

Research Article

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Uniform topology on EQ -algebras

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Abstract: In this paper, we use filters of an EQ -algebra E to induce a uniform structure (E, \mathcal{K}) , and then the part \mathcal{K} induce a uniform topology \mathcal{T} in E . We prove that the pair (E, \mathcal{T}) is a topological EQ -algebra, and some properties of (E, \mathcal{T}) are investigated. In particular, we show that (E, \mathcal{T}) is a first-countable, zero-dimensional, disconnected and completely regular space. Finally, by using convergence of nets, the convergence of topological EQ -algebras is obtained.

Keywords: Uniform space, Topological EQ -algebra, Filter, Converge sequence

MSC: 06F99, 54E15

1 Introduction

EQ -algebras were proposed by Novák [1] with the introduction of developing an algebraic structure of truth values for fuzzy type theory (FTT). It has three basic binary operations (meet, multiplication and a fuzzy equality) and a top element. In Novák et al. [2], the study of EQ -algebras has been further deepened. Moreover, the axioms originally introduced in [1] have been slightly modified. Motivated by the assumption that the truth values in FTT were either an $IMTL$ -algebra, a BL -algebra or an MV -algebra, all the algebras above are special kinds of residuated lattices with monoidal operation (multiplication) and its residuum. The latter is a natural interpretation of implication in fuzzy logic; the equivalence is then interpreted by the biresiduum, a derived operation. From the algebraic point of view, the class of EQ -algebras generalizes, in a certain sense, the class of residuated lattices and so, they may become an interesting class of algebraic structures as such. Some interesting consequences of EQ -algebras were obtained (see [3-5]).

The concept of a uniform space can be considered either as axiomatizations of some geometric notions, close to but quite independent of the concept of a topological space, or as convenient tools for an investigation of topological space. In the paper [6], Haveski et al. considered a collection of filters and used the congruence relation with respect to filters to define a uniformity and turned the BL -algebra into a uniform topological space. Then Ghorbani and Hasankhani [7] defined uniform topology and quotient topology on a quotient residuated lattice and proved these topologies coincide. Furthermore, many mathematicians have endowed a number of algebraic structures associated with logical systems with topology and have found some of their properties. In [8], Hoo introduced topological MV -algebras and obtained some interesting results. Hoo's mainly work reveals that the essential ingredients are the existence of an adjoint pair of operations and the fact that ideals of MV -algebras correspond to their congruences. Nganou and Tebu [9] generalized Hoo's work to FL_{ew} -algebras. They considered a similar approach to study FL_{ew} -algebras. Ciungu [10] investigated some concepts of convergence in the class of perfect BL -algebras. Mi Ko and Kim [11] studied relationships between closure operators and BL -algebras. In [12, 13], Borzooei et

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al. studied metrizable on (semi)topological BL -algebras and the relationship between separation axioms and (semi)topological quotient BL -algebras. As EQ -algebras are the generalizations of residuated lattices which the adjoin property failed, our study of uniform topologies in EQ -algebras is meaningful.

This paper is organized as follows: In Section 2, we recall some facts about EQ -algebras and topologies, which are needed in the sequel. In Section 3, in order to induce uniform topology, we use the class of filters of EQ -algebras to construct uniform structures. In Section 4, using the given concept of topological EQ -algebras, we show that EQ -algebras with the uniform topology are topological EQ -algebras, and also some properties are obtained.

2 Preliminaries

In this section, we summarize some definitions and results about EQ -algebras, which will be used in the following sections of this paper.

Definition 2.1 ([2, 14]). *An EQ -algebra is an algebra $E = (E, \wedge, \otimes, \sim, 1)$ of type $(2,2,2,0)$ satisfying the following axioms:*

- (E1) $(E, \wedge, 1)$ is a \wedge -semilattice with top element 1. We put $x \leq y$ if and only if $x \wedge y = x$;
- (E2) $(E, \otimes, 1)$ is a monoid and \otimes is isotone in both arguments with respect to \leq ;
- (E3) $x \sim x = 1$, (reflexivity axiom);
- (E4) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$, (substitution axiom);
- (E5) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$, (congruence axiom);
- (E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$, (monotonicity axiom);
- (E7) $x \otimes y \leq x \sim y$. (boundedness axiom).

For the convenience of readers, we mention some basic properties of the operations on EQ -algebras in the following proposition.

Proposition 2.2 ([2, 14]). *Let E be an EQ -algebra, $x \rightarrow y := (x \wedge y) \sim x$ and $\bar{x} := x \sim 1$. Then the following properties hold for all $x, y, z \in E$:*

- (i) $x \otimes y \leq x \wedge y \leq x, y$;
- (ii) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$;
- (iii) $x \sim y \leq x \rightarrow y$;
- (iv) $x \rightarrow x = 1$;
- (v) $(x \sim y) \otimes (y \sim z) \leq x \sim z$;
- (vi) $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$;
- (vii) $x \leq \bar{x}, \bar{1} = 1$;
- (viii) $x \otimes (x \sim y) \leq \bar{y}$;
- (ix) $(x \rightarrow y) \otimes (y \rightarrow x) \leq x \sim y \leq (x \rightarrow y) \wedge (y \rightarrow x)$;
- (x) if $x \leq y \rightarrow z$, then $x \otimes y \leq \bar{z}$;
- (xi) if $x \leq y \leq z$, then $z \sim x \leq z \sim y$ and $x \sim z \leq x \sim y$.

Definition 2.3 ([2, 14]). *Let E be an EQ -algebra. We say that it is*

- (i) separated if for all $a, b \in E$, $a \sim b = 1$ implies $a=b$;
- (ii) good if for all $a \in E$, $a \sim 1 = a$.

Definition 2.4 ([2, 14]). *Let E be an EQ -algebra. A subset F of E is called an EQ -filter (filter for short) of E if for all $a, b, c \in E$ we have that:*

- (i) $1 \in F$;
- (ii) if $a, a \rightarrow b \in F$; then $b \in F$,
- (iii) if $a \rightarrow b \in F$, then $a \otimes c \rightarrow b \otimes c \in F$ and $c \otimes a \rightarrow c \otimes b \in F$.

Remark 2.5. Note that Definition 2.4 differs from the original definition of filters (see [2, Definition 4]). In Definition 2.4, we do not need this condition: (i i)' if $a, b \in F$, then $a \otimes b \in F$ (see [2, Definition 4]) because it follows from the other conditions. In fact, let F be a filter of an EQ-algebra E . First, we show that F satisfies the condition that if $x \in F$ and $x \leq y$, then $y \in F$. From $x \wedge y = x$ it follows that $x \rightarrow y = 1$. By Definition 2.4 (i) and (ii), it follows that $y \in F$. Let $a, b \in F$. From Proposition 2.2 (vii), it follows that $b \leq 1 \rightarrow b$. From Definition 2.4 (iii), it then follows that $(a \otimes 1) \rightarrow (a \otimes b) \in F$. Hence, by Definition 2.4 (i) and (ii), $a \otimes b \in F$.

Proposition 2.6 ([2, 14]). Let F be a filter of an EQ-algebra E . For all $a, b, a', b', c, e, f \in F$ such that $a \sim b$ and $a' \sim b' \in F$, the following holds:

- (i) if $e \in F$ and $e \leq f$, then $f \in F$;
- (ii) if $e, e \sim f \in F$, then $f \in F$;
- (iii) $a \leftrightarrow b \in F, (a \rightarrow b) \otimes (b \rightarrow c) \in F$, where $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$;
- (iv) $(a \wedge a') \sim (b \wedge b') \in F$;
- (v) $(a \otimes c) \sim (b \otimes c) \in F$ and $(c \otimes a) \sim (c \otimes b) \in F$,
- (vi) $(a \sim a') \sim (b \sim b') \in F$.

As is usually done, given a filter F of an EQ-algebra E , we can define a binary relation on E by

$$a \equiv_F b \text{ if and only if } a \sim b \in F.$$

From Proposition 2.6, we immediately have the following theorem.

Theorem 2.7 ([2, 14]). Let F be a filter of an EQ-algebra E . The relation \equiv_F is a congruence relation on E .

Definition 2.8 ([15]). A poset (D, \leq) is called an upward directed set if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

We recall some basic notions of general topology which will be needed in the sequel.

Recall that a set A with a family \mathcal{T} of its subsets is called a *topological space*, denoted by (A, \mathcal{T}) , if $A, \emptyset \in \mathcal{T}$, the intersection of any finite number of the members of \mathcal{T} is in \mathcal{T} , and the arbitrary union of members of \mathcal{T} is in \mathcal{T} . The members of \mathcal{T} are called *open sets* of A , and the complement of an open set U , $A - U$, is a *closed set*. A subfamily $\{U_\alpha\}_{\alpha \in I}$ of \mathcal{T} is called a *base* of \mathcal{T} if for each $x \in U \in \mathcal{T}$ there is an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$. A subset P of A is a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. Let \mathcal{T}_x denote the totality of all neighborhoods of x in A , then subfamily \mathcal{V}_x of \mathcal{T}_x is a *fundamental system of neighborhoods* of x , if for each U_x in \mathcal{T}_x , there exists a V_x in \mathcal{V}_x such that $V_x \subseteq U_x$. If every point x in A has a countable fundamental system of neighborhoods, then we say that the space (A, \mathcal{T}) satisfies the *first axiom of countability* or is *first-countable*. A topological space (A, \mathcal{T}) is a *zero-dimensional space* if \mathcal{T} has a clopen base. A topological space (A, \mathcal{T}) is called a *regular space* if for any closed subset C of A and $x \in A$ such that $x \notin C$, then there exist disjoint open sets U, V such that $x \in U$ and $C \subseteq V$, or equivalently, for any open subset U containing x , there exists open subset V such that $x \in V \subseteq \overline{V} \subseteq U$. A topological space (A, \mathcal{T}) is called a *completely regular space*, if for every $x \in X$ and every closed set $F \subset A$ such that $x \notin F$ there exists a continuous function $f : A \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. Let (A, \mathcal{T}) and (B, \mathcal{V}) be two topological spaces, a mapping f of A to B is *continuous* if $f^{-1}(U) \in \mathcal{T}$ for any $U \in \mathcal{V}$. The mapping f from (A, \mathcal{T}) to (B, \mathcal{V}) is called a *homeomorphism* if f is bijective, and f and f^{-1} are continuous, or equivalently, if f is bijective, continuous and open (closed). The mapping f from (A, \mathcal{T}) to (B, \mathcal{V}) is called a *quotient map* if f is surjective, and $V \in \mathcal{V}$ if and only if $f^{-1}(V) \in \mathcal{T}$. A topological space (A, \mathcal{T}) is *compact* if each open cover of A is reducible to a finite subcover, and *locally compact* if for every $x \in A$ there exists a neighborhood U of x such that \overline{U} is a compact subspace of A .

Let (X, \mathcal{T}) be a topological space. We have following separation axioms in (X, \mathcal{T}) :

- (1) T_0 : For each $x, y \in X$ and $x \neq y$, there is at least one of them has a neighborhood excluding the other.
- (2) T_1 : For each $x, y \in X$ and $x \neq y$ each has neighborhood not containing the other.
- (3) T_2 : For each $x, y \in X$ and $x \neq y$ both have disjoint neighborhoods $U, V \in \tau$ such that $x \in U$ and $y \in V$.

A topological space satisfying T_i is called a T_i -space, for any $i = 0, 1, 2$. A T_2 -space is also known as a Hausdorff space.

Let $(A, *)$ be an algebra of type 2 and \mathcal{T} be a topology on A . Then $(A, *, \mathcal{T})$ is called a *left (right) topological algebra*, if for all $a \in A$ the map $*$: $A \rightarrow A$ is defined by $x \mapsto a * x$ ($x \mapsto x * a$) is continuous, or equivalently, for any $x \in A$ and any open subset V containing $a * x$ ($x * a$) there exists an open subset W containing x such that $a * W \subseteq V$ ($W * a \subseteq V$). A right and left topological algebra $(A, *, \mathcal{T})$ is called a *semitopological algebra*. Moreover, if the operation $*$ is continuous, or equivalently, for each $x, y \in A$ and each open subset W containing $x * y$, there exist two open subsets V_1 and V_2 containing x and y respectively, such that $V_1 * V_2 \subseteq W$, then $(A, *, \mathcal{T})$ is called a *topological algebra*.

3 Uniformity in EQ -algebras

From now on, we write E instead of the EQ -algebra $\langle E, \wedge, \otimes, \sim, 1 \rangle$ for convenience, unless otherwise stated.

Let X be a nonempty set and U, V be any subsets of $X \times X$. We have the following notation:

- (1) $U \circ V = \{(x, y) \in X \times X : (x, z) \in U, (y, z) \in V, \text{ for some } z \in X\}$;
- (2) $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$;
- (3) $\Delta = \{(x, x) \in X \times X : x \in X\}$.

Definition 3.1 ([16]). *By a uniformity on X we shall mean a nonempty collection \mathcal{K} of subsets of $X \times X$ which satisfies the following conditions:*

- (U1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$;
- (U2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$;
- (U3) if $U \in \mathcal{K}$, then there exists $V \in \mathcal{K}$ such that $V \circ V \subseteq U$;
- (U4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$;
- (U5) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair (X, \mathcal{K}) is then called a *uniform structure (uniform space) on X* .

In the following we use the filters of EQ -algebras to induce uniform structures.

Theorem 3.2. *Let Λ be an arbitrary family of filters of E which is closed under intersection. If $U_F = \{(x, y) \in E \times E : x \equiv_F y\}$ and $\mathcal{K}^* = \{U_F : F \in \Lambda\}$, then \mathcal{K}^* satisfies conditions (U1)-(U4).*

Proof. (U1): Since F is a filter of E , we have $x \equiv_F x$, for any $x \in E$. Hence $\Delta \subseteq U_F$ for all $U_F \in \mathcal{K}^*$.

(U2): For any $U_F \in \mathcal{K}^*$, we have

$$(x, y) \in (U_F)^{-1} \Leftrightarrow (y, x) \in U_F \Leftrightarrow y \equiv_F x \Leftrightarrow x \equiv_F y \Leftrightarrow (x, y) \in U_F.$$

(U3): For any $U_F \in \mathcal{K}^*$, the transitivity of \equiv_F implies that $U_F \circ U_F \subseteq U_F$.

(U4): For any $U_F, U_J \in \mathcal{K}^*$, we claim that $U_F \cap U_J = U_{F \cap J}$. If $(x, y) \in U_F \cap U_J$, then $x \equiv_F y$ and $x \equiv_J y$. Hence $x \sim y \in F$ and $x \sim y \in J$. Then $x \sim y \in F \cap J$ and so $(x, y) \in U_{F \cap J}$. Conversely, let $(x, y) \in U_{F \cap J}$. Then $x \equiv_{F \cap J} y$, hence $x \sim y \in F \cap J$, and thus $x \sim y \in F, x \sim y \in J$. Therefore $x \equiv_F y$ and $x \equiv_J y$, which means that $(x, y) \in U_F \cap U_J$. So $U_F \cap U_J = U_{F \cap J}$. Since $F, J \in \Lambda$, then $F \cap J \in \Lambda$ and so $U_F \cap U_J \in \mathcal{K}^*$. \square

Theorem 3.3. *Let $\mathcal{K} = \{U \subseteq E \times E : \exists U_F \in \mathcal{K}^* \text{ s.t. } U_F \subseteq U\}$, where \mathcal{K}^* comes from Theorem 3.2. Then \mathcal{K} satisfies a uniformity on E and the pair (E, \mathcal{K}) is a uniform structure.*

Proof. By Theorem 3.2, the collection \mathcal{K} satisfies the conditions (U1)-(U4). It suffices to show that \mathcal{K} satisfies (U5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq E \times E$. Then there exists $U_F \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. \square

Let $x \in E$ and $U \in \mathcal{K}$. Define $U[x] := \{y \in E : (x, y) \in U\}$. Clearly, if $V \subseteq U$, then $V[x] \subseteq U[x]$.

Theorem 3.4. *Let E be an EQ -algebra. Then*

$$\mathcal{T} = \{G \subseteq E : (\forall x \in G) (\exists U \in \mathcal{K}) \text{ s.t. } U[x] \subseteq G\}$$

is a topology on E , where \mathcal{K} comes from Theorem 3.3.

Proof. Clearly, \emptyset and the set E belong to \mathcal{T} . It is clear that \mathcal{T} is closed under arbitrary union. Finally to show that \mathcal{T} is closed under finite intersection, let $G, H \in \mathcal{T}$ and suppose that $x \in G \cap H$. Then there exist $U, V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. If $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$, hence $G \cap H \in \mathcal{T}$. Thus \mathcal{T} is a topology on E . \square

Note that for any x in E , $U[x]$ is a neighborhood of x .

Definition 3.5. *Let Λ be an arbitrary family of filters of an EQ -algebra E which is closed under intersection. Then the topology \mathcal{T} comes from Theorem 3.4 is called a uniform topology on E induced by Λ .*

We denote the uniform topology \mathcal{T} obtained from an arbitrary family of filters Λ by \mathcal{T}_Λ , and if $\Lambda = \{F\}$, we denote it by \mathcal{T}_F .

Example 3.6. *Let $E = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:*

\otimes	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

\sim	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

We can easily check that $\langle E, \wedge, \otimes, \sim, 1 \rangle$ is an EQ -algebra. Consider the filter $F = \{b, 1\}$, and $\Lambda = \{F\}$. Therefore as in Theorem 3.2, we construct $\mathcal{K}^* = \{U_F\} = \{(x, y) : x \equiv_F y\} = \{(0, 0), (a, a), (b, b), (b, 1), (1, b), (1, 1)\}$. We can check that (E, \mathcal{K}^*) is a uniform space, where $\mathcal{K} = \{U : U_F \subseteq U\}$. Neighborhoods are $U_F[0] = \{0\}$, $U_F[a] = \{a\}$, $U_F[b] = \{b, 1\}$, $U_F[1] = \{b, 1\}$. From above we get that $\mathcal{T}_F = \{\emptyset, \{0\}, \{a\}, \{b, 1\}, \{0, a\}, \{0, b, 1\}, \{a, b, 1\}, \{0, a, b, 1\}\}$. Thus (E, \mathcal{T}_F) is a uniform topological space.

4 Topological properties of the space (E, \mathcal{T}_Λ)

Note that from Theorem 3.4 giving the Λ family of filters of an EQ -algebra E which is closed under intersection. We can induce a uniform topology \mathcal{T}_Λ on E . In this section we study topological properties on (E, \mathcal{T}_Λ) .

Let E be an EQ -algebra and C, D be subsets of E . Then we define $C * D$ as follows: $C * D = \{x * y : x \in C, y \in D\}$, where $*$ $\in \{\wedge, \otimes, \sim\}$.

Definition 4.1. *Let \mathcal{U} be a topology on E . Then (E, \mathcal{U}) is called a topological EQ -algebra(TEQ -algebra for short) if the operations \wedge, \otimes and \sim are continuous with respect to \mathcal{U} .*

Recall that a topological space (X, \mathcal{U}) is a discrete space if for any $x \in X$, $\{x\}$ is an open set.

Example 4.2. *Every EQ -algebra with a discrete topology is a TEQ -algebra.*

Theorem 4.3. *The pair (E, \mathcal{T}_Λ) is a TEQ -algebra.*

Proof. By Definition 4.1, it suffices to show that $*$ is continuous, where $*$ $\in \{\wedge, \otimes, \sim\}$. Indeed, assume that $x * y \in G$, where $x, y \in E$ and G is an open subset of E . Then there exist $U \in \mathcal{K}$, $U[x * y] \subseteq G$, and a filter F such that

$U_F \in \mathcal{K}^*$ and $U_F \subseteq U$. We claim that the following relation holds:

$$U_F[x] * U_F[y] \subseteq U_F[x * y] \subseteq U[x * y].$$

Let $h * k \in U_F[x] * U_F[y]$. Then $h \in U_F[x]$ and $k \in U_F[y]$ we get that $x \equiv_F h$ and $y \equiv_F k$. Hence $x * y \equiv_F h * k$. From that we obtain $(x * y, h * k) \in U_F \subseteq U$. Hence $h * k \in U_F[x * y] \subseteq U[x * y]$. Then $h * k \in G$. Clearly, $U_F[x]$ and $U_F[y]$ are neighborhoods of x and y , respectively. Therefore, the operation $*$ is continuous. \square

Example 4.4. In Example 3.6, it is easy to check that (E, \mathcal{T}_F) is a TEQ -algebra.

Theorem 4.5. Let Λ be a family of filters of E which is closed under intersection. Any filter in the collection Λ is a clopen subset of E for the topology \mathcal{T}_Λ .

Proof. Let F be a filter of E in Λ and $y \in F^c$. Then $y \in U_F[y]$ and we get $F^c \subseteq \bigcup\{U_F[y] : y \in F^c\}$. We claim that for all $y \in F^c$, $U_F[y] \subseteq F^c$. If $z \in U_F[y]$, then $z \equiv_F y$. Hence $z \sim y \in F$. If $z \in F$, by Lemma 2.6 (i), we get that $y \in F$, which is a contradiction. So $z \in F^c$ and we get $\bigcup\{U_F[y] : y \in F^c\} \subseteq F^c$. Hence $F^c = \bigcup\{U_F[y] : y \in F^c\}$. Since $U_F[y]$ is open for all $y \in E$, it follows that F is a closed subset of E . We show that $F = \bigcup\{U_F[y] : y \in F\}$. If $y \in F$, then $y \in U_F[y]$ and we get $F \subseteq \bigcup\{U_F[y] : y \in F\}$. Let $y \in F$. If $z \in U_F[y]$, then $z \equiv_F y$ and so $y \sim z \in F$. Since $y \in F$, by Lemma 2.6 (i), $z \in F$, and we get $\bigcup\{U_F[y] : y \in F\} \subseteq F$. So F is also an open subset of E . \square

Theorem 4.6. Let Λ be a family of filters of E which is closed under intersection. For any $x \in E$ and $F \in \Lambda$, $U_F[x]$ is a clopen subset of E for the topology \mathcal{T}_Λ .

Proof. First we show that $(U_F[x])^c$ is open. If $y \in (U_F[x])^c$, then $y \sim x \in F^c$. We claim that $U_F[y] \subseteq (U_F[x])^c$. If $z \in U_F[y]$, then $z \in (U_F[x])^c$, otherwise $z \in U_F[x]$, we get that $z \sim y \in F$ and $z \sim x \in F$. Since F is a filter, we get that $(x \sim z) \otimes (z \sim y) \in F$. By $(x \sim z) \otimes (z \sim y) \leq x \sim y$ and F is a filter, it follows that $x \sim y \in F$, which is a contradiction. Hence $U_F[y] \subseteq (U_F[x])^c$ for all $y \in (U_F[x])^c$, and so $U_F[x]$ is closed. It is clear that $U_F[x]$ is open. So $U_F[x]$ is a clopen subset of E . \square

A topological space X is connected if and only if X has only X and \emptyset as clopen subsets. Therefore we have the following corollary.

Corollary 4.7. The space (E, \mathcal{T}_Λ) is not, in general, a connected space.

Proof. It clearly follows from Theorem 4.6. \square

Theorem 4.8. $\mathcal{T}_\Lambda = \mathcal{T}_J$, where $J = \bigcap\{F : F \in \Lambda\}$.

Proof. Let \mathcal{K} and \mathcal{K}^* be as in Theorems 3.2 and 3.3. Now consider $\Lambda_0 = \{J\}$, define $(\mathcal{K}_0)^* = \{U_J\}$ and $\mathcal{K}_0 = \{U : U_J \subseteq U\}$. Let $G \in \mathcal{T}_\Lambda$. So for each $x \in G$, there is $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq F$ we get that $U_J \subseteq U_F$ for any filter F of Λ . Since $U \in \mathcal{K}$, there exists $F \in \Lambda$ such that $U_F \subseteq U$. Hence $U_J[x] \subseteq U_F[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, we get that $G \in \mathcal{T}_J$. So $\mathcal{T}_\Lambda \subseteq \mathcal{T}_J$. Conversely, let $H \in \mathcal{T}_J$. Then for any $x \in H$ there is $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. Hence $U_J[x] \subseteq H$. Since Λ is closed under intersection, so $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in \mathcal{T}_\Lambda$. Therefore, $\mathcal{T}_J \subseteq \mathcal{T}_\Lambda$. \square

Corollary 4.9. Let F and J be filters of E and $F \subseteq J$. Then J is clopen in the topological space (E, \mathcal{T}_F) .

Proof. Consider $\Lambda = \{F, J\}$. Then by Theorem 4.8, $\mathcal{T}_\Lambda = \mathcal{T}_F$. Hence by Theorem 4.5, J is clopen in the topological space (E, \mathcal{T}_F) . \square

Remark 4.10. Let Λ be a family of filters of E which is closed under intersection and $J = \bigcap\{F : F \in \Lambda\}$. We have the following statements:

- (i) By Theorem 4.8, we know that $\mathcal{T}_\Lambda = \mathcal{T}_J$. For any $U \in \mathcal{K}$, $x \in E$, we can get that $U_J[x] \subseteq U[x]$. Hence \mathcal{T}_Λ is equivalent to $\{A \subseteq E : \forall x \in A, U_J[x] \subseteq A\}$. So $A \subseteq E$ is open set if and only if for all $x \in A$, $U_J[x] \subseteq A$ if and only if $A = \bigcup_{x \in A} U_J[x]$;
- (ii) For all $x \in E$, by (i), we know that $U_J[x]$ is the smallest neighborhood of x ;
- (iii) Let $\mathcal{B}_J = \{U_J[x] : x \in E\}$. By (i) and (ii), it is easy to check that \mathcal{B}_J is a base of \mathcal{T}_J ;
- (iv) For all $x \in E$, $\{U_J[x]\}$ is a denumerable fundamental system of neighborhoods of x .

Lemma 4.11. *If F is a filter of E , then for all $x \in E$, $U_F[x]$ is a clopen compact set in the topological space (E, \mathcal{T}_F) .*

Proof. By Theorem 4.6, it is enough to show that $U_F[x]$ is a compact set. Let $U_F[x] \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of E . Since $x \in U_F[x]$, there exists $\alpha \in I$ such that $x \in O_\alpha$. Then $U_F[x] \subseteq O_\alpha$. Hence $U_F[x]$ is compact. Therefore $U_F[x]$ is a clopen compact set in the topological space (E, \mathcal{T}_F) . \square

Theorem 4.12. *Let Λ be a family of filters of E which is closed under intersection. Then (E, \mathcal{T}_Λ) is a first-countable, zero-dimensional, disconnected and completely regular space.*

Proof. By Theorem 4.8, it suffices to show that (E, \mathcal{T}_J) is a first-countable, zero-dimensional, disconnected and completely regular space. Let $x \in E$. By Remark 4.10 (iv), $\{U_J[x]\}$ is a denumerable fundamental system of neighborhoods of x , so (E, \mathcal{T}_J) is first-countable. Let $\mathcal{B}_J = \{U_J[x] : x \in E\}$. By Remark 4.10 (iii) and Theorem 4.6, we get that \mathcal{B}_J is a clopen basis of (E, \mathcal{T}_J) , hence (E, \mathcal{T}_J) is a zero-dimensional space. By Corollary 4.7, we get that (E, \mathcal{T}_J) is a disconnected space. By Lemma 4.11 and Remark 4.10 (ii), $U_J[x]$ is a compact neighborhood of x . Hence (E, \mathcal{T}_J) is a locally compact space. Let $x \in E$ and V be a neighborhood of x . By Remark 4.10 (ii) and Lemma 4.11, there exists closed neighborhood $U_J[x]$ of x such that $U_J[x] \subseteq V$. Therefore, (E, \mathcal{T}_J) is a regular space. Since (E, \mathcal{T}_J) is a locally compact space, we get that it is completely regular. \square

Theorem 4.13. *Let Λ be a family of filters of E which is closed under intersection. Then (E, \mathcal{T}_Λ) is a discrete space if and only if there exists $F \in \Lambda$ such that $U_F[x] = \{x\}$ for all $x \in E$.*

Proof. Let \mathcal{T}_Λ be a discrete topology on E . If for any $F \in \Lambda$, there exists $x \in E$ such that $U_F[x] \neq \{x\}$. Let $J = \bigcap \Lambda$. Then $J \in \Lambda$, there exists $x_0 \in E$ such that $U_J[x_0] \neq \{x_0\}$. It follows that there exists $y_0 \in U_F[x_0]$ and $x_0 \neq y_0$. By Remark 4.10 (ii), $U_J[x_0]$ is the smallest neighborhood of x_0 . Hence $\{x_0\}$ is not an open subset of E , which is a contradiction. Conversely, for any $x \in E$, there exists $F \in \Lambda$ such that $U_F[x] = \{x\}$. Hence $\{x\}$ is an open set of E . Therefore, (E, \mathcal{T}_Λ) is a discrete space. \square

Theorem 4.14. *Let Λ be a family of filters of E which is closed under intersection, $J = \bigcap \Lambda$ and E be a separated EQ -algebra. Then the following conditions are equivalent:*

- (i) (E, \mathcal{T}_J) is a discrete space;
- (ii) $J = \{1\}$.

Proof. (i) \Rightarrow (ii): By Theorem 4.13, we have $U_J[1] = \{1\}$. We show that $J \subseteq U_J[1]$. Let $x \in J$. By Proposition 2.2 (vii), we get that $x \leq x \sim 1$. Since J is a filter and $x \in J$, hence $x \sim 1 \in J$. So $x \in U_J[1]$. It follows that $J \subseteq U_J[1]$. Since $U_J[1] = \{1\}$ and $1 \in J$. Therefore, $J = \{1\}$.

(ii) \Rightarrow (i): Let $J = \{1\}$. Since E is separated, we can get that $U_J[x] = \{x\}$. It follows that (E, \mathcal{T}_J) is discrete. \square

Corollary 4.15. *Let Λ be a family of filters of E which is closed under intersection, $J = \bigcap \Lambda$ and E be a separated EQ -algebra. Then (E, \mathcal{T}_J) is a Hausdorff space if and only if $J = \{1\}$.*

Proof. Let (E, \mathcal{T}_J) be a Hausdorff space. First we show that for any $x \in E$, $U_J[x] = \{x\}$. If there exists $x \neq y \in U_J[x]$, then $y \in U_J[x] \cap U_J[y]$. By Remark 4.10 (ii), $U_J[x]$ and $U_J[y]$ are the smallest neighborhoods of x and y , respectively. Hence for any neighborhood U of x and neighborhood V of y , we have that $y \in U_J[x] \cap U_J[y] \subseteq$

$U \cap V \neq \emptyset$, which is a contradiction. Hence by Theorems 4.13 and 4.14, $J = \{1\}$. The other side of the proof directly follows from Theorem 4.14. \square

Definition 4.16. Let E_1 and E_2 be EQ -algebras. A mapping $\varphi : E_1 \rightarrow E_2$ is called an EQ -morphism from E_1 to E_2 if

$$\varphi(x * y) = \varphi(x) * \varphi(y)$$

for any $*$ \in $\{\wedge, \otimes, \sim\}$. If, in addition, the mapping φ is bijective, then we call φ an EQ -isomorphism. Note that $\varphi(1) = 1$ when φ is an EQ -morphism.

Proposition 4.17. Let $\varphi : E_1 \rightarrow E_2$ be an EQ -morphism. Then the following properties hold:

- (i) if F is a filter of E_2 , then the set $\varphi^{-1}(F)$ is a filter of E_1 ;
- (ii) if φ is surjective and F is a filter of E_1 , then $\varphi(F)$ is a filter of E_2 .

Proof. It is easy to prove by definition of filters. \square

Lemma 4.18. Let E_1 and E_2 be EQ -algebras and F be a filter of E_2 . If $\varphi : E_1 \rightarrow E_2$ is an EQ -isomorphism, then

$$(a, b) \in U_{\varphi^{-1}(F)} \text{ if and only if } (\varphi(a), \varphi(b)) \in U_F, \text{ for any } a, b \in E.$$

Proof. For any $(a, b) \in U_{\varphi^{-1}(F)} \Leftrightarrow a \sim b \in \varphi^{-1}(F) \Leftrightarrow \varphi(a) \sim \varphi(b) \in F \Leftrightarrow (\varphi(a), \varphi(b)) \in U_F$. \square

Theorem 4.19. Let E_1 and E_2 be EQ -algebras and F be a filter of E_2 . If $\varphi : E_1 \rightarrow E_2$ is an EQ -isomorphism, then the following properties hold:

- (i) for any $a \in E_1$, $\varphi(U_{\varphi^{-1}(F)}[a]) = U_F[\varphi(a)]$;
- (ii) for any $b \in E_2$, $\varphi^{-1}(U_F[b]) = U_{\varphi^{-1}(F)}[\varphi^{-1}(b)]$.

Proof. (i) Let $b \in \varphi(U_{\varphi^{-1}(F)}[a])$. Then there exists $c \in U_{\varphi^{-1}(F)}[a]$ such that $b = \varphi(c)$. It follows that $a \sim c \in \varphi^{-1}(F) \Rightarrow \varphi(a) \sim \varphi(c) \in F \Rightarrow \varphi(a) \sim b \in F \Rightarrow b \in U_F[\varphi(a)]$.

Conversely, $b \in U_F[\varphi(a)] \Rightarrow \varphi(a) \sim b \in F \Rightarrow \varphi^{-1}(\varphi(a) \sim b) \in \varphi^{-1}(F) \Rightarrow a \sim \varphi^{-1}(b) \in \varphi^{-1}(F) \Rightarrow \varphi^{-1}(b) \in U_{\varphi^{-1}(F)}[a] \Rightarrow b \in \varphi(U_{\varphi^{-1}(F)}[a])$.

(ii) $a \in \varphi^{-1}(U_F[b]) \Leftrightarrow \varphi(a) \in U_F[b] \Leftrightarrow \varphi(a) \sim b \in F \Leftrightarrow \varphi^{-1}(\varphi(a) \sim b) \in \varphi^{-1}(F) \Leftrightarrow a \sim \varphi^{-1}(b) \in \varphi^{-1}(F) \Leftrightarrow a \in U_{\varphi^{-1}(F)}[\varphi^{-1}(b)]$. \square

Theorem 4.20. Let E_1 and E_2 be EQ -algebras and F be a filter of E_2 . If $\varphi : E_1 \rightarrow E_2$ is an EQ -isomorphism, then φ is a continuous map from $(E_1, \mathcal{T}_{\varphi^{-1}(F)})$ to (E_2, \mathcal{T}_F) .

Proof. Let $A \in \mathcal{T}_F$. By Remark 4.10 (i), we can get that $A = \bigcup_{a \in A} U_F[a]$. It follows that $\varphi^{-1}(A) = \varphi^{-1}(\bigcup_{a \in A} U_F[a]) = \bigcup_{a \in A} \varphi^{-1}(U_F[a])$. We claim that if $b \in \varphi^{-1}(U_F[a])$, then $U_{\varphi^{-1}(F)}[b] \subseteq \varphi^{-1}(U_F[a])$. Indeed, let $c \in U_{\varphi^{-1}(F)}[b]$, we get that $c \sim b \in \varphi^{-1}(F)$, so $\varphi(c) \sim \varphi(b) \in F$. Since $\varphi(b) \in U_F[a]$, we get that $\varphi(c) \sim a \in F$. It follows that $\varphi(c) \in U_F[a]$. Thus we have that $\varphi(c) \in U_F[a]$. So $c \in \varphi^{-1}(U_F[a])$. Hence $\varphi^{-1}(U_F[a]) = \bigcup_{b \in \varphi^{-1}(U_F[a])} U_{\varphi^{-1}(F)}[b] \in \mathcal{T}_{\varphi^{-1}(F)}$. Therefore, $\varphi^{-1}(A) = \bigcup_{a \in A} \varphi^{-1}(U_F[a]) \in \mathcal{T}_{\varphi^{-1}(F)}$. So φ is a continuous map. \square

Theorem 4.21. Let E_1 and E_2 be EQ -algebras and F be a filter of E_2 . If $\varphi : E_1 \rightarrow E_2$ is an EQ -isomorphism, then φ is a quotient map from $(E_1, \mathcal{T}_{\varphi^{-1}(F)})$ to (E_2, \mathcal{T}_F) .

Proof. From Theorem 4.20 we get that φ is a continuous surjective map. It is enough to show that φ is an open map. Let A be an open set of $(E_1, \mathcal{T}_{\varphi^{-1}(F)})$. We claim that $\varphi(A)$ is an open set of (E_2, \mathcal{T}_F) . Let $a \in \varphi(A)$. We shall show that $U_F[a] \subseteq \varphi(A)$. Indeed, for any $b \in U_F[a]$, we get that $b \sim a \in F$. By Lemma 4.15, we have $\varphi^{-1}(a) \sim \varphi^{-1}(b) \in \varphi^{-1}(F)$. Hence $\varphi^{-1}(b) \in U_{\varphi^{-1}(F)}[\varphi^{-1}(a)]$. Since $a \in \varphi(A)$ and φ is injective we get that

$\varphi^{-1}(a) \in A$. By Remark 4.10 (i), it follows that $U_{\varphi^{-1}(F)}[\varphi^{-1}(a)] \subseteq A$. So $\varphi^{-1}(b) \in A$, we get that $b \in \varphi(A)$. Therefore, $U_F[a] \subseteq \varphi(A)$. So φ is a quotient map. \square

Corollary 4.22. *Let E_1 and E_2 be EQ-algebras and F be a filter of E_2 . If $\varphi : E_1 \rightarrow E_2$ is an EQ-isomorphism, then φ is a homeomorphism map from $(E_1, \mathcal{T}_{\varphi^{-1}(F)})$ to (E_2, \mathcal{T}_F) .*

Proof. It clearly follows from Theorem 4.21. \square

Recall that a uniform space (X, \mathcal{K}) is totally bounded if for each $U \in \mathcal{K}$, there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n U[x_i]$.

Theorem 4.23. *Let F be a filter of E . Then the following conditions are equivalent:*

- (1) *the topological space (E, \mathcal{T}_F) is compact;*
- (2) *the topological space (E, \mathcal{T}_F) is totally bounded;*
- (3) *there exists $P = \{x_1, \dots, x_n\} \subseteq E$ such that for all $a \in E$ there exists $x_i \in P$ such that $a \equiv_F x_i$.*

Proof. (1) \Rightarrow (2): The proof is straightforward.

(2) \Rightarrow (3): Since (E, \mathcal{T}_F) is totally bounded, there exist $x_1, \dots, x_n \in E$ such that $E = \bigcup_{i=1}^n U_F[x_i]$. Now let $a \in E$. Then there exists x_i such that $a \in U_F[x_i]$, therefore $a \sim x_i \in F$ i.e. $a \equiv_F x_i$.

(3) \Rightarrow (1): For any $a \in E$, by hypothesis, there exists $x_i \in P$ such that $a \sim x_i \in F$. We can get that $a \in \bigcup_F[x_i]$, hence $E = \bigcup_{i=1}^n U_F[x_i]$. Now let $E = \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of E . Then for any $x_i \in E$ there exists $\alpha_i \in I$ such that $x_i \in O_{\alpha_i}$. Since O_{α_i} is an open set, $U_F[x_i] \subseteq O_{\alpha_i}$, so we have that

$$E = \bigcup_{i \in I} U_F[x_i] \subseteq \bigcup_{i=1}^n O_{\alpha_i}.$$

Therefore $E = \bigcup_{i=1}^n O_{\alpha_i}$, whence (E, \mathcal{T}_F) is compact. \square

Theorem 4.24. *If F is a filter of E such that F^c is a finite set, then the topological space (E, \mathcal{T}_F) is compact.*

Proof. Let $E = \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open subset of E . Let $F^c = \{x_1, \dots, x_n\}$. Then there exist $\alpha, \alpha_1, \dots, \alpha_n \in I$ such that $1 \in O_\alpha, x_1 \in O_{\alpha_1}, \dots, x_n \in O_{\alpha_n}$. Then $U_F[1] \subseteq O_\alpha$, but $U_F[1] = F$. Hence $E = \bigcup_{i=1}^n O_{\alpha_i} \cup O_\alpha$. \square

Theorem 4.25. *If F is a filter of E , then F is a compact set in the topological space (E, \mathcal{T}_F) .*

Proof. Let $F \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each O_α is open set of E . Since $1 \in F$, there is $\alpha \in I$ such that $1 \in O_\alpha$. Then $F = U_F[1] \subseteq O_\alpha$. Hence F is a compact set in the topological space (E, \mathcal{T}_F) . \square

Our next target is to establish the convergence of EQ-algebras using the convergence of nets.

Definition 4.26. *Let E be an EQ-algebra and (D, \leq) be an upward directed set. If for any $\alpha \in D$ we have $a_\alpha \in E$, then we call $\{a_\alpha\}_{\alpha \in D}$ a net of E .*

Definition 4.27. *Let $\{a_\alpha\}_{\alpha \in D}$ be a net of E . In the topological space (E, \mathcal{T}_F) , say that $\{a_\alpha\}_{\alpha \in D}$*

- (i) *converges to the point a of E if for any neighborhood U of a , there exists $d_0 \in D$ such that $a_\alpha \in U$ for any $\alpha \geq d_0$;*
- (ii) *Cauchy sequence if there exists $d_0 \in D$ such $a_\alpha \equiv_F a_\beta$ for any $\alpha, \beta \geq d_0$.*

A net $\{a_\alpha\}_{\alpha \in D}$, which converges to a is said to be convergent. For simplicity, we write $\lim a_\alpha = a$ and we say that a is a limit of $\{a_\alpha\}_{\alpha \in D}$.

Example 4.28. Consider the TEQ -algebra (E, \mathcal{T}_F) in Example 4.4. Clearly, (\mathbb{N}, \leq) is an upward directed set, where \mathbb{N} is a natural number set. We define $\{a_n\}_{n \in \mathbb{N}}$ as $a_0 = 0, a_1 = a, a_2 = b, a_n = 1, n \geq 3$. It is easy to check that $\{a_n\}_{n \in \mathbb{N}}$ is a net of E . Let $n_0 = 3$. For any neighborhood U of 1, if $n \geq 3$, then $1 \in U$. Therefore, $\lim a_n = 1$.

Theorem 4.29. Let $\{a_\alpha\}_{\alpha \in D}$ and $\{b_\alpha\}_{\alpha \in D}$ be nets of E and F be a filter of E . Then in the topological space (E, \mathcal{T}_F) we have:

- (i) if $\lim b_\alpha = b$ and $\lim a_\alpha = a$, for some $a, b \in E$, then the sequence $\{a_\alpha * b_\alpha\}_{\alpha \in D}$ is convergent and $\lim a_\alpha * b_\alpha = a * b$, for any operation $* \in \{\wedge, \otimes, \sim\}$;
- (ii) any convergent sequence of E is a Cauchy sequence.

Proof. (i) Let $\lim a_\alpha = a, \lim b_\alpha = b$ and $* \in \{\wedge, \otimes, \sim\}$, for some $a, b \in E$. For any neighborhood W of $a * b$ we get that $U_F[a * b] \subseteq W$. Clearly, $U_F[a]$ and $U_F[b]$ are neighborhoods of a and b , respectively. By hypothesis, there exist $d_1, d_2 \in D$ such that $a_\alpha \in U_F[a]$ and $b_\alpha \in U_F[b]$, for any $\alpha \geq d_1$ and $\alpha \geq d_2$. Since D is an upward directed set, then there exists $d_0 \in D$ such that $d_0 \geq d_1$ and $d_0 \geq d_2$. By Theorem 4.3, we get that $U_F[a] * U_F[b] \subseteq U_F[a * b]$. So $a_\alpha \in U_F[a]$ and $b_\alpha \in U_F[b]$, for any $\alpha \geq d_0$. It follows that $a_\alpha * b_\alpha \in U_F[a] * U_F[b] \subseteq U_F[a * b] \subseteq W$, for any $\alpha \geq d_0 \in D$. Therefore, $\lim a_\alpha * b_\alpha = a * b$.

(ii) Let $\{a_\alpha\}_{\alpha \in D}$ be a net of E and $\lim a_\alpha = a$. For the neighborhood $U_F[a]$ of a , there exists $d \in D$ such that $a_\alpha \in U_F[a]$, for any $\alpha \geq d$. So if $\alpha, \beta \geq d$, then $a_\alpha, a_\beta \in U_F[a]$ that is $a_\alpha \equiv_F a$ and $a_\beta \equiv_F a$. It follows that $a_\alpha \equiv_F a_\beta$. Therefore, $\{a_\alpha\}_{\alpha \in D}$ is a Cauchy sequence. \square

5 Conclusion

It is well known that EQ -algebras play an important role in investigating the algebraic structures of logical systems. In this study, we endowed an EQ -algebra with uniform topology \mathcal{T}_Λ and proposed the concept of the topological EQ -algebra. We then stated and proved special properties of (E, \mathcal{T}_Λ) . Especially, we proved that (E, \mathcal{T}_Λ) is a first-countable, zero-dimensional, disconnected and completely regular space. From the category point of view, the role of isomorphism in algebra is the same as the role of homeomorphism in topology. Hence we also studied the relationship between isomorphism(algebraic invariant) and homeomorphism(topological invariant) in topological EQ -algebras. Finally, we investigated the convergent properties of topological EQ -algebras.

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