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Research Article

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On non-Hopfian groups of fractions

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Abstract: The group of fractions of a semigroup S, if exists, can be written as $G = SS^{-1}$. If S is abelian, then G must be abelian. We say that a semigroup identity is transferable if being satisfied in S it must be satisfied in $G = SS^{-1}$. One of problems posed by G.Bergman in 1981 asks whether the group G must satisfy every semigroup identity which is satisfied in S, that is whether every semigroup identity is transferable. The first non-transferable identities were constructed in 2005 by S.V.Ivanov and A.M.Storozhev.

A group G is called Hopfian if each epimorphizm $G \to G$ is the automorphism. The residually finite groups are Hopfian, however there are many problems concerning the Hopfian property e.g. of infinite Burnside groups, of finitely generated relatively free groups [11, Problem 15]. We prove here that if $G = SS^{-1}$ is an n-generator group of fractions of a relatively free semigroup S, satisfying m-variable ($m < n$) non-transferable identity, then G is the non-Hopfian group.

Keywords: Positive law, Hopfian group

MSC: 20F99, 20F05

1 Useful facts and definitions [2, 10]

- A semigroup S is called *cancellative* if for all $a, b, c \in S$ either of equalities $ca = cb,\ ac = bc$ implies $a = b$. All semigroups below are assumed to be cancellative.
- A semigroup S satisfies left (resp. right) *Ore condition* if for arbitrary $a, b \in S$ there are $a', b' \in S$ such that $aa' = bb'$ ($resp.\ a'a = b'b$). The Ore conditions can be defined as $aS \cap bS \neq \emptyset$, ($resp.\ Sa \cap Sb \neq \emptyset$).
- If a semigroup S satisfies both Ore conditions it embeds in the group G of its fractions $G = SS^{-1} = S^{-1}S$.
- A subsemigroup S of a group G is called *generating* if elements of S generate G as a group. The group of fractions of S is unique in the sense that if a group G contains S as a generating semigroup then G is naturally isomorphic to $S^{-1}S = SS^{-1}$.
- A group G satisfies an *identity* $u(x_1, ..., x_m) \equiv v(x_1, ..., x_m)$ if for every elements $g_1, ..., g_m$ in G the equality $u(g_1, ..., g_m) = v(g_1, ..., g_m)$ holds.
- An identity in a semigroup has a form $u(x_1, ..., x_n) \equiv v(x_1, ..., x_n)$ where the words u and v are written without inverses of variables, $u, v \in \mathcal{F}$. Such an identity in a group is called a semigroup identity or a positive identity. By the word "identity" we mean a non-trivial identity.

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2 GB-Problem

Proposition 2.1. *If a generating semigroup S of a group $G = F/N$ satisfies an identity, then for every $s, t \in S$ there exist $s', t' \in S$ such that $ss' = tt'$ (left Ore condition), $G = SS^{-1}$ and $F = \mathcal{F}\mathcal{F}^{-1}N$.*

Proof. An n-variable identity $a(x_1, ..., x_n) \equiv b(x_1, ..., x_n)$ in S implies a 2-variable identity if we replace the i-th variable by xy^i. In view of the cancellation property, it can be written as $xu(x, y) = yv(x, y)$. For $x = s,\ y = t,\ s' := u(s, t),\ t' := v(s, t)$, we have $ss' = tt'$. Since each $g \in G$ is a product of elements in $S \cup S^{-1}$, and for every $s, t \in S$, $s^{-1}t = s't'^{-1}$, we obtain $G = SS^{-1}$. Since by assumption $G = F/N$, we have $F = \mathcal{F}\mathcal{F}^{-1}N$. □

Remark. *Note, that if we use the last letters on both sides of the identity, we get right Ore condition and the equality $G = S^{-1}S$.*

In 1981 G. Bergman [3], [4] posed the following natural question mentioned later in a different form in [5, Question 11.1]. It asks **whether the group of fractions $G = SS^{-1}$ of a semigroup S must satisfy every semigroup identity satisfied by S.**

Note that the behavior of the identities may depend on additional properties of the group. A group G is called **semigroup respecting** (**S-R** group) if all of the identities holding in any generating semigroup S of G, hold in G. It is shown that extensions of soluble groups by locally finite groups of finite exponent are S-R groups [6, Theorem C]. For more results see [2].

Definition 2.2. *We say that a semigroup identity $a \equiv b$ is* **transferable** *if being satisfied in S, it is necessary satisfied in $G = SS^{-1}$.*

It is clear that the abelian identity is transferable. The nilpotent semigroup identities in S found by A. I. Mal'tsev [7] are transferable. The identities $x^n y^n = y^n x^n$ are transferable [8].

So the Problem is **whether every semigroup identity is transferable**. This question we address as the GB-Problem. A group $G = SS^{-1}$ where S satisfies a non-transferable identity we call GB-counterexample.

The first non-transferable identity and hence GB-counterexample (actually a family of them) was found in 2005 by S. Ivanov and A. Storozhev [9]. To speak of GB-counterexamples we suggest a new approach to the GB-Problem.

3 Another approach to the GB-Problem

Let $F := F_n$ be a free group and $\mathcal{F}$ be a free semigroup with the unity, both freely generated by the set $X = \{x_1, x_2, x_3, ...x_n\}$.

By [10, Construction 12.3], if $S = \mathcal{F}/\rho$ is a semigroup and $G = F/N$ is its group of fractions then there is a natural homomorphism $\varphi : F \to F/N$ such that $\mathcal{F}^{\varphi} = \mathcal{F}/\rho$, that is

$$F \longrightarrow F/N =: G$$

$$\mathcal{F} \longrightarrow \mathcal{F}/\rho =: S.$$

Each relation $a = b$ in S is defined by the pair $(a, b) \in \rho$. We consider the set

$$A := N \cap \mathcal{F}\mathcal{F}^{-1} = \{ab^{-1}, (a, b) \in \rho\},$$

which consists of words corresponding to relations $a = b$ in S.

Proposition 3.1. *Let S be a generating semigroup in a group $G = F/N$, $A = N \cap \mathcal{F}\mathcal{F}^{-1}$ and A^F be the normal closure of the set A in F. Then if S satisfies an identity, the following equality holds*

$$F = \mathcal{F}\mathcal{F}^{-1}A^F.$$

Proof. There is the natural homomorphism $F \to F/A^F$, such that $\mathcal{F} \to \mathcal{F}/\mu$, where the congruence μ consists of pairs (a,b) for $ab^{-1} \in A^F \cap \mathcal{F}\mathcal{F}^{-1}$.

$$A^F \cap \mathcal{F}\mathcal{F}^{-1} = \{ab^{-1}, (a,b) \in \mu\}.$$

Since by [10, Corollary 12.8] $A^F \cap \mathcal{F}\mathcal{F}^{-1} = A$, we have $\mu = \rho$. So S is the generating semigroup in the group F/A^F. Then, if S satisfies an identity, we have by Proposition 2.1 $F/A^F = SS^{-1}$, which implies $F = \mathcal{F}\mathcal{F}^{-1}A^F$. □

Lemma 3.2. *Let S be a generating semigroup in a group $G = F/N$, and $A = N \cap \mathcal{F}\mathcal{F}^{-1}$. If S satisfies an identity, then $N = A^F$.*

Proof. Since $F = \mathcal{F}\mathcal{F}^{-1}A^F$. and $A^F \subseteq N$, we have by Dedekind's law

$$N = N \cap F = N \cap \mathcal{F}\mathcal{F}^{-1}A^F = (N \cap \mathcal{F}\mathcal{F}^{-1})A^F = A\,A^F = A^F.$$ □

4 Positive endomorphisms in F

The words in $\mathcal{F}$ are called *positive words*. We say that an endomorphism in F is a positive endomorphism if it maps generators to positive words $X \to \mathcal{F}$. The set of positive endomorphisms in F we denote by End^+, the set of all endomorphisms - by End. The positive endomorphisms act as endomorphisms of $\mathcal{F}$.

Lemma 4.1. *Let $G = SS^{-1} = F/N$. The normal subgroup N is positively invariant if and only if S is a relatively free semigroup.*

Proof. If N is positively invariant subgroup in F then the set A=$N \cap \mathcal{F}\mathcal{F}^{-1}$ is also positively invariant. If $a = b$ is a relation in S, then the word ab^{-1} is in A. Since A is positively invariant, the relation is the identity in S. So if N is positively invariant, then S is the relatively free semigroup.

Conversely, if S is a relatively free semigroup then each relation $a = b$ in S is the identity, hence the set A is positively invariant and by Lemma 3.2, the normal subgroup $N = A^F$ is positively invariant. □

We recall now the following facts similar to those in ([11] 12.21, 12.22).

Notation. *Let $w := ab^{-1} \in \mathcal{F}\mathcal{F}^{-1}$. By w^{End} we denote the End-invariant subgroup (fully invariant subgroup) corresponding to the word w in F which is generated by all images of w under endomorphisms in F.*

By w^{End^+} we denote the End^+-invariant subgroup (positively invariant subgroup) corresponding to the word w in F which is generated by all images of w under positive endomorphisms in F. This subgroup need not be normal.

Corollary 4.2. *If $G = SS^{-1} = F/N$, then:*

(i) *S satisfies an identity $a \equiv b$ if $(ab^{-1})^{End^+} \subseteq N$.*

(ii) *G satisfies an identity $a \equiv b$ if $(ab^{-1})^{End} \subseteq N$.*

(iii) *G is an* **S-R** *group if* **for each identity** *ab^{-1}:*

$$the\ inclusion\ (ab^{-1})^{End^+} \subseteq N\ \ implies\ \ (ab^{-1})^{End} \subseteq N.$$

(iv) *A semigroup identity $a \equiv b$ is* **transferable** *if* **for every** *$N \lhd F$:*

$$the\ inclusion\ (ab^{-1})^{End^+} \subseteq N\ implies\ \ (ab^{-1})^{End} \subseteq N.$$

In particular we obtain

Corollary 4.3. *A semigroup identity $a \equiv b$ is* **transferable** *if*

$$\left((ab^{-1})^{End^+}\right)^F = (ab^{-1})^{End}.$$

If S is a relatively free semigroup satisfying only transferable identities, then its group of fractions is the relatively free group.

If S is a relatively free semigroup defined by one identity $a \equiv b$, then $A = (ab^{-1})^{End^+}$ and by Lemma 3.2 the group of fractions of S is $G_0 := F/N_0$, where

$$N_0 = A^F = \left((ab^{-1})^{End^+}\right)^F.$$

Corollary 4.4. *For every semigroup S satisfying an identity $a \equiv b$ its group of fractions $G = F/N$ is a quotient group of $G_0 = F/N_0$.*

5 GB-counterexamples and non-Hopfian groups

Definition 5.1. *A group $G = SS^{-1} = F/N$ is a GB-***counterexample** *if S satisfies a* **non-transferable identity** *$a \equiv b$. By another words, if N contains $(ab^{-1})^{End^+}$, but does not contain $(ab^{-1})^{End}$.*

The first and the only known family of the GB-counterexamples was found by S.V.Ivanov and A.M.Storozhev [9]. Their groups $G = SS^{-1}$ do not satisfy any identity, while S satisfies a 2-variable non-transferable identity with sufficiently large parameters, similar to that introduced by A. Yu. Ol'shanskii in [12].

Our aim is to show which GB-counterexamples are non-Hopfian groups.

Definition 5.2. *A group G is called Hopfian if each epimorphizm $G \to G$ is the automorphism, or in other words, if G is not isomorphic to its proper quotient.*

Remark. *If G is a non-Hopfian group, there is $K \lhd G$ such that $G \cong G/K$. Then for every quotient G/M we have*

$$G/M \cong G/KM \cong (G/M)/(KM/M),$$

which implies that G/M is the non-Hopfian group unless $K \subseteq M$.

It may help to see that the group of fractions $G = F/N$ of a semigroup S is non-Hopfian since by Corollary 4.4, it is the quotient of $G_0 = F/N_0$, the group of fractions of a relatively free semigroup.

So we are interested in the question: **when a relatively free semigroup satisfying at least one non-transferable identity, must have a non-Hopfian group of fractions?**

It is not known whether each non-transferable identity implies a 2-variable non-transferable identity. So we assume that S satisfies a non-transferable identity of the form $a(x_1, ..., x_m) \equiv b(x_1, ..., x_m)$. We show that if S is the n-generator semigroup and $n > m$, then the group of fractions of S is the non-Hopfian group. For this purpose we start with the following "Common denominator Lemma".

Lemma 5.3. *Let a generating semigroup S of a group G satisfy an identity. Then for all $g_1, g_2, ..., g_m$ in G there are $s_1, s_2, ..., s_m$, and r in S such that $g_i = s_i r^{-1}$, $i = 1, 2, ..., m$.*

Proof. In view of Proposition 2.1, $G = SS^{-1}$. So for $m = 1$ the statement is clear. Let $g_i = t_i q^{-1}$ for $i \le m-1$ and $g_m = ab^{-1}$. By left Ore condition there exist $q', b' \in S$ such that $qq' = bb'$. We denote $r := qq' = bb'$, $s_i := t_i q'$ and $s_m := ab'$. Then

$$g_i = t_i q^{-1} = t_i(q'q'^{-1})q^{-1} = (t_i q')(q'^{-1}q^{-1}) = s_i r^{-1},$$
$$g_m = ab^{-1} = a(b'b'^{-1})b^{-1} = (ab')(b'^{-1}b^{-1}) = s_m r^{-1},$$

as required. ☐

Theorem 5.4. *Let $G = F/N = SS^{-1}$ be an n-generator group of fractions of a relatively free semigroup S, satisfying a non-transferable identity $a(x_1, ..., x_m) \equiv b(x_1, ..., x_m)$, $n > m$. Then G is the non-Hopfian group.*

Proof. In view of Lemma 4.1 we can assume that N is the normal, positively invariant subgroup in F, such that by Corollary 4.2,

$$(ab^{-1})^{End^+} \subseteq N, \quad (ab^{-1})^{End} \nsubseteq N. \tag{1}$$

where $a := a(x_1, ..., x_m)$, and $b := b(x_1, ..., x_m)$ are the words in $\mathcal{F}$.

Let α be an automorphism in the free group F, which maps $x_1 \to x_1$ and $x_i \to x_i x_1^{-1}$ for $i > 1$. Then

$$F/N \cong F/N^{\alpha}.$$

Note that α^{-1} maps $x_i \to x_i x_1$, $i > 1$, hence α^{-1} is the positive endomorphism. Since N is the positively invariant subgroup in F, we have $N^{\alpha^{-1}} \subseteq N$, and hence

$$N \subseteq N^{\alpha}. \tag{2}$$

To show that the inclusion (2) is proper, assume the contrary, that $N = N^{\alpha}$. By assumption the word $w := ab^{-1}$, where $w := w(x_1, x_2, \ldots, x_m) \in N$ defines the identity $a \equiv b$ in S but not in G, so the condition (1) holds. It means that for some words $g_2, g_3, ..., g_{m+1}$ in F, the value of w is not in N:

$$w(g_2, g_3, \ldots g_{m+1}) \notin N. \tag{3}$$

Together with the word $w(x_1, x_2, \ldots, x_m)$, N must contain the word $w(x_2, x_3, \ldots, x_{m+1})$. Since we assumed that (2) is: $N^{\alpha} = N$, we have

$$w(x_2 x_1^{-1}, x_3 x_1^{-1}, \ldots, x_{m+1} x_1^{-1}) \in N.$$

By Lemma 5.3 for every $g_2, g_3, ..., g_{m+1}$ in G there are $s_2, s_3, ..., s_{m+1}$, and r in S such that $g_i = s_i r^{-1}$. The map $x_1 \to r$ and $x_i \to s_i$ is positive. So since N is positively invariant, we get $w(g_2, g_3, \ldots g_{m+1}) \in N$, which contradicts (3).

So the inclusion (2) is proper $N \subsetneq N^{\alpha}$. Then

$$(F/N)\big/(N^{\alpha}/N) \cong F/N^{\alpha} \cong F/N,$$

that is F/N is isomorphic to its quotient, which proves that $G = F/N$ is the non-Hopfian group. □

Corollary 5.5.

1. *The n-generator ($n > 2$) GB-counterexamples constructed in* [9] *by S.V.Ivanov and A.M.Storozhev are the non-Hopfian groups.*
2. *An infinitely generated GB-counterexample must be the non-Hopfian group.*

6 Remark: When $A^F = \langle A \rangle$

In the book [10], just after Theorem 12.10 there is a wrong statement saying that *it is shown in* [13] that *the right Ore condition implies that $\langle A \rangle = A^F$.* Our next Theorem describes the conditition for this equality to hold.

Theorem 6.1. *Let $G = F/N$ where N is positively invariant, $A = N \cap \mathcal{F}\mathcal{F}^{-1}$. The equality $\langle A \rangle = A^F$ holds if and only if G is a relatively free group of finite exponent.*

Proof. If $\langle A \rangle = A^F$ then by Lemma 3.2, $N = \langle A \rangle$ is generated as a subgroup by elements from the set A. Since N is normal, together with ab^{-1} where $a, b \in \mathcal{F}$, it must contain the word $(ab^{-1})^b = b^{-1}a$ as a product of words $st^{-1} \in A$. That is $b^{-1}a = s_1 t_1^{-1} s_2 t_2^{-1} ... s_n t_n^{-1}$, $s, t \in \mathcal{F}$, and hence

$$a = b s_1 t_1^{-1} s_2 t_2^{-1} ... s_n t_n^{-1} \in \langle A \rangle.$$

We can write $a = x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}, k_i > 0$. Since by assumption N is positively invariant, N contains x^n, $n = \sum_i k_i$, $x \in X$. Then N contains x^{-n}. Now, since $x^{-1} = x^{n-1}x^{-n} \in \mathcal{F}N$, we have $\mathcal{F}^{-1} \subseteq \mathcal{F}N$. Then by Proposition 3.1

and Lemma 3.2, $F = \mathcal{F}N$. Hence each endomorphism $\phi \in End$ coincides modulo N with a positive endomorphism $\varphi \in End^+$. If $w \in N$, then $w^\phi \in w^\varphi N = N$. So N is fully invariant and must contain F^n, which implies that G is a relatively free group of finite exponent.

Conversely, let x^n belong to N. To show that $A^F = \langle A \rangle$ it suffices to check that for every $ab^{-1} \in A$ and each generator x, the word $x^{-1}(ab^{-1})x$ is a product of elements in A. So let $ab^{-1} \in A$, then for each $c \in \mathcal{F}$, $(ca)(cb)^{-1} \in A$, hence $(x^{n-1}a)(x^{n-1}b)^{-1} \in A$. Now, since $x^{\pm n} \in N \cap \mathcal{F}\mathcal{F}^{-1} =: A$, we get $x^{-1}(ab^{-1})x = x^{-n}(x^{n-1}a)(x^{n-1}b)^{-1}x^n \in \langle A \rangle$. This implies that the subgroup $\langle A \rangle$ is normal, which finishes the proof. □

Let $G = F/N$ be a group constructed by S.V.Ivanov and A.M.Storozhev in [9]. Since $G = SS^{-1}$ where S satisfies a positive identity, N is positively invariant and S satisfies the left Ore condition. However the equality $\langle A \rangle = A^F$ does not hold because otherwise G would have a finite exponent, which is not so.

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