(\tau)}{d\tau}|_{\tau=\tau_j} > 0$.

Proof. Differentiate the two sides of Eq.(7) with respect to τ . For the sake of simplicity, we denote τ_j and ω_0 by τ , ω , respectively, then

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + Ae^{-\lambda\tau} + 2B\lambda + De^{-\lambda\tau}}{A\lambda^2e^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D\lambda^2e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{3\lambda^2 + Ae^{-\lambda\tau} + 2B\lambda + De^{-\lambda\tau}}{-\lambda^4 - B\lambda^3} - \frac{\tau}{\lambda}. \end{aligned}$$

Thus

$$\begin{aligned} Re\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\lambda=i\omega} &= Re\left[\frac{(3(i\omega)^2 + A(\cos(\omega\tau) - i\sin(\omega\tau)) + 2B(i\omega) + D(\cos(\omega\tau) - i\sin(\omega\tau)))}{-(i\omega)^4 - B(i\omega)^3} - \frac{\tau}{i\omega}\right] \\ &= 2(A + D)^2\omega^8 + (3C^2 + A^2B^2 + 2AB^2D + B^2D^2)\omega^6 + 2B^2C^2\omega^4 > 0. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Since the multiplicity of roots with positive real roots of Eq.(7) can change only if a root appears on or crosses the imaginary axis as time delay τ varies, similarly to Lemma 2.4 and Theorem 2.5 in [24], we have the following results

Theorem 2.2. *Suppose that (H) hold, then the following statements are true.*

- (I) *All roots of Eq.(7) have negative real parts and the $E^* : (x_0^*, y_0^*, z_0^*)$ of system (4) is absolutely stable, if $a_4 \geq 0$ and one of the following conditions is satisfied: (i) $D_0 > 0$ and $z_1^* \leq 0$, (ii) $D_0 = 0$ and $z_2^* \leq 0$, (iii) $D_0 < 0$ and $z_3^* \leq 0$;*
- (II) *All roots of Eq.(9) have negative real parts and the $E^* : (x_0^*, y_0^*, z_0^*)$ of system (4) is asymptotically stable for $\tau \in [0, \tau^0)$, if $a_4 < 0$ or $a_4 \geq 0$ and one of the following conditions is satisfied: (i) $D_0 > 0, z_1^* > 0$ and $h(z_1^*) < 0$, (ii) $D_0 = 0, z_2^* > 0$ and $h(z_2^*) < 0$, (iii) $D_0 < 0, z_3^* \leq 0$ and $h(z_3^*) < 0$;*
- (III) $\tau = \tau^j$ ($j = 0, 1, 2, \dots$) are Hopf bifurcation values for system (4) if the conditions as stated in (II) are satisfied.

3 Direction of Hopf bifurcations and stability of bifurcating periodic solutions

In the previous section, we have already obtained that, under certain conditions, the system (4) can undergo Hopf bifurcation at the positive equilibrium $E^* : (x_0^*, y_0^*, z_0^*)$ when τ takes some critical values $\tau = \tau^j$, ($j = 0, 1, 2, \dots$). In this section, by employing the normal form theory and center manifold theorem introduced by Hassard et al. [23], we shall present the formula determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions of (4).

Let $u_i(t) = \tilde{u}_i(\tau t)$, $\tau = \tau^j + \mu$, $\mu \in R$, then $\mu = 0$ is the Hopf bifurcation value for system (4), dropping the bars for simplification of notations, system (4) becomes the following functional differential equation in $C = C([-1, 0], R^3)$,

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{12}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3$, and $L_\mu : C \rightarrow R^3, f : R \times C \rightarrow R^3$ are given by

$$L_\mu(\phi) = (\tau^j + \mu) \begin{pmatrix} M & N & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau^j + \mu) \begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \\ 0 & G & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \tag{13}$$

and

$$f(\mu, \phi) = (\tau^j + \mu) \begin{pmatrix} a_{11}\phi_1^2(0) + a_{12}\phi_1(0)\phi_2(0) \\ a_{21}\phi_1(-1)\phi_2(0) + a_{22}\phi_2(0)\phi_3(0) \\ a_{31}\phi_2(-1)\phi_3(0) \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} M &= -bx_0^*, & N &= -cx_0^*, & a_{11} &= -b, & a_{12} &= -c, & P &= c_1y_0^*, \\ Q &= -e_1y_0^*, & a_{21} &= c_1, & a_{22} &= -e_1, & G &= e_2z_0^*, & a_{31} &= e_2. \end{aligned}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{15}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau^j + \mu) \begin{pmatrix} M & N & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta) + (\tau^j + \mu) \begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \\ 0 & G & 0 \end{pmatrix} \delta(\theta + 1), \tag{16}$$

where δ is the Dirac delta function. For $\phi \in C([-1, 0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then system (12) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{17}$$

where $u_t(\theta) = u(t + \theta)$ for $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 \psi(-t)d\eta(t, 0), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{18}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in section 2, we know that $\pm i\omega_0\tau^j$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of A^* . We first need to compute the eigenvectors of $A(0)$ and A^* corresponding to $i\omega_0\tau^j$ and $-i\omega_0\tau^j$, respectively. Suppose $q(\theta) = (1, q_1, q_2)^T e^{i\omega_0\tau^j\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau^j$, then $A(0)q(0) = i\omega_0\tau^j q(0)$. From the definition of $A(0)$ and (13), (15), (16) we have

$$\tau^j \begin{pmatrix} M & N & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix} q(0) + \tau^j \begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \\ 0 & G & 0 \end{pmatrix} q(-1) = i\omega_0\tau^j q(0).$$

For $q(-1) = q(0)e^{-i\omega_0\tau^j}$, then we obtain

$$q_1 = \frac{i\omega_0 - M}{N}, \quad q_2 = \frac{G(i\omega_0 - M)e^{-i\omega_0\tau^j}}{i\omega_0 N}$$

Similarly, we can obtain the eigenvector $q^*(s) = D(1, q_1^*, q_2^*)e^{i\omega_0\tau^j s}$ of A^* corresponding to $-i\omega_0\tau^j$, where

$$q_1^* = \frac{-i\omega_0 - M}{Pe^{-i\omega_0\tau^j}}, \quad q_2^* = \frac{(i\omega_0 + M)Q}{i\omega_0 Pe^{-i\omega_0\tau^j}}.$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1$, we need to determine the value of D . By (18), we have

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta)d\eta(\theta)q(\xi)d\xi \\ &= \bar{D}\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - (1, \bar{q}_1^*, \bar{q}_2^*) \int_{-1}^0 \theta e^{i\omega_0\tau^j\theta} d\eta(\theta)(1, q_1, q_2)^T\} \\ &= \bar{D}\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* + \tau^j(P\bar{q}_1^* + Gq_1\bar{q}_2^*)e^{-i\omega_0\tau^j}\}. \end{aligned}$$

Therefore, we can choose D as

$$D = [1 + \bar{q}_1q_1^* + \bar{q}_2q_2^* + \tau^j(Pq_1^* + G\bar{q}_1q_2^*)e^{i\omega_0\tau^j}]^{-1}.$$

Next we will compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (17) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{19}$$

On the center manifold C_0 , we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{20}$$

z and \bar{z} are local coordinates for center manifold C_0 in the direction of q^* and \bar{q}^* . Note that W is real if u_t is real. We only consider real solutions for solutions $u_t \in C_0$ of (17). Since $\mu = 0$, we have

$$\dot{z}(t) = i\omega_0\tau^j z + g(z, \bar{z}), \tag{21}$$

where

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \dots \tag{22}$$

Following from (19), (20), (14) and using the algorithms given in [23], we can get the coefficients which will be used to determine the important quantities

$$\begin{aligned} g_{20} &= 2\tau^j \bar{D}(a_{11} + a_{12}q_1 + a_{21}\bar{q}_1^*q_1e^{-i\omega_0\tau^j} + a_{22}\bar{q}_1^*q_1q_2 + a_{31}\bar{q}_2^*q_1q_2e^{-i\omega_0\tau^j}), \\ g_{11} &= 2\tau^j \bar{D}a_{11} + 2\tau^j \bar{D}a_{12}Re\{q_1\} + 2\tau^j \bar{D}a_{21}\bar{q}_1^*Re\{q_1e^{i\omega_0\tau^j}\} \\ &\quad + 2\tau^j \bar{D}a_{22}\bar{q}_1^*Re\{q_1\bar{q}_2\} + 2\tau^j \bar{D}a_{31}\bar{q}_2^*Re\{\bar{q}_1q_2e^{i\omega_0\tau^j}\}, \\ g_{02} &= 2\tau^j \bar{D}(a_{11} + a_{12}\bar{q}_1 + a_{21}\bar{q}_1^*\bar{q}_1e^{i\omega_0\tau^j} + a_{22}\bar{q}_1^*\bar{q}_1\bar{q}_2 + a_{31}\bar{q}_2^*\bar{q}_1\bar{q}_2e^{-i\omega_0\tau^j}), \\ g_{21} &= \tau^j \bar{D}(2a_{11} + a_{12}\bar{q}_1)W_{20}^{(1)}(0) + 2\tau^j \bar{D}(2a_{11} + a_{12}q_1)W_{11}^{(1)}(0) \\ &\quad + 2\tau^j \bar{D}(a_{12} + a_{21}\bar{q}_1^*e^{-i\omega_0\tau^j} + a_{22}\bar{q}_1^*q_2)W_{11}^{(2)}(0) \\ &\quad + \tau^j \bar{D}(a_{12} + a_{21}\bar{q}_1^*e^{i\omega_0\tau^j} + a_{22}\bar{q}_1^*\bar{q}_2)W_{20}^{(2)}(0) + \tau^j \bar{D}a_{21}\bar{q}_1^*\bar{q}_1W_{20}^{(1)}(-1) \\ &\quad + 2\tau^j \bar{D}a_{21}\bar{q}_1^*q_1W_{11}^{(1)}(-1) + 2\tau^j \bar{D}(a_{22}\bar{q}_1^*q_1 + a_{31}\bar{q}_2^*q_1e^{-i\omega_0\tau^j})W_{11}^{(3)}(0) \end{aligned}$$

$$\begin{aligned}
 & +\tau^j \bar{D}(a_{22}\bar{q}_1^* \bar{q}_1 + a_{31}\bar{q}_2^* \bar{q}_1 e^{i\omega_0\tau^j})W_{20}^{(3)}(0) + \tau^j \bar{D}a_{31}\bar{q}_2^* \bar{q}_2 W_{20}^{(2)}(-1) \\
 & + 2\tau^j \bar{D}a_{31}\bar{q}_2^* q_2 W_{11}^{(2)}(-1),
 \end{aligned} \tag{23}$$

where

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau^j}q(0)e^{i\omega_0\tau^j\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau^j}\bar{q}(0)e^{-i\omega_0\tau^j\theta} + E_1e^{2i\omega_0\tau^j\theta}, \tag{24}$$

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau^j}q(0)e^{i\omega_0\tau^j\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau^j}\bar{q}(0)e^{-i\omega_0\tau^j\theta} + E_2, \tag{25}$$

and $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$, $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$ are both constant vectors.

Similarly to the algorithms given in [25], we can obtain

$$\begin{aligned}
 E_1^{(1)} &= \frac{2}{M_1} \begin{vmatrix} a_{11} + a_{12}q_1 & -N & 0 \\ a_{21}q_1e^{-i\omega_0\tau^j} + a_{22}q_1q_2 & 2i\omega_0 & -Q \\ a_{31}q_1q_2e^{-i\omega_0\tau^j} & -Ge^{-2i\omega_0\tau^j} & 2i\omega_0 \end{vmatrix}, \\
 E_1^{(2)} &= \frac{2}{M_1} \begin{vmatrix} 2i\omega_0 - M & a_{11} + a_{12}q_1 & 0 \\ -Pe^{-2i\omega_0\tau^j} & a_{21}q_1e^{-i\omega_0\tau^j} + a_{22}q_1q_2 & -Q \\ 0 & a_{31}q_1q_2e^{-i\omega_0\tau^j} & 2i\omega_0 \end{vmatrix}, \\
 E_1^{(3)} &= \frac{2}{M_1} \begin{vmatrix} 2i\omega_0 - M & -N & a_{11} + a_{12}q_1 \\ -Pe^{-2i\omega_0\tau^j} & 2i\omega_0 & a_{21}q_1e^{-i\omega_0\tau^j} + a_{22}q_1q_2 \\ 0 & -Ge^{-2i\omega_0\tau^j} & a_{31}q_1q_2e^{-i\omega_0\tau^j} \end{vmatrix},
 \end{aligned}$$

where

$$M_1 = \begin{vmatrix} 2i\omega_0 - M & -N & 0 \\ -Pe^{-2i\omega_0\tau^j} & 2i\omega_0 & -Q \\ 0 & -Ge^{-2i\omega_0\tau^j} & 2i\omega_0 \end{vmatrix}.$$

Similarly, we get

$$\begin{aligned}
 E_2^{(1)} &= \frac{1}{M_2} \begin{vmatrix} 2a_{11} + a_{12}(q_1 + \bar{q}_1) & -N & 0 \\ a_{21}(q_1e^{i\omega_0\tau^j} + \bar{q}_1e^{-i\omega_0\tau^j}) + a_{22}(q_1\bar{q}_2 + \bar{q}_1q_2) & 0 & -Q \\ a_{31}(q_1\bar{q}_2e^{-i\omega_0\tau^j} + \bar{q}_1q_2e^{i\omega_0\tau^j}) & -G & 0 \end{vmatrix}, \\
 E_2^{(2)} &= \frac{1}{M_2} \begin{vmatrix} -M & 2a_{11} + a_{12}(q_1 + \bar{q}_1) & 0 \\ -P & a_{21}(q_1e^{i\omega_0\tau^j} + \bar{q}_1e^{-i\omega_0\tau^j}) + a_{22}(q_1\bar{q}_2 + \bar{q}_1q_2) & -Q \\ 0 & a_{31}(q_1\bar{q}_2e^{-i\omega_0\tau^j} + \bar{q}_1q_2e^{i\omega_0\tau^j}) & 0 \end{vmatrix}, \\
 E_2^{(3)} &= \frac{1}{M_2} \begin{vmatrix} -M & -N & 2a_{11} + a_{12}(q_1 + \bar{q}_1) \\ -P & 0 & a_{21}(q_1e^{i\omega_0\tau^j} + \bar{q}_1e^{-i\omega_0\tau^j}) + a_{22}(q_1\bar{q}_2 + \bar{q}_1q_2) \\ 0 & -G & a_{31}(q_1\bar{q}_2e^{-i\omega_0\tau^j} + \bar{q}_1q_2e^{i\omega_0\tau^j}) \end{vmatrix},
 \end{aligned}$$

where

$$M_2 = \begin{vmatrix} -M & -N & 0 \\ -P & 0 & -Q \\ 0 & -G & 0 \end{vmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (24) and (25). Furthermore, we can compute g_{21} by (23). Thus we can compute the following values:

$$\begin{aligned}
 c_1(0) &= \frac{i}{2\omega_0\tau^j}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\
 \mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\frac{d\lambda(\tau^j)}{d\tau}\}}, \\
 \beta_2 &= 2Re\{C_1(0)\}, \\
 T_2 &= -\frac{Im\{C_1(0)\} + \mu_2 Im\{\frac{d\lambda(\tau^j)}{d\tau}\}}{\omega_0\tau^j}, \quad (k = 0, 1, 2, \dots).
 \end{aligned}$$

Theorem 3.1.

- (i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical(subcritical) and the bifurcating periodic solutions exist for $\tau > \tau^j$ ($\tau < \tau^j$);
- (ii) β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable(unstable);
- (iii) T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$), then the period increase (decrease).

4 Numerical simulations

In this section, we give some numerical simulations supporting our theoretical predictions. As an example, we consider the following system

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(0.9 - 0.6x(t) - 0.6y(t - \tau_1)), \\ \frac{dy(t)}{dt} = y(t)(-0.6 + 0.9x(t - \tau_2 - \tau_3) - 0.9z(t - \tau_2)), \\ \frac{dz(t)}{dt} = z(t)(-0.7 + 0.9y(t - \tau_1 - \tau_3)), \end{cases} \quad (26)$$

Obviously, the hypothesis (H) is satisfied because $ae_2c_1 - d_2cc_1 - d_1be_2 = 0.027 > 0$ and therefore system (3) has only one positive equilibrium $E^* : (0.7222222222, 0.7777777778, 0.0555555556)$. In addition, $h(z)$ has the following form

$$h(z) = z^4 + 0.1897872903z^3 - 0.1140921027z^2 - 0.0004600555558z - 4.622436920 * 10^{-7}. \quad (27)$$

Theorem 4.1. *The positive equilibrium $E^* : (0.7222222222, 0.7777777778, 0.0555555556)$ of system (26) is asymptotically stable when $\tau_1 + \tau_2 + \tau_3 < \tau_0 = 1.966861973$ while it is unstable when $\tau_1 + \tau_2 + \tau_3 > \tau_0 = 1.966861973$, and system (26) can undergo a Hopf bifurcation at the positive equilibrium $E^* : (0.7222222222, 0.7777777778, 0.0555555556)$ when $\tau_1 + \tau_2 + \tau_3$ passes through the critical values $\tau^j = 1.966861973 + 0.5084240855j\pi$ ($j = 0, 1, 2, \dots$) (see Figs. 1-8).*

Fig. 1. The trajectory graph of system (26) in $t - x$ plane with $\tau_1 = 0.5, \tau_2 = 0.4$ and $\tau_3 = 0.3$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

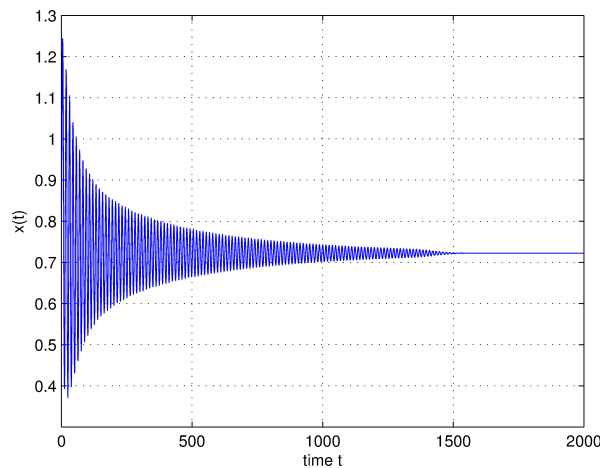


Fig. 2. The trajectory graph of system (26) in $t - y$ plane with $\tau_1 = 0.5$, $\tau_2 = 0.4$ and $\tau_3 = 0.3$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

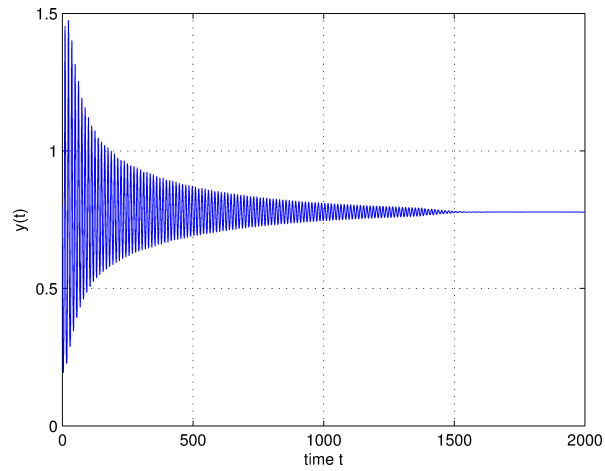


Fig. 3. The trajectory graph of system (26) in $t - z$ plane with $\tau_1 = 0.5$, $\tau_2 = 0.4$ and $\tau_3 = 0.3$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

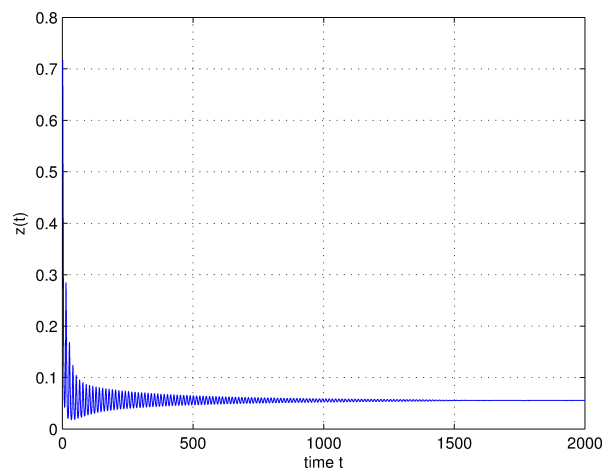


Fig. 4. The phase graph of system (26) in $x - y - z$ plane with $\tau_1 = 0.5$, $\tau_2 = 0.4$ and $\tau_3 = 0.3$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

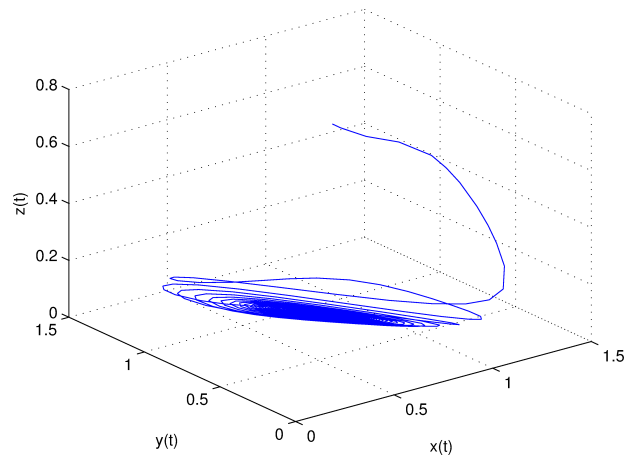


Fig. 5. The trajectory graph of system (26) in $t - x$ plane with $\tau_1 = 0.9, \tau_2 = 0.7$ and $\tau_3 = 0.4$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

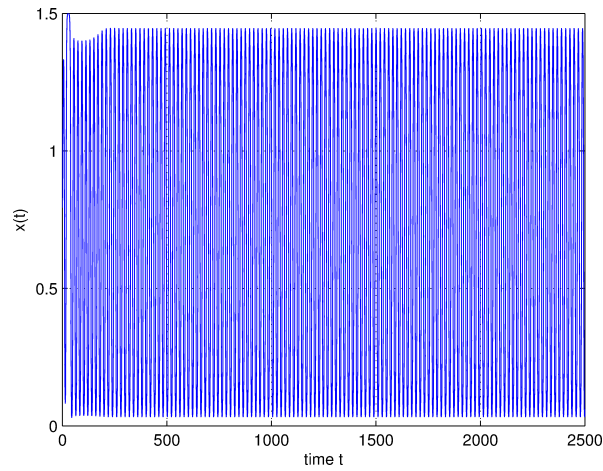


Fig. 6. The trajectory graph of system (26) in $t - y$ plane with $\tau_1 = 0.9, \tau_2 = 0.7$ and $\tau_3 = 0.4$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

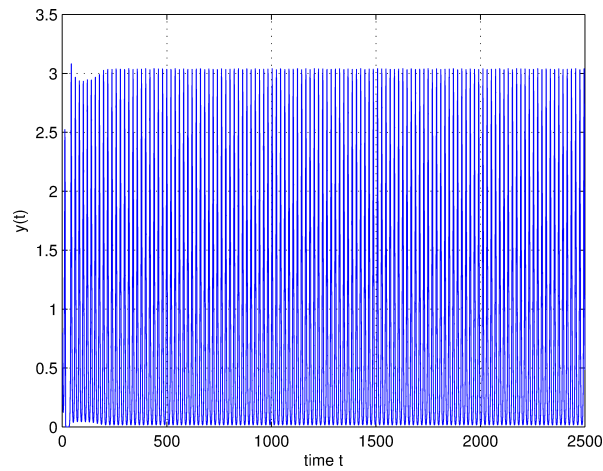


Fig. 7. The trajectory graph of system (26) in $t - z$ plane with $\tau_1 = 0.9, \tau_2 = 0.7$ and $\tau_3 = 0.4$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.

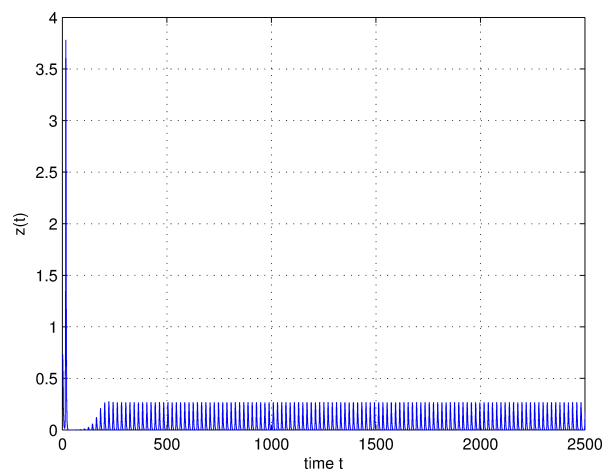
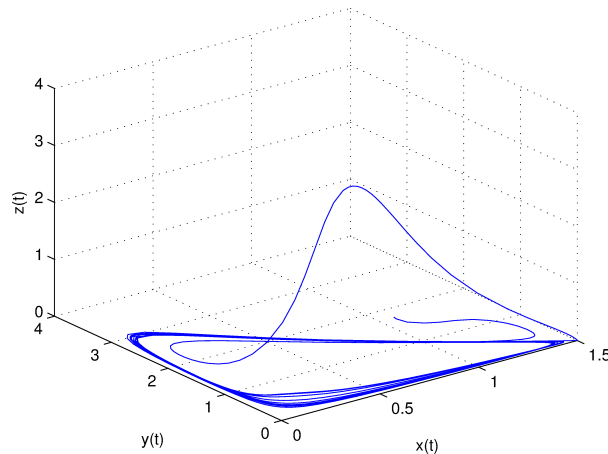


Fig. 8. The phase graph of system (26) in $x - y - z$ plane with $\tau_1 = 0.9$, $\tau_2 = 0.7$ and $\tau_3 = 0.4$. The initial value is $(x(0) = 0.8, y(0) = 0.8, z(0) = 0.7)$.



Competing interests

The authors declare that they have no competing interests.

Authors's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgement: This paper was supported by the Natural Science Foundation of Hunan Province (Nos:2017JJ2130, 2017JJ3131), and Social Science Foundation of Hunan Province (No:16YBA243), and Key project of Education Department for Hunan Province.

The authors would like to thank the referees for their valuable suggestions and comments, which led to the improvement of the manuscript.

References

- [1] Liu J., Li Y.K., Zhao L.L., On a periodic predator-prey system with time delays on time scales, *Comm. Nonl. Sci. Numer. Simu.* 14 (2009) 3432-3438.
- [2] Li Y.K., Zhang T.W., On the existence and stability of a unique almost periodic sequence solution in discrete predator-prey models with time delays, *Appl. Math. Modelling* 35 (2011) 5448-5459.
- [3] Li Y.K., Positive periodic solutions of periodic neutral Lotka-Volterra system with distributed delays, *Chaos, Solitons Fractals* 37 (2008) 288-298.
- [4] Yin F.Q., Li Y.K., Positive periodic solutions of a single species model with feedback regulation and distributed time delays, *Appl. Math. Comput.* 153 (2004) 475-484.
- [5] Li Y.K., Zhu L.F., Positive periodic solutions for a class of higher-dimensional state-dependent delay functional differential equations with feedback control, *Appl. Math. Comput.* 159 (2004) 783-795.
- [6] Li Y.K., Kuang Y., Periodic solutions of periodic delay Lotka-Volterra equations and systems, *J. Math. Anal. Appl.* 255 (2001) 260-280.
- [7] Li Y.K., Periodic solutions for delay Lotka-Volterra competition systems, *J. Math. Anal. Appl.* 246 (2000) 230-244.
- [8] Li Y.K., Zhao K.H., Ye Y., Multiple positive periodic solutions of n species delay competition systems with harvesting terms, *Nonlinear Anal. RWA* 12 (2011) 1013-1022.
- [9] Cai S.M., Hao J.J., He Q.B., Liu Z.R., Exponential synchronization of complex delayed dynamical networks via pinning periodically intermittent control, *Phys. Lett. A* 375 (2011) 1965-1971.
- [10] Zhou J., Liu Z.R., Chen G.R., Dynamics of periodic delayed neural networks, *Neural Networks* 17 (2004) 87-101.
- [11] Sun H.J., Cao H.J., Bifurcations and chaos of a delayed ecological model, *Chaos, Solitons Fractals* 33 (2007) 1383-1393.

- [12] Li J.B., He X.Z., Liu Z.R., Hamiltonian symmetric groups and multiple periodic solutions of differential delay equations, *Nonlinear Anal.* 35 (1999) 457-474.
- [13] Fang H., Li J.B., On the existence of periodic solutions of a neutral delay model of single-species population growth, *J. Math. Anal. Appl.* 259 (2001) 8-17.
- [14] Baek H., Lee H.H., Permanence of a three-species food chain system with impulsive perturbations, *Kyungpook Math. J.* 48 (2008) 503-514.
- [15] Cui G.H., Yan X.P., Stability and bifurcation analysis on a three-species food chain system with two delays, *Comm. Nonl. Sci. Numer. Simu.* 16 (2011) 3704-3720.
- [16] Boudjellaba H., Sari T., Stability loss delay in harvesting competing populations, *J. Differ. Equat.* 152 (1999) 394-408.
- [17] Patra B., Maiti A., Samanta G., Effect of time-delay on a ratio-dependent food chain model, *Nonlinear. Anal. Model Contr.* 14 (2009) 199-216.
- [18] Kar T., Batabyalb A., Persistence and stability of a two prey one predator system, *Int. J. Eng. Sci. Tech.* 2 (2010) 174-190.
- [19] Xu C.J., Tang X.H., Liao M.X., Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments, *Appl. Math. Comput.* 216 (2010) 2920-2936.
- [20] Hu G.P., Li W.T., Hopf bifurcation analysis for a delayed predator-prey system with diffusion effects, *Nonlinear Anal. RWA* 11 (2010) 819-826.
- [21] Kuang Y., *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.
- [22] Yan X.P., Li W.T., Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *Appl. Math. Comput.* 177 (2006) 427-445.
- [23] Hassard B., Kazarinoff N., Wan Y., *Theory and Applications of Hopf bifurcation*, Cambridge Univ. Press, Cambridge, 1981.
- [24] Hu G.P., Li W.T., Yan X.P., Hopf bifurcation in a predator-prey system with multiple delays, *Chaos, Solitons Fractals* 42 (2009) 1273-1285.
- [25] Zhang T.L., Liu J.L., Teng Z.D., Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure, *Nonlinear Anal. RWA* 11 (2010) 293-306.