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# Reduction of a Schwarz-type boundary value problem for biharmonic monogenic functions to Fredholm integral equations

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**Abstract:** We consider a commutative algebra  $\mathbb{B}$  over the field of complex numbers with a basis  $\{e_1, e_2\}$  satisfying the conditions  $(e_1^2 + e_2^2)^2 = 0$ ,  $e_1^2 + e_2^2 \neq 0$ . Let  $D$  be a bounded simply-connected domain in  $\mathbb{R}^2$ . We consider (1-4)-problem for monogenic  $\mathbb{B}$ -valued functions  $\Phi(xe_1 + ye_2) = U_1(x, y)e_1 + U_2(x, y)ie_1 + U_3(x, y)e_2 + U_4(x, y)ie_2$  having the classic derivative in the domain  $D_\zeta = \{xe_1 + ye_2 : (x, y) \in D\}$ ; to find a monogenic in  $D_\zeta$  function  $\Phi$ , which is continuously extended to the boundary  $\partial D_\zeta$ , when values of two component-functions  $U_1, U_4$  are given on the boundary  $\partial D$ . Using a hypercomplex analog of the Cauchy type integral, we reduce the (1-4)-problem to a system of integral equations on the real axes. We establish sufficient conditions under which this system has the Fredholm property and the unique solution. We prove that a displacements-type boundary value problem of 2-D isotropic elasticity theory is reduced to (1-4)-problem with appropriate boundary conditions.

**Keywords:** Biharmonic equation, Biharmonic algebra, Monogenic function, Schwarz-type boundary value problem, Displacements-type boundary value problem

**MSC:** 30G35, 31A30, 74B05

## 1 Biharmonic monogenic functions and Schwarz-type boundary value problems for them

An associative commutative two-dimensional algebra  $\mathbb{B}$  with the unit 1 over the field of complex numbers  $\mathbb{C}$  is called *biharmonic* (see [1, 2]) if in  $\mathbb{B}$  there exists a basis  $\{e_1, e_2\}$  satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0. \quad (1)$$

Such a basis  $\{e_1, e_2\}$  is also called *biharmonic*.

In the paper [2] I. P. Mel'nichenko proved that there exists the unique biharmonic algebra  $\mathbb{B}$ , and he constructed all biharmonic bases in  $\mathbb{B}$ . Note that the algebra  $\mathbb{B}$  is isomorphic to four-dimensional over the field of real numbers  $\mathbb{R}$  algebras considered by A. Douglis [3] and L. Sobrero [4].

In what follows, we consider a biharmonic basis  $\{e_1, e_2\}$  with the following multiplication table (see [1]):

$$e_1 = 1, \quad e_2^2 = e_1 + 2ie_2,$$

where  $i$  is the imaginary complex unit. We consider also a basis  $\{1, \rho\}$  (see [2]), where a nilpotent element  $\rho = 2e_1 + 2ie_2$  satisfies the equality  $\rho^2 = 0$ .

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We use the euclidian norm  $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$  in the algebra  $\mathbb{B}$ , where  $a = z_1 e_1 + z_2 e_2$  and  $z_1, z_2 \in \mathbb{C}$ . Consider a *biharmonic plane*  $\mu\{e_1, e_2\} := \{\zeta = x e_1 + y e_2 : x, y \in \mathbb{R}\}$  which is a linear span of the elements  $e_1, e_2$  over the field of real numbers  $\mathbb{R}$ . With a domain  $D$  of the Cartesian plane  $xOy$  we associate the congruent domain  $D_\zeta := \{\zeta = x e_1 + y e_2 : (x, y) \in D\}$  in the biharmonic plane  $\mu\{e_1, e_2\}$  and the congruent domain  $D_z := \{z = x + iy : (x, y) \in D\}$  in the complex plane  $\mathbb{C}$ . Its boundaries are denoted by  $\partial D, \partial D_\zeta$  and  $\partial D_z$ , respectively. Let  $\overline{D_\zeta}$  (or  $\overline{D_z}$ ) be the closure of domain  $D_\zeta$  (or  $D_z$ ). In what follows,  $\zeta = x e_1 + y e_2, z = x + iy$  and  $x, y \in \mathbb{R}$ .

We say that a function  $\Phi : D_\zeta \rightarrow \mathbb{B}$  is *monogenic* in a domain  $D_\zeta$  and denote  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$ , if the derivative  $\Phi'(\zeta)$  exists at every point  $\zeta \in D_\zeta$ :

$$\Phi'(\zeta) := \lim_{h \rightarrow 0, h \in \mu\{e_1, e_2\}} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}.$$

Every  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  has the derivative of any order in  $D_\zeta$  (cf., e.g., [5, 6]) and satisfies the equalities

$$(\Delta_2)^2 \Phi(\zeta) \equiv \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \Phi(\zeta) = \Phi^{(4)}(\zeta) (e_1^2 + e_2^2)^2 = 0 \quad \forall \zeta \in D_\zeta.$$

due to the conditions (1). Therefore, we shall also term such a function  $\Phi$  by *biharmonic monogenic* function in  $D_\zeta$ .

Any function  $\Phi: D_\zeta \rightarrow \mathbb{B}$  has an expansion of the type

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) i e_1 + U_3(x, y) e_2 + U_4(x, y) i e_2, \tag{2}$$

where  $U_l: D \rightarrow \mathbb{R}, l = 1, 2, 3, 4$ , are real-valued component-functions. We shall use the following notation:  $U_l[\Phi] := U_l, l = 1, 2, 3, 4$ .

All component-functions  $U_l, l = 1, 2, 3, 4$ , in the expansion (2) of any  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  are biharmonic functions, i.e., satisfy the biharmonic equation  $(\Delta_2)^2 U(x, y) = 0$  in  $D$ . At the same time, every biharmonic in a simply-connected domain  $D$  function  $U(x, y)$  is the first component  $U_1 \equiv U$  in the expression (2) of a certain function  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  and, moreover, all such functions  $\Phi$  are found in [5, 6] in an explicit form.

It is proved in [1] that  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  if and only if its each real-valued component-function in (2) is real differentiable in  $D$  and the following analog of the Cauchy – Riemann condition is fulfilled:

$$\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2 \quad \forall \zeta \in D_\zeta. \tag{3}$$

Every  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  is expressed via two corresponding analytic functions  $F: D_z \rightarrow \mathbb{C}, F_0: D_z \rightarrow \mathbb{C}$  of the complex variable  $z$  in the form (cf., e.g., [5–7]):

$$\Phi(\zeta) = F(z) e_1 - \left( \frac{iy}{2} F'(z) - F_0(z) \right) \rho \quad \forall \zeta \in D_\zeta. \tag{4}$$

The equality (4) establishes one-to-one correspondence between functions  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  and pairs of complex-valued analytic functions  $F, F_0$  in  $D_z$ .

In what follows, we assume that the domain  $D_z$  is a bounded and simply-connected, and in this case we shall say that the domain  $D_\zeta$  is also bounded and simply connected.

V.F. Kovalev [8] considered the following boundary value problem for biharmonic monogenic functions: to find a function  $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  which is continuously extended onto the closure  $\overline{D_\zeta}$  when values of two component-functions in (2) are given on  $\partial D_\zeta$ , i.e., the following boundary conditions are satisfied:

$$U_k(x, y) = u_k(\zeta), \quad U_m(x, y) = u_m(\zeta) \quad \forall \zeta \in \partial D_\zeta, \quad 1 \leq k < m \leq 4,$$

where  $u_k$  and  $u_m$  are given functions.

We shall call such a problem by the  $(k-m)$ -problem. V.F. Kovalev [8] called it by a *biharmonic Schwartz problem* owing to its analogy with the classical Schwartz problem on finding an analytic function of the complex variable when values of its real part are given on the boundary of domain.

It was established in [8] that all biharmonic Schwartz problems are reduced to the main three problems: the (1-2)-problem or the (1-3)-problem or the (1-4)-problem.

It is shown (see [8–10]) that the fundamental biharmonic problem (cf, e.g., [11, p.146]) is reduced to the (1-3)-problem. In [9], we investigated the (1-3)-problem for cases where  $D_\zeta$  is either an upper half-plane or a unit disk in the biharmonic plane. Its solutions were found in explicit forms with using of some integrals analogous to the classic Schwarz integral. Similar results are obtained in [12, 13] for the (1-4)-problem.

In [10], using a hypercomplex analog of the Cauchy type integral, we reduced the (1-3)-problem to a system of integral equations and established sufficient conditions under which this system has the Fredholm property. It was made for the case where the boundary of domain belongs to a class being wider than the class of Lyapunov curves that was usually required in the plane elasticity theory (cf., e.g., [11, 14–16]).

In this paper we develop a method for reducing (1-4)-problem to a system of the Fredholm integral equations. The obtained results are appreciably similar to respective results for (1-3)-problem in [10], however, in contrast to (1-3)-problem, which is solvable in a general case if and only if a certain natural condition is satisfied, the (1-4)-problem is solvable unconditionally.

Let us underscore that (1-4)-problem is used in [17, 18] for solving a displacements-type boundary value problem on finding some partial derivatives of displacements by their limiting values in the case of 2-D isotropic elasticity theory.

Note that in the papers [4, 19–21] for investigations of biharmonic functions there are other approaches, which involve commutative finite-dimensional algebras over the field  $\mathbb{R}$  and suitable “analytic” hypercomplex functions.

## 2 Solving process of (1-4)-problem via analytic functions of the complex variable

A method for solving the (1-4)-problem by means of its reduction to classical Schwartz boundary value problems for analytic functions of the complex variable is delivered in [12, 13]. Let us formulate some results of such a kind.

For a continuous function  $u : \partial D_\zeta \rightarrow \mathbb{R}$ , by  $\widehat{u}$  we denote the function defined on  $\partial D_z$  by the equality  $\widehat{u}(z) = u(\zeta)$  for all  $z = x + iy \in \partial D_z$ .

The following theorem is proved in the papers [12, 13].

**Theorem 2.1.** *Let  $u_l : \partial D_\zeta \rightarrow \mathbb{R}$ ,  $l \in \{1, 4\}$ , are continuous functions and, moreover, a function  $yF'(z)$  has a continuous extendibility to the boundary  $\partial D_z$ , where  $F$  is a solution of the classical Schwartz problem with the boundary condition*

$$\operatorname{Re} F(t) = \widehat{u}_1(t) - \widehat{u}_4(t) \quad \forall t \in \partial D_z.$$

*Then a solution of the (1-4)-problem is expressed by the formula (4), where a function  $F_0$  is a solution of the classical Schwartz problem with the boundary condition*

$$\operatorname{Re} F_0(t) = \frac{1}{2} \left( \widehat{u}_4(t) - \operatorname{Im} t \operatorname{Im} \lim_{z \rightarrow t, z \in D_z} F'(z) \right) \quad \forall t \in \partial D_z.$$

A particular case of Theorem 2.1 is the following theorem.

**Theorem 2.2.** *A general solution of the homogeneous (1-4)-problem with  $u_1 = u_4 \equiv 0$  is expressed by the formula  $\Phi(\zeta) = a_1 i e_1 + a_2 e_2$ , where  $a_1$  and  $a_2$  are arbitrary real constants.*

Note, that Theorem 2.2 is proved in [12, 13] for an arbitrary (bounded or unbounded) simply connected domain  $D_\zeta$ .

### 3 Scheme for reducing the (1-4)-problem to a system of Fredholm integral equations

Let us develop a method for solving the inhomogeneous (1-4)-problem without an essential in Theorem 2.1 assumption that a function  $yF'(z)$  has a continuous extendibility to the boundary  $\partial D_z$ .

Let the boundary  $\partial D_\zeta$  be a closed smooth Jordan curve. We assume that every boundary function  $u_l, l \in \{1, 4\}$ , satisfies the Dini condition

$$\int_0^1 \frac{\omega(u_l, \eta)}{\eta} d\eta < \infty \tag{5}$$

with the modulus of continuity  $\omega(u_l, \varepsilon) := \sup_{\zeta_1, \zeta_2 \in \partial D_\zeta: \|\zeta_1 - \zeta_2\| \leq \varepsilon} \|u_l(\zeta_1) - u_l(\zeta_2)\|$ .

We seek solutions in a class of functions represented in the form

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\partial D_\zeta} (\varphi_1(\tau)e_1 + \varphi_4(\tau)ie_2)(\tau - \zeta)^{-1} d\tau \quad \forall \zeta \in D_\zeta, \tag{6}$$

where the functions  $\varphi_l: \partial D_\zeta \rightarrow \mathbb{R}, l \in \{1, 4\}$ , satisfy Dini conditions of the type (5).

It is proved in Theorem 4.2 [10] that the integral (6) has limiting values  $\Phi^+(\zeta_0) := \lim_{\zeta \rightarrow \zeta_0, \zeta \in D_\zeta} \Phi(\zeta)$ ,  $\Phi^-(\zeta_0) := \lim_{\zeta \rightarrow \zeta_0, \zeta \in \mu\{e_1, e_2\} \setminus \overline{D_\zeta}} \Phi(\zeta)$  in every point  $\zeta_0 \in \partial D_\zeta$  that are represented by the Sokhotski–Plemelj formulas:

$$\Phi^+(\zeta_0) = \frac{1}{2} \varphi(\zeta_0) + \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau, \tag{7}$$

$$\Phi^-(\zeta_0) = -\frac{1}{2} \varphi(\zeta_0) + \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau, \tag{8}$$

where  $\varphi(\tau) := \varphi_1(\tau)e_1 + \varphi_4(\tau)ie_2$  and a singular integral is understood in the sense of its Cauchy principal value:

$$\int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \rightarrow 0} \int_{\{\tau \in \partial D_\zeta: \|\tau - \zeta_0\| > \varepsilon\}} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau.$$

We use a conformal mapping  $z = \tau(t)$  of the upper half-plane  $\{t \in \mathbb{C} : \text{Im } t > 0\}$  onto the domain  $D_z$ . Denote  $\tau_1(t) := \text{Re } \tau(t), \tau_2(t) := \text{Im } \tau(t)$ .

Inasmuch as the mentioned conformal mapping is continued to a homeomorphism between the closures of corresponding domains, the function

$$\tilde{\tau}(s) := \tau_1(s)e_1 + \tau_2(s)e_2 \quad \forall s \in \overline{\mathbb{R}}$$

generates a homeomorphic mapping of the extended real axis  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$  onto the curve  $\partial D_\zeta$ .

Introducing the function

$$g(s) := g_1(s)e_1 + g_4(s)ie_2 \quad \forall s \in \overline{\mathbb{R}},$$

where  $g_l(s) := \varphi_l(\tilde{\tau}(s)), l \in \{1, 4\}$ , we rewrite the equality (7) in the form (cf. [10])

$$\Phi^+(\zeta_0) = \frac{1}{2} g(t) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s)k(t, s) ds + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s) \frac{1 + st}{(s - t)(s^2 + 1)} ds,$$

where

$$k(t, s) = k_1(t, s)e_1 + ipk_2(t, s),$$

$$k_1(t, s) := \frac{\tau'(s)}{\tau(s) - \tau(t)} - \frac{1 + st}{(s - t)(s^2 + 1)},$$

$$k_2(t, s) := \frac{\tau'(s)(\tau_2(s) - \tau_2(t))}{2(\tau(s) - \tau(t))^2} - \frac{\tau_2'(s)}{2(\tau(s) - \tau(t))},$$

and a correspondence between the points  $\zeta_0 \in \partial D_\zeta \setminus \{\tilde{\tau}(\infty)\}$  and  $t \in \mathbb{R}$  is given by the equality  $\zeta_0 = \tilde{\tau}(t)$ .

Now, we single out components  $U_l[\Phi^+(\zeta_0)]$ ,  $l \in \{1, 4\}$ , and after the substitution them into the boundary conditions of the (1-4)-problem, we shall obtain the following system of integral equations for finding the functions  $g_1$  and  $g_4$ :

$$\begin{aligned} U_1[\Phi^+(\zeta_0)] &\equiv \frac{1}{2} g_1(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(s) (\operatorname{Im} k_1(t, s) + 2\operatorname{Re} k_2(t, s)) ds - \frac{1}{\pi} \int_{-\infty}^{\infty} g_4(s) \operatorname{Re} k_2(t, s) ds = \tilde{u}_1(t), \\ U_4[\Phi^+(\zeta_0)] &\equiv \frac{1}{2} g_4(t) + \frac{1}{\pi} \int_{-\infty}^{\infty} g_1(s) \operatorname{Re} k_2(t, s) ds + \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} g_4(s) (\operatorname{Im} k_1(t, s) - 2\operatorname{Re} k_2(t, s)) ds = \tilde{u}_4(t) \quad \forall t \in \mathbb{R}, \end{aligned} \tag{9}$$

where  $\tilde{u}_l(t) := u_l(\tilde{\tau}(t))$ ,  $l \in \{1, 4\}$ .

Let  $C(\overline{\mathbb{R}})$  denote the Banach space of functions  $g_*: \overline{\mathbb{R}} \rightarrow \mathbb{C}$  that are continuous on the extended real axis  $\overline{\mathbb{R}}$  with the norm  $\|g_*\|_{C(\overline{\mathbb{R}})} := \sup_{t \in \overline{\mathbb{R}}} |g_*(t)|$ .

In Theorem 6.13 [10] there are conditions which are sufficient for compactness of integral operators in equations of the system (9) in the space  $C(\overline{\mathbb{R}})$ .

To formulate such conditions, consider the conformal mapping  $\sigma(T)$  of the unit disk  $\{T \in \mathbb{C} : |T| < 1\}$  onto the domain  $D_z$  such that  $\tau(t) = \sigma\left(\frac{t-i}{t+i}\right)$  for all  $t \in \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$ .

Thus, it follows from Theorem 6.13 [10] that if the conformal mapping  $\sigma(T)$  have the nonvanishing continuous contour derivative  $\sigma'(T)$  on the unit circle  $\Gamma := \{T \in \mathbb{C} : |T| = 1\}$ , and its modulus of continuity

$$\omega_\Gamma(\sigma', \varepsilon) := \sup_{T_1, T_2 \in \Gamma, |T_1 - T_2| \leq \varepsilon} |\sigma'(T_1) - \sigma'(T_2)|$$

satisfies a condition of the type (5), then the integral operators in the system (9) are compact in the space  $C(\overline{\mathbb{R}})$ .

Let  $\mathcal{D}(\overline{\mathbb{R}})$  denote the class of functions  $g_* \in C(\overline{\mathbb{R}})$  whose moduli of continuity

$$\begin{aligned} \omega_{\mathbb{R}}(g_*, \varepsilon) &= \sup_{\tau_1, \tau_2 \in \mathbb{R}: |\tau_1 - \tau_2| \leq \varepsilon} |g_*(\tau_1) - g_*(\tau_2)|, \\ \omega_{\mathbb{R}, \infty}(g_*, \varepsilon) &= \sup_{\tau \in \mathbb{R}: |\tau| \geq 1/\varepsilon} |g_*(\tau) - g_*(\infty)| \end{aligned}$$

satisfy the Dini conditions

$$\int_0^1 \frac{\omega_{\mathbb{R}}(g_*, \eta)}{\eta} d\eta < \infty, \quad \int_0^1 \frac{\omega_{\mathbb{R}, \infty}(g_*, \eta)}{\eta} d\eta < \infty.$$

Since the functions  $\varphi_1, \varphi_4$  in the expression (6) of a solution of the (1-4)-problem have to satisfy conditions of the type (5), it is necessary to require that corresponding functions  $g_1, g_4$  satisfying the system (9) should belong to the class  $\mathcal{D}(\overline{\mathbb{R}})$ . In the next theorem we state a condition on the conformal mapping  $\sigma(T)$ , under which all solutions of the system (9) satisfy the mentioned requirement.

**Theorem 3.1.** *Let every function  $u_l: \partial D_\zeta \rightarrow \mathbb{R}$ ,  $l \in \{1, 4\}$ , satisfy the condition (5). Let the conformal mapping  $\sigma(T)$  have the nonvanishing continuous contour derivative  $\sigma'(T)$  on the circle  $\Gamma$ , and its modulus of continuity satisfy the condition*

$$\int_0^2 \frac{\omega_\Gamma(\sigma', \eta)}{\eta} \ln \frac{3}{\eta} d\eta < \infty.$$

Then the following assertions are true:

- (i) the system of Fredholm integral equations (9) has the unique solution in  $C(\overline{\mathbb{R}})$ ;
- (ii) all functions  $g_1, g_4 \in C(\overline{\mathbb{R}})$  satisfying the system (9) belong to the class  $\mathcal{D}(\overline{\mathbb{R}})$ , and the corresponding functions  $\varphi_1, \varphi_4$  in (6) satisfy Dini conditions of the type (5).

*Proof.* In such a way as in Theorem 7.2 [10], we establish that all functions  $g_1, g_4 \in C(\overline{\mathbb{R}})$  satisfying the system (9) belong to the class  $\mathcal{D}(\overline{\mathbb{R}})$ . Thus, corresponding functions  $\varphi_1, \varphi_4$  in (6) satisfy Dini conditions of the type (5).

To prove that the system (9) has the unique solution, consider the homogeneous system of equations (9) with  $\tilde{u}_1 = \tilde{u}_4 \equiv 0$ .

Let  $g_1 = g_1^0, g_4 = g_4^0$  be a solution of such a system. Consider the function

$$\varphi_0(\tau) := g_1^0(s)e_1 + g_4^0(s)ie_2, \quad \tau = \tilde{\tau}(s), \quad \forall s \in \mathbb{R}$$

corresponding to this solution.

Then the function, which is defined for all  $\zeta \in D_\zeta$  by the formula

$$\Phi_0(\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi_0(\tau)(\tau - \zeta)^{-1} d\tau, \tag{10}$$

is a solution of the homogeneous (1-4)-problem with  $u_1 = u_4 \equiv 0$  that is corresponding to the homogeneous system of equations (9). Taking into account Theorem 2.2, we have the equality  $\Phi_0(\zeta) = a_1 ie_1 + a_2 e_2$  for all  $\zeta \in D_\zeta$ . Therefore, from the Sokhotski–Plemelj formulas (7), (8) we obtain the following relations:

$$\varphi_0(\zeta_0) = \Phi_0^+(\zeta_0) - \Phi_0^-(\zeta_0) = a_1 ie_1 + a_2 e_2 - \Phi_0^-(\zeta_0) \quad \forall \zeta_0 \in \partial D_\zeta,$$

which imply the equality

$$\Phi_0^-(\zeta_0) = -g_1^0(t)e_1 + a_1 ie_1 + a_2 e_2 - g_4^0(t)ie_2, \quad \zeta_0 = \tilde{\tau}(t), \quad \forall t \in \mathbb{R}. \tag{11}$$

Thus, the function (10) is monogenic in  $\mu\{e_1, e_2\} \setminus \overline{D_\zeta}$  and satisfies the equalities

$$U_1[i\Phi_0^-(\zeta_0)] = -a_1, \quad U_4[i\Phi_0^-(\zeta_0)] = a_2 \quad \forall \zeta_0 \in \partial D_\zeta.$$

Now, taking into account Theorem 2.2, which is undoubtedly true for unbounded domain  $\mu\{e_1, e_2\} \setminus D_\zeta$ , one can easily conclude that  $\Phi_0$  is constant in  $\mu\{e_1, e_2\} \setminus D_\zeta$ . Moreover,  $\Phi_0(\zeta) \equiv 0$  in  $\mu\{e_1, e_2\} \setminus D_\zeta$  because the function (10) is vanishing at infinity. At last, after a substitution of 0 into the left-hand side of equality (11) we conclude that  $g_1^0 = g_4^0 \equiv 0$ , i.e. the homogeneous system of equations (9) has only the trivial solution.

Therefore, by the Fredholm theory, the non-homogeneous system of equations (9) has a unique solution. □

## 4 A displacements-type boundary value problem

A relation between the (1-4)-problem and a displacements-type boundary value problem of the plane elasticity theory is considered in [17, 18].

Consider the following displacements-type boundary value  $(u_x, v_y)$ -problem: to find in a domain  $D$  of the Cartesian plane  $xOy$  the first order partial derivatives  $\mathcal{V}_1 := \frac{\partial u}{\partial x}, \mathcal{V}_2 := \frac{\partial v}{\partial y}$  for displacements  $u = u(x, y), v = v(x, y)$  of an elastic isotropic body occupying  $D$ , when their limiting values are given on  $\partial D$ :

$$\lim_{(x,y) \rightarrow (x_0,y_0), (x,y) \in D} \mathcal{V}_k(x, y) = h_k(x_0, y_0) \quad \forall (x_0, y_0) \in \partial D, \quad k = 1, 2,$$

where  $h_k: \partial D \rightarrow \mathbb{R}, k = 1, 2$ , are given functions.

It is well-known the equalities (cf., e.g., [16, p. 5], [22, pp. 8–9]):

$$2\mu \mathcal{V}_k(x, y) = -\mathcal{W}_k(x, y) + \kappa_0 (\mathcal{W}_1 + \mathcal{W}_2) =: C_k(x, y) \quad \forall (x, y) \in D, \quad k = 1, 2, \tag{12}$$

where  $\kappa_0 := \frac{\lambda+2\mu}{2(\lambda+\mu)}$ ,  $\mu$  and  $\lambda$  are Lamé constants (cf., e.g., [22, p. 2]),  $\mathcal{W}_1 := \frac{\partial^2 W}{\partial x^2}$ ,  $\mathcal{W}_2 := \frac{\partial^2 W}{\partial y^2}$ , and  $W: D \rightarrow \mathbb{R}$  is a biharmonic function in the domain  $D$ .

Taking into account the equalities (12), one can conclude that the  $(u_x, v_y)$ -problem is equivalent to finding the functions  $C_k(x, y)$ ,  $k = 1, 2$ , in the domain  $D$ , when their boundary values satisfy the conditions

$$C_k(x_0, y_0) = 2\mu h_k(x_0, y_0) \quad \forall (x_0, y_0) \in \partial D, \quad k = 1, 2. \quad (13)$$

Assuming that  $W$  has continuous partial derivatives till the second order up to the boundary  $\partial D$ , we deduce that in this case  $(u_x, v_y)$ -problem is reduced to finding the functions  $\mathcal{W}_k$ ,  $k = 1, 2$ , in the domain  $D$ , when their limiting values are given on  $\partial D$ .

Consider in the domain  $D_\zeta$  a function  $\Phi_* \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$  for which  $U_1[\Phi_*] = W$ . Then it follows from the Cauchy–Riemann condition (3) that for the biharmonic monogenic function  $\Phi := \Phi_*''$  the following equalities hold:

$$U_1[\Phi(\zeta)] = \mathcal{W}_1(x, y), \quad U_4[\Phi(\zeta)] = \frac{1}{2}(\mathcal{W}_1(x, y) - \mathcal{W}_2(x, y)).$$

Thus, in such a way  $(u_x, v_y)$ -problem is reduced to (1-4)-problem with appropriate boundary conditions, and Theorem 3.1 makes it possible to find the elastic equilibrium for an isotropic body occupying  $D$ . In this case, for finding stresses an additional assumption on the function  $W$  is required, but finding displacements is free of any assumptions (see [18]).

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## References

- [1] Kovalev V. F., Mel'nichenko I. P., Biharmonic functions on the biharmonic plane, Reports Acad. Sci. USSR, ser. A., 1981, No. 8, 25 – 27 (in Russian).
- [2] Mel'nichenko I. P., Biharmonic bases in algebras of the second rank, Ukr. Mat. Zh., 1986, 38(2), 224 – 226 (in Russian); English transl. (Springer) in Ukr. Math. J., 1986, 38(2), 252 – 254.
- [3] Douglis A., A function-theoretic approach to elliptic systems of equations in two variables, Communications on Pure and Applied Mathematics, 1953, 6(2), 259 – 289.
- [4] Sobrero L., Nuovo metodo per lo studio dei problemi di elasticità, con applicazione al problema della piastra forata, Ricerche di Ingegneria, 1934, 13(2), 255 – 264 (in Italian).
- [5] Gryshchuk S. V., Plaksa S. A., Monogenic functions in a biharmonic algebra, Ukr. Mat. Zh., 2009, 61(12), 1587 – 1596 (in Russian); English transl. (Springer) in Ukr. Math. J., 2009, 61(12), 1865 – 1876.
- [6] Gryshchuk S. V., Plaksa S. A., Basic Properties of Monogenic Functions in a Biharmonic Plane, in: “Complex Analysis and Dynamical Systems V”. Contemporary Mathematics, Vol. 591, Amer. Math. Soc., Providence, RI, 2013, 127 – 134.
- [7] Gryshchuk S. V., Plaksa S. A., Monogenic functions in a biharmonic plane (Russian. English summary), Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky, 2009, No. 12, 13 – 20.
- [8] Kovalev V. F., Biharmonic Schwarz problem, Preprint No. 86.16. Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1986 (in Russian).
- [9] Gryshchuk S. V., Plaksa S. A., Schwartz-type integrals in a biharmonic plane, International Journal of Pure and Applied Mathematics, 2013, 83(1), 193 – 211.
- [10] Gryshchuk S. V., Plaksa S. A., Monogenic functions in the biharmonic boundary value problem, Math. Methods Appl. Sci., 2016, 39(11), 2939 – 2952.
- [11] Muskhelishvili N. I., Some basic problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending, Springer Science+Business Media Dordrecht, Originally published by Noordhoff International Publishing in 1977, 1977.
- [12] Gryshchuk S. V., Plaksa S. A., A Schwartz-type boundary value problem, ArXiv preprint /arXiv: arXiv:1610.00436v1 [math.AP], 2016.
- [13] Gryshchuk S. V., Plaksa S. A., A Schwartz-type boundary value problem, Lobachevskii Journal of Mathematics, (in press).
- [14] Bogan Yu. A., On Fredholm integral equations in two-dimensional anisotropic theory of elasticity, Sib. Zh. Vychisl. Mat., 2001, 4(1), 21 – 30 (in Russian).
- [15] Bogan Yu. A., On the Principal Boundary Value Problem in the Two-Dimensional Anisotropic Theory of Elasticity, Journal of Elasticity, 2011, 103(2), 269 – 280.

- [16] Mikhlin S. G., The plane problem of the theory of elasticity, Trans. Inst. of seismology, Acad. Sci. USSR, No. 65, Acad. Sci. USSR Publ. House, Moscow–Leningrad, 1935 (in Russian).
- [17] Gryshchuk S. V.,  $\mathbb{B}$ -valued monogenic functions and their applications to boundary value problems in displacements of 2-D Elasticity, ArXiv preprint /arXiv:1601.01626v1 [math.AP]/, 2016.
- [18] Gryshchuk S. V.,  $\mathbb{B}$ -valued monogenic functions and their applications to boundary value problems in displacements of 2-D Elasticity, in: "Analytic Methods of Analysis and Differential Equations: AMADE 2015", S. V. Rogosin and M. V. Dubatovskaya, *Belarusian State University, Minsk, Belarus (Eds.)*, Cambridge Scientific Publishers Ltd, 45 Margett Street, Cottenham, Cambridge CB24 8QY, UK, 2016 (to appear), ISBN (paperback) 978-1-908106-56-8, 37 – 47.
- [19] Robert P. Gilbert and Wolfgang L. Wendland., Analytic, Generalized, Hyperanalytic Function Theory and an Application to Elasticity, Proceedings of the Royal Society of Edinburgh, Section: A Mathematics, 1975, 73, 317 – 331.
- [20] Tsoi Sun Bon, The Neumann problem for the biharmonic equation, *Differentsial'nye Uravneniya*, 1991, 27(1), 169 – 172 (in Russian).
- [21] Edenhofer J. A., Solution of the Biharmonic Dirichlet Problem by means of Hypercomplex Analytic Functions, in: "Functional Theoretic Methods for Partial Differential Equations" (Proc. of Intern. Symposium Held at Darmstand, Germany, April 12 – 15, 1976), Ser. Lecture Notes in Mathematics, Vol. 561, V. E. Meister, W.L. Wendland, N. Weck (Eds.), Springer-Verlag, 1976, 192 – 202.
- [22] Lu J., *Complex Variable Methods in Plane Elasticity /Series in Pure Mathematics. Vol. 22/*, World Scientific, Singapore, 1995.